MATH 5453

Foundation of Fluid Dynamics

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Monday 12 noon Emmanuel Center SR7

Tuesday 12 noon " ——————— "

Wednesday 2 pm EC Stoner SR(8.60) —— red route - staircase 4 → down 2 floors room behind

- Homework theory 20%
- Homework numerics 20%
- Homework experimental 20%
- Exam 40%

3 Homework sheets

?

Chapter I: Continuum hypothesis and kinematics

Fluid: substance that flows and that conforms to the shape of its container.

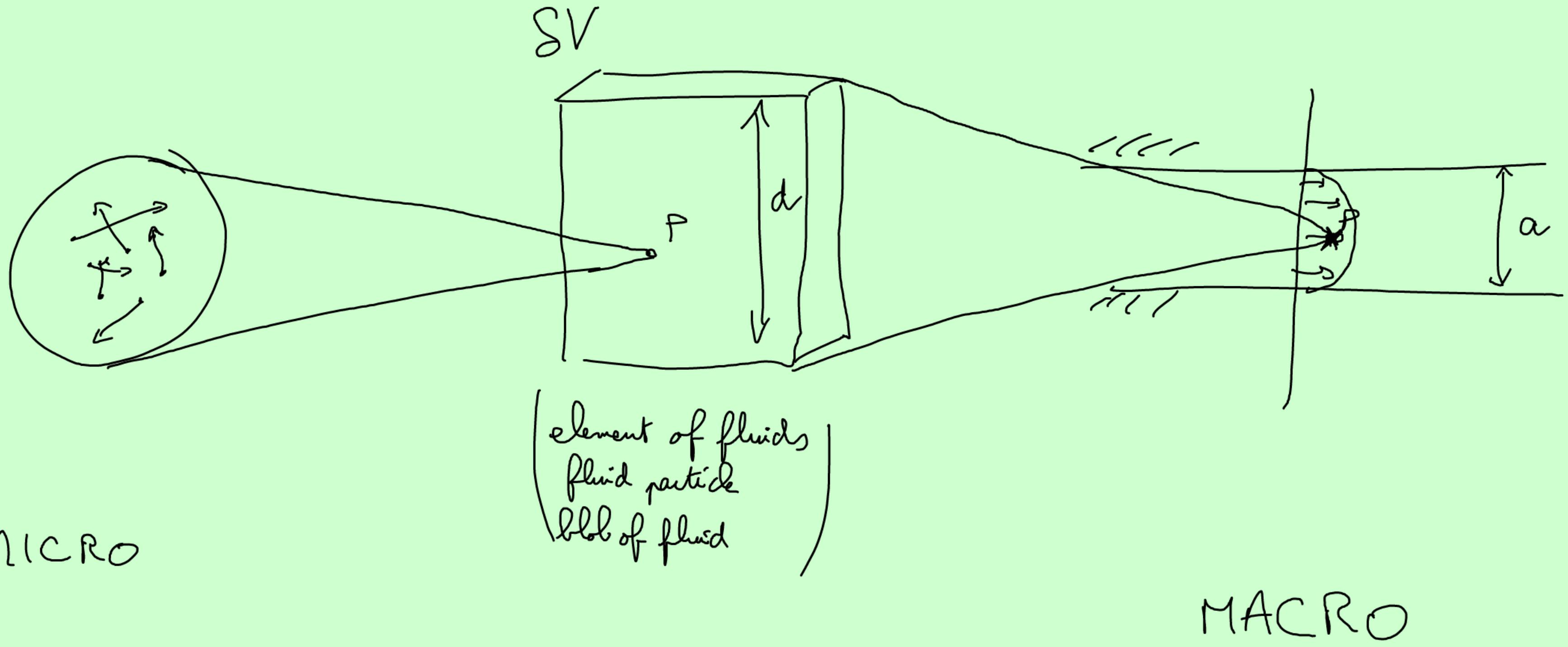
Dynamics: study of forces and torques when there is motion

Continuum hypothesis

$1\,cm^3$ of water contains 10^{23} particles

size of the particle : $1\,\text{Å} = 10^{-10}\,m$

mean free path $l_p = 10^{-8}\,m$



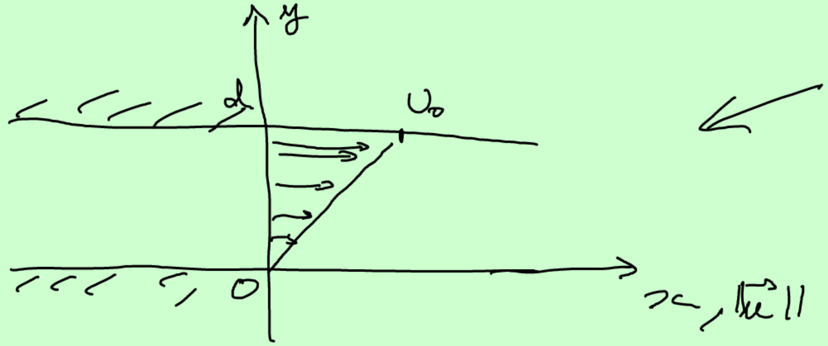
CONTINUUM HYPOTHESIS

$l_p \ll d \;,\; d \ll a$

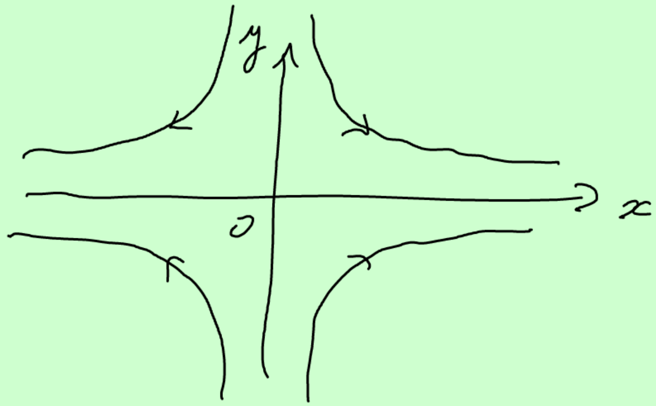
$$\vec{u}(P,t) = \frac{\sum_{\text{particles in } \delta V} \vec{u}_{\text{particle}}}{\sum_{\text{particle in } \delta V} 1} \qquad \underbrace{\rho}_{\text{density}}(P,t) = \frac{\sum_{\text{particles in } \delta V} m_{\text{particles}}}{\delta V}$$

Simple flows

- Plane Couette flow : $\vec{u} = \frac{U_0 y}{d} \vec{e_x}$



- Stagnation flow : $\vec{u} = E x \vec{e_x} - E y \vec{e_y}$, $E > 0$



Time derivatives

- $\frac{\partial f}{\partial t}(\vec{x}, t)$: rate of change of f with time at $\vec{x}$ EULER
- $\frac{df}{dt}(\vec{x}(t), t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$

$$= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

$$= \frac{\partial f}{\partial t} + \underbrace{(\vec{u}\cdot\vec{\nabla})}_{\text{transport}} f$$: rate of change of f with time as we follow a particle passing through $\vec{x}$ at time t. LAGRANGE

Particle paths

(2)

Visualization of the trajectory of a single particle (passive tracer).

Release a particle at $t=0$ and $\vec{x}=\vec{x}_0$

$$\frac{d\vec{x}}{dt} = \vec{u}(\vec{x},t)$$

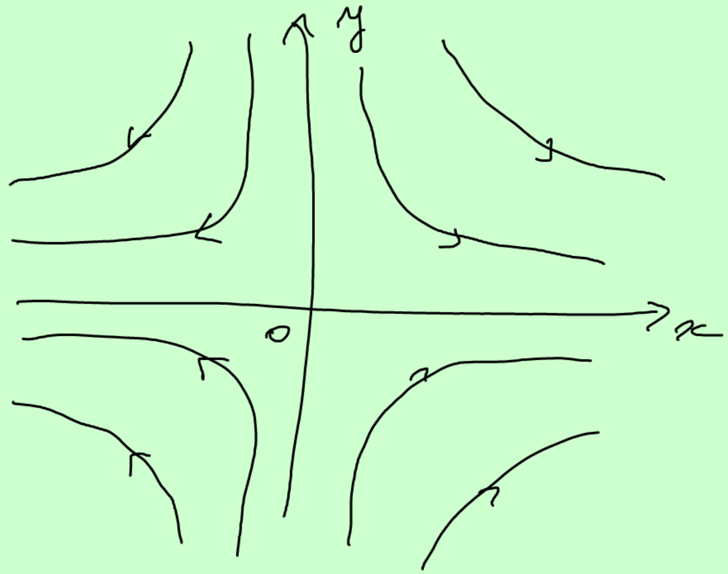
Example: Stagnation flow $\vec{u} = Ex\,\vec{e_x} - Ey\,\vec{e_y}$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = Ex \\ \dfrac{dy}{dt} = -Ey \end{cases}$$

$$\Rightarrow \begin{cases} x = x_0 e^{Et} \\ y = y_0 e^{-Et} \end{cases}$$

$$\Rightarrow xy = x_0 y_0$$

$$\Rightarrow \quad y = \frac{x_0 y_0}{x}$$



<u>Streamline</u> : line everywhere tangent to the velocity

$$d\vec{x} = \vec{u}\, ds$$

element of displacement along a streamline

arclength along the streamline

$$\Rightarrow \quad \frac{d\vec{x}}{ds} = \vec{u}$$

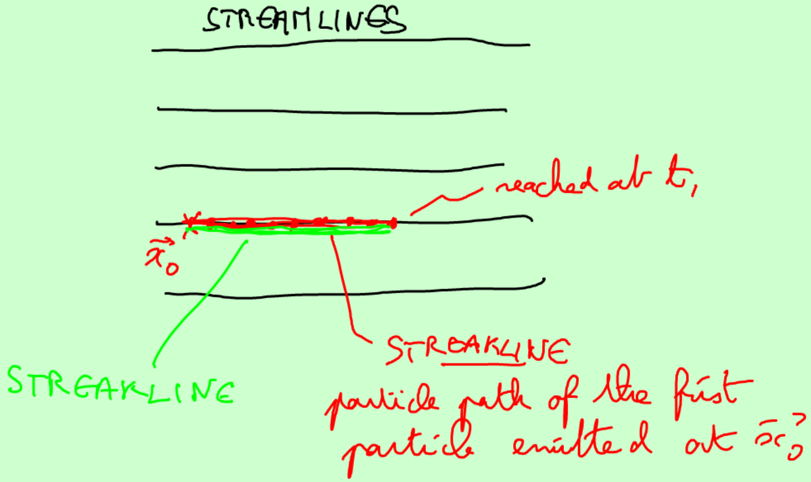
$$\Rightarrow \quad \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = ds$$

If the flow is steady, streamlines and particle paths coincide.

Streakline: Continuous release of dye from $\vec{x} = \vec{x}_0$ from $t = t_0$ to $t = \bar{t}$. The line thereby drawn is a streakline.

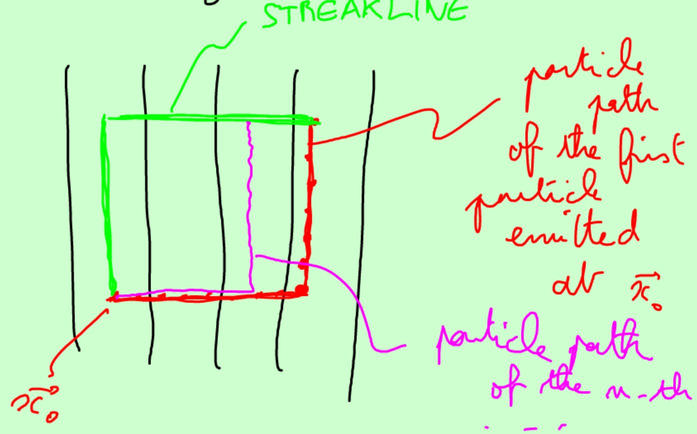
$t < t_1$

$\vec{u} = u\,\vec{e_x} \qquad u > 0$

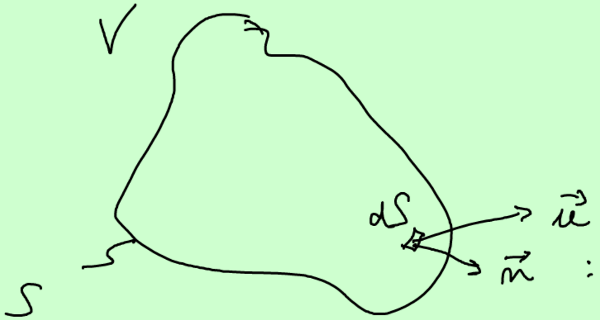


$t > t_1$

$\vec{u} = v\,\vec{e_y} \qquad v > 0$



Chapter 2: Mass conservation



V is a fixed control volume

$\vec{n}$: unit outward pointing normal total mass in V : $M_V = \int_V \rho \, dV$

Rate of change of mass in V : $\dfrac{dM_V}{dt} = \dfrac{d}{dt} \int_V \rho \, dV$

$$= \int_V \frac{\partial \rho}{\partial t} \, dV$$

M_V may change with time due to the flow of fluid through S :

$$\frac{dM_V}{dt} = - \int_S \rho \, \vec{u} \cdot \vec{n} \, dS$$

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t} \, dV = - \int_S \rho \, \underbrace{\vec{u} \cdot \vec{n} \, dS}$$

divergence theorem

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t} \, dV = - \int_V \vec{\nabla} \cdot (\rho \vec{u}) \, dV$$

$$\Rightarrow \int_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) \right] dV = 0$$

We can take V to be the "blob" of fluid, so that the integrand has to vanish pointwise.

$$\Rightarrow \begin{cases} \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0 \\ \underbrace{\dfrac{\partial \rho}{\partial t} + \rho\vec{\nabla}\cdot\vec{u} + (\vec{u}\cdot\vec{\nabla})\rho}\, = 0 \\ \dfrac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{u} = 0 \end{cases}$$

continuity equation
mass conservation

$\bar{\bar{B}}$ $\bar{\bar{A}}$

$\underline{\underline{u}}$

Incompressible fluid: a fluid which has constant density ③

$$\frac{D\rho}{Dt} = 0 \Rightarrow \rho\vec{\nabla}\cdot\vec{u} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{u} = 0}$$ incompressibility {condition / constraint}

We know that $\vec{\nabla}\cdot(\underbrace{\vec{\nabla}\times\vec{\varphi}}_{\vec{u}}) = 0$

$$\Rightarrow \vec{u} = \vec{\nabla}\times\vec{\varphi}$$ streamfunction

<u>Examples in two dimensions</u>: $\vec{u} = u(x,y)\,\vec{e_x} + v(x,y)\,\vec{e_y}$

$$\hookrightarrow \vec{\varphi} = \varphi(x,y)\,\vec{e_z}$$

$$u = \frac{\partial \varphi}{\partial y} \qquad v = -\frac{\partial \varphi}{\partial x}$$

$$\vec{\nabla}\times\vec{\varphi} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix}$$

streamline : line where ψ is constant

- $d\psi = 0 \Rightarrow \dfrac{\partial \psi}{\partial x} dx + \dfrac{\partial \psi}{\partial y} dy = 0$

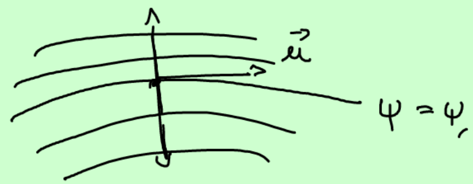
$\Rightarrow -v\, dx + u\, dy = 0$

$\Rightarrow \vec{u} \times \vec{dl} = \vec{0} \qquad \vec{dl} = dx\, \vec{e_x} + dy\, \vec{e_y}$

element of displacement along the streamline

$\Rightarrow \vec{u} \mathbin{/\!/} \vec{dl}$ (streamline)

- $\vec{u} \cdot \vec{\nabla}\psi = u \dfrac{\partial \psi}{\partial x} + v \dfrac{\partial \psi}{\partial y}$



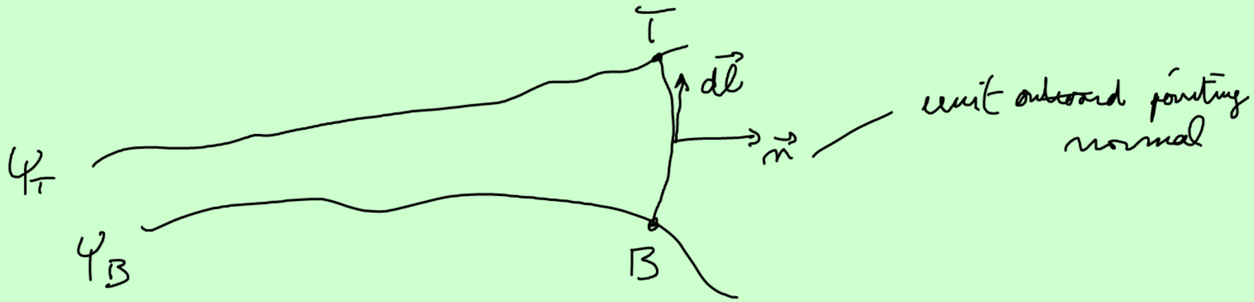
$= -uv + vu$

$= 0$

$\Rightarrow \vec{u} \perp$ direction of fastest increase of ψ

$\Rightarrow \vec{u}$ tangent to the streamline

flux between streamlines :



$$Q = \int_B^T \vec{u}\cdot\underbrace{\vec{n}\,ds}$$

$\vec{dl} = dx\,\vec{e_x} + dy\,\vec{e_y}$
element of displacement
along the (arbitrary)
TB line

$$\vec{n}\,ds = dy\,\vec{e_x} - dx\,\vec{e_y}$$

$$= ds\left(\frac{\partial y}{\partial s}\vec{e_x} - \frac{\partial x}{\partial s}\vec{e_y}\right)$$

$$\Big\downarrow$$

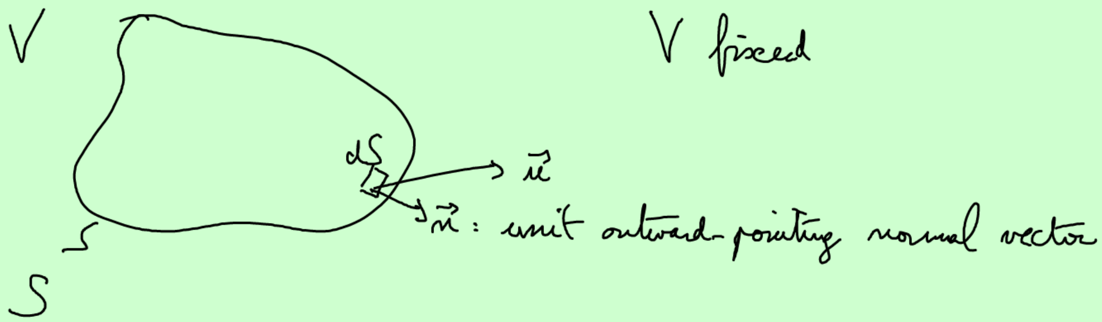
$$Q = \int_B^T \left[u\frac{\partial y}{\partial s} - v\frac{\partial x}{\partial s}\right] ds$$

$$= \int_B^T \left[\frac{\partial \psi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial s}\right] ds$$

$$= \int_{\psi_B}^{\psi_T} d\psi$$

$$= \psi_T - \psi_B$$

Chapter 3: The Navier-Stokes equation



V fixed

momentum density: $\vec{q} = \rho \vec{u}$

Total momentum in V: $\int_V \rho \vec{u} \, dV$

Rate of change of the total momentum in V with time: $\dfrac{d}{dt} \int_V \rho \vec{u} \, dV = \int_V \dfrac{\partial (\rho \vec{u})}{\partial t} \, dV$

$$\underbrace{\begin{pmatrix}\text{rate of change of}\\ \text{momentum in } V\end{pmatrix}}_{\vec{\mathrm{I}}} = \underbrace{\begin{pmatrix}\text{flow of momentum}\\ \text{through } S\end{pmatrix}}_{\vec{\mathrm{II}}} + \underbrace{\begin{pmatrix}\text{net force acting}\\ \text{on } V\end{pmatrix}}_{\vec{\mathrm{III}}}$$

↑ Newton second law

$$\vec{I} = \int_V \frac{\partial}{\partial t}(\rho\vec{u})\,dV \longrightarrow I_i = \int_V \frac{\partial}{\partial t}(\rho u_i)\,dV$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\vec{II} = -\int_S \vec{q}\,\vec{u}\cdot\vec{n}\,dS \longrightarrow II_i = -\int_S q_i\, u_j n_j\, dS$$

$$-\int_S \rho\,\vec{u}\,\vec{u}\cdot\vec{n}\,dS \qquad = -\int_S \rho\, u_i u_j n_j\, dS$$

divergence theorem

$$= -\int_V \vec{\nabla}\cdot(\rho\vec{u}\vec{u})\,dV \qquad = -\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)\,dV$$

4

$$\vec{III} = \int_V \vec{F}_{body}\,dV + \int_S \vec{F}_{local}\,dS$$

force densities

$$III_i = \int_V F_{body_i}\,dV + \int_S F_{local_i}\,dS$$

$$F_{body_i} = \rho\, g_i + \dots$$

$$F_{local_i} = \tau_{ji}\, n_j \longrightarrow \int_S F_{local_i}\,dS = \int_S \tau_{ji}\, n_j\, dS$$

$$= \int_V \frac{\partial}{\partial x_j} \tau_{ji} \, dV \quad \text{(by the divergence theorem)}$$

$\bar{\bar{\tau}}$ = total stress tensor

$$\int_S \overrightarrow{F_{local}} \, dS = \int_V \vec{\nabla} \cdot \bar{\bar{\tau}} \, dV$$

$$\Rightarrow \int_V \frac{\partial}{\partial t}(\rho u_i) \, dV = - \int_V \frac{\partial}{\partial x_j}(\rho u_i u_j) \, dV + \int_V \rho g_i \, dV + \int_V \frac{\partial}{\partial x_j} \tau_{ji} \, dV$$

$$\Rightarrow \frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho g_i + \frac{\partial}{\partial x_j} \tau_{ji}$$

because we can take V as small as the "blob" of fluid and therefore, the integrand has to vanish pointwise.

$$\Rightarrow \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} + \underbrace{u_i \frac{\partial \rho}{\partial t} + u_i \frac{\partial(\rho u_j)}{\partial x_j}}_{=0 \text{ because mass is conserved}} = \rho g_i + \frac{\partial}{\partial x_j} \tau_{ji}$$

$$\Rightarrow \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho g_i + \frac{\partial \tau_{ji}}{\partial x_j}$$

$$\boxed{\rho \frac{\partial \vec{u}}{\partial t} + \rho \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \rho \vec{g} + \vec{\nabla} \cdot \bar{\bar{\tau}}}$$

equation of motion
conservation of momentum

Parenthesis on the velocity gradient: velocity gradient tensor $\bar{K}$

$$K_{ij} = \frac{\partial u_i}{\partial x_j}$$

$$= \underbrace{E_{ij}}_{\text{symmetric}} + \underbrace{\Omega_{ij}}_{\text{antisymmetric}}$$

symmetric: $E_{ij} = E_{ji}$, strain rate tensor

antisymmetric: $\Omega_{ij} = -\Omega_{ji}$, vorticity tensor

$$E_{ij} = \frac{1}{2} \left(K_{ij} + K_{ji} \right)$$

$$\Omega_{ij} = \frac{1}{2} \left(K_{ij} - K_{ji} \right)$$

Example in 2D: $\vec{u} = u\, \vec{e_x} + v\, \vec{e_y}$

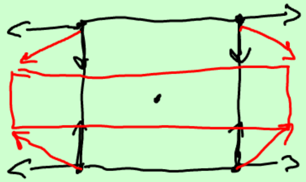
incompressible $\partial_x u + \partial_y v = 0$

$$\bar{\bar{K}} = \begin{pmatrix} \underline{\partial_x u} & \underline{\partial_y u} \\ \underline{\partial_x v} & -\partial_x u \end{pmatrix}$$

3 unknowns

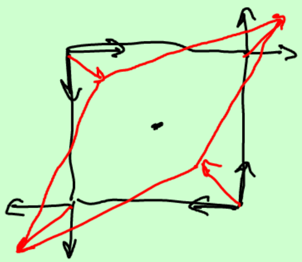
$$\bar{\bar{E}} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \qquad \bar{\bar{\Omega}} = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$$

- $\bar{\bar{E}}_a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \partial_x u = 1, \quad \partial_y v = -1$



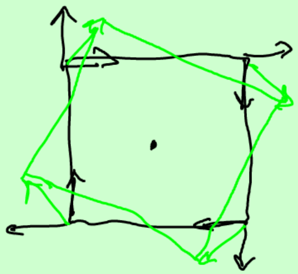
DEFORMATION

- $\bar{\bar{E}}_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \partial_y u = 1, \quad \partial_x v = 1$



DEFORMATION

- $\bar{\bar{\Omega}}_c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad \partial_y u = 1, \quad \partial_x v = -1$



SOLID BODY ROTATION

velocity gradient $\bar{\bar{K}}$ — $\bar{\bar{E}}$ symmetric, strain rate tensor $\rightarrow$ deformations (force)

— $\bar{\bar{\Omega}}$ anti-symmetric, vorticity tensor $\rightarrow$ solid body rotations (torques)

Vorticity: $\vec{\omega} = \vec{\nabla} \times \vec{u}$

$$\omega_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} u_k \implies \varepsilon_{ilm}\, \omega_i = \varepsilon_{ilm}\, \varepsilon_{ijk} \frac{\partial}{\partial x_j} u_k$$

$$= \left(\delta_{lj}\, \delta_{mk} - \delta_{lk}\, \delta_{mj} \right) \frac{\partial}{\partial x_j} u_k$$

$$= \frac{\partial u_m}{\partial x_l} - \frac{\partial u_l}{\partial x_m}$$

$$= 2\, \Omega_{ml}$$

$$\implies \Omega_{ml} = \frac{1}{2} \varepsilon_{ilm}\, \omega_i$$

$$= \frac{1}{2} \varepsilon_{klm}\, \omega_k$$

$$\implies \Omega_{ij} = \frac{1}{2} \varepsilon_{kji}\, \omega_k$$

$$= -\frac{1}{2} \varepsilon_{ijk}\, \omega_k$$

$$\bar{\bar{\Omega}} = \left(\quad \vec{\omega}\,! \quad \right)$$

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \vec{\nabla}) \vec{u} = \rho \vec{g} + \vec{\nabla} \cdot \bar{\bar{\tau}}$$

total stress tensor

Constitutive equations:

* Ideal fluid: $\vec{F}_{local} = -P\vec{n}$

total pressure

$$\Rightarrow F_{local_i} = \underbrace{\tau_{ji} n_j = -P n_i}$$

$$\Rightarrow \tau_{ji} = -P \delta_{ij}$$

$$\bar{\bar{\tau}} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

$$\Rightarrow \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial}{\partial x_j} u_i = \rho g_i + \frac{\partial}{\partial x_j}(-P\delta_{ji})$$

$$\Rightarrow \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial}{\partial x_j} u_i = \rho g_i - \frac{\partial P}{\partial x_i}$$

$$\boxed{\rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \vec{\nabla}) \vec{u} = \rho \vec{g} - \vec{\nabla} P}$$ Euler equation

+ incompressibility

* Newtonian fluids: • fluid for which the relationship between strain rate and stress is linear
• isotropic fluid

$$\rightarrow \underbrace{\tau_{ji}}_{\text{total stress tensor}} = -P\,\delta_{ij} + \underbrace{\sigma_{ji}}_{\text{viscous stress tensor}}$$

$$\sigma_{ji} = 2\underbrace{\mu}_{\text{dynamic viscosity}} \Big[\underbrace{\epsilon_{ij}}_{\text{strain rate tensor}} - \frac{1}{3}\,\delta_{ij}\, \underbrace{\frac{\partial u_k}{\partial x_k}}_{0 \text{ incompressible fluid}} \Big] + \cdots$$

dynamic viscosity — not to be confused with $\nu = \frac{\mu}{\rho}$ kinematic viscosity

$$\boxed{\sigma_{ji} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

$$\boxed{\tau_{ji} = -P\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

$$\Rightarrow \rho\frac{\partial u_i}{\partial t} + \rho\, u_j \frac{\partial u_i}{\partial x_j} = \rho g_i - \frac{\partial P}{\partial x_i} + \overbrace{\mu\left(\frac{\partial^2 u_i}{\partial x_j^2} + \underbrace{\frac{\partial^2 u_j}{\partial x_i \partial x_j}}_{=0 \text{ by incompressibility}}\right)}^{\text{constant}}$$

$$\Rightarrow \quad \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho g_i - \frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

$$\boxed{\rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \vec{\nabla}) \vec{u} = \rho \vec{g} - \vec{\nabla} P + \mu \vec{\nabla}^2 \vec{u}}$$ Navier-Stokes equation

+ incompressibility ($\vec{\nabla} \cdot \vec{u} = 0$)

Different kinds of pressure: no flow, no forces (except the weight) $\longrightarrow \vec{0} = \rho \vec{g} - \vec{\nabla} P_H$

$$\Rightarrow P_H = \rho \vec{g} \cdot \vec{x} + P_0$$

hydrostatic pressure — reference pressure

We can write $P = P_H + p$

total pressure — hydrostatic pressure — dynamic pressure

The Navier-Stokes equation becomes:

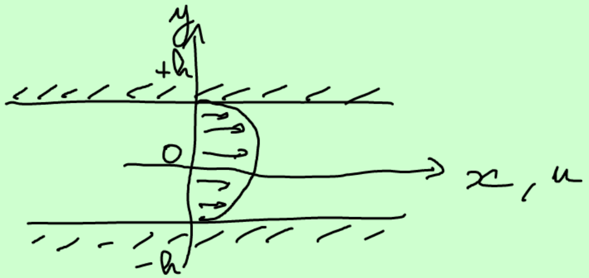
$$\boxed{\rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \vec{\nabla}) \vec{u} = - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u}}$$

Chapter 4: Some exact solutions of the Navier–Stokes equation

Plane Poiseuille flow

POISEUILLE

(0) Make a sketch



$\leftarrow \quad \vec{\nabla} p = -G\,\vec{e}_x \quad , \quad G > 0$

(1) Equations:

2D Navier–Stokes :

$$\begin{cases} \rho \cancel{\dfrac{\partial u}{\partial t}} + \rho\, u \dfrac{\partial u}{\partial x} + \cancel{\rho\, v \dfrac{\partial u}{\partial y}} = -\dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u}{\partial x^2} + \mu \dfrac{\partial^2 u}{\partial y^2} \\[2ex] \cancel{\rho \dfrac{\partial v}{\partial t}} + \rho\, u \cancel{\dfrac{\partial v}{\partial x}} + \rho\, v \cancel{\dfrac{\partial v}{\partial y}} = -\dfrac{\partial p}{\partial y} + \mu \cancel{\dfrac{\partial^2 v}{\partial x^2}} + \mu \cancel{\dfrac{\partial^2 v}{\partial y^2}} \end{cases}$$

Incompressibility :

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} = 0$$

(2) Boundary conditions: no-slip :

$$\begin{cases} y = -h & u = 0 \ , \ v = 0 \\ y = +h & u = 0 \ , \ v = 0 \end{cases}$$

impenetrability condition

(3) Hypotheses : * one-dimensional flow : $v \equiv 0$

* steady flow : $\dfrac{\partial}{\partial t} = 0$

(c) Solution:

$$\begin{cases} \cancel{\rho u \dfrac{\partial u}{\partial x}} = -\dfrac{\partial p}{\partial x} + \cancel{\mu \dfrac{\partial^2 u}{\partial x^2}} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ 0 = -\dfrac{\partial p}{\partial y} \\ \dfrac{\partial u}{\partial x} = 0 \end{cases}$$

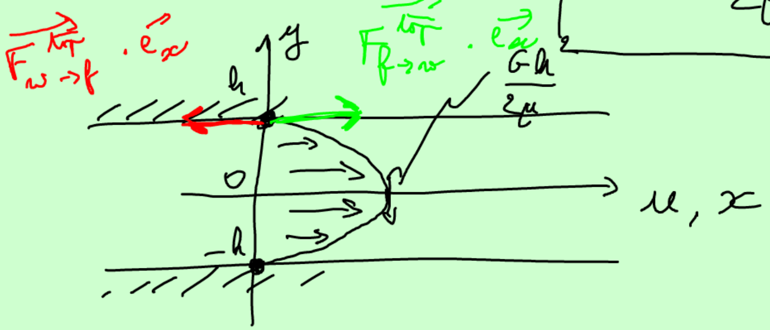
$$\begin{cases} 0 = -\dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ 0 = -\dfrac{\partial p}{\partial y} \longrightarrow p(x) \\ \dfrac{\partial u}{\partial x} = 0 \longrightarrow u(y) \end{cases}$$

$$\underbrace{\frac{\partial p}{\partial x}}_{f(x)} = \underbrace{\mu \frac{\partial^2 u}{\partial y^2}}_{f(y)} = -G \ , \quad G > 0$$

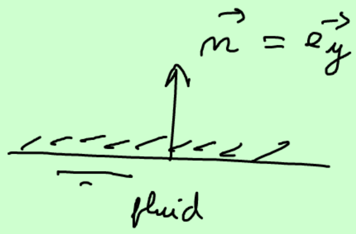
$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{G}{\mu}$$

$$\Rightarrow u = -\frac{G}{2\mu} y^2 + k_1 y + k_2$$

BC: $y = \pm h$, $u = 0$ $\longrightarrow$ $\boxed{u = \dfrac{G}{2\mu}\left(h^2 - y^2\right)}$



force density exerted by the wall onto the fluid at $y = +h$



$$\vec{F}^{top}_{w \to f} = \vec{n} \cdot \bar{\bar{\tau}} \Big|_{y=+h}$$

$$= \vec{e_y} \cdot \bar{\bar{\tau}} \Big|_{y=+h}$$

$$= \tau_{yx}\Big|_{y=+h} \vec{e_x} + \tau_{yy}\Big|_{y=+h} \vec{e_y}$$

$$= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\Big|_{y=+h} \vec{e_x} + \left[-P + \mu\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right)\right]\Big|_{y=+h} \vec{e_y}$$

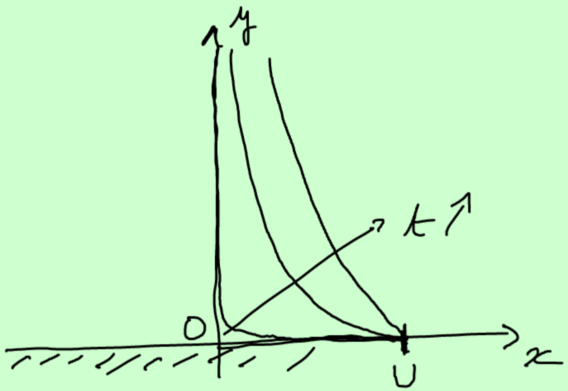
$$= \mu \frac{\partial u}{\partial y}\Big|_{y=+h} \vec{e_x} - P\,\vec{e_y}$$

$$= \underline{-Gh}\,\vec{e_x} - P\,\vec{e_y}$$

$$\vec{F}^{top}_{f \to w} = -\vec{F}^{top}_{w \to f}$$

Stokes first problem

(0)



(1)
$$\begin{cases} \rho \dfrac{\partial u}{\partial t} + \rho u \dfrac{\partial u}{\partial x} + \cancel{\rho v \dfrac{\partial u}{\partial y}} = -\dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u}{\partial x^2} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ \rho \cancel{\dfrac{\partial v}{\partial t}} + \rho u \cancel{\dfrac{\partial v}{\partial x}} + \rho v \cancel{\dfrac{\partial v}{\partial y}} = -\dfrac{\partial p}{\partial y} + \mu \cancel{\dfrac{\partial^2 v}{\partial x^2}} + \mu \cancel{\dfrac{\partial^2 v}{\partial y^2}} \\ \dfrac{\partial u}{\partial x} + \cancel{\dfrac{\partial v}{\partial y}} = 0 \end{cases}$$

(2) BC: $y = 0$, $u = U$, $v = 0$

$y \to \infty$, $u \to 0$, $v \to 0$

IC: $t = 0$, $u = 0$, $v = 0$

(3) Hypothesis: * one-dimensional : $v \equiv 0$

* no force resulting from the dynamic pressure : $\vec{\nabla} p = \vec{0}$

(4) Solution :
$$\begin{cases} \rho \dfrac{\partial u}{\partial t} + \cancel{\rho u \dfrac{\partial u}{\partial x}} = -\dfrac{\partial p}{\partial x} + \cancel{\mu \dfrac{\partial^2 u}{\partial x^2}} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ 0 = -\dfrac{\partial p}{\partial y} \longrightarrow p(x, t) \\ \dfrac{\partial u}{\partial x} = 0 \longrightarrow u(y, t) \end{cases}$$

$$\Rightarrow \rho \frac{\partial u}{\partial t} = \cancel{-\frac{\partial p}{\partial x}} + \mu \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}}$$

$\nu = \dfrac{\mu}{\rho}$ kinematic viscosity heat equation

diffusion equation

2 methods : • separation of variables $u(y,t) = Y(y)T(t)$ ...

• <u>similarity solution</u>

similarity variable: $\eta = y\, t^{a}$

$u(y,t) = f(\eta)$

(7)

$$\left.\begin{aligned}
\partial_t &= \partial_\eta \frac{\partial \eta}{\partial t} = a\, y\, t^{a-1}\, \partial_\eta \\
\partial_y &= \partial_\eta \frac{\partial \eta}{\partial y} = t^{a}\, \partial_\eta \\
\partial_{yy} &= \partial_y\left(\partial_y\right) = t^{2a}\, \partial_{\eta\eta}
\end{aligned}\right\}
\quad \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \Rightarrow a\, y\, t^{a-1} \underset{\frac{\partial f}{\partial \eta}}{f'} = \nu\, t^{2a} f''$$

$$\Rightarrow f'' = \frac{a\, y\, t^{-a-1}}{\nu} f'$$

For this equation to be an equation of η only, we need $t^{a} = t^{-a-1} \Rightarrow a = -\frac{1}{2}$

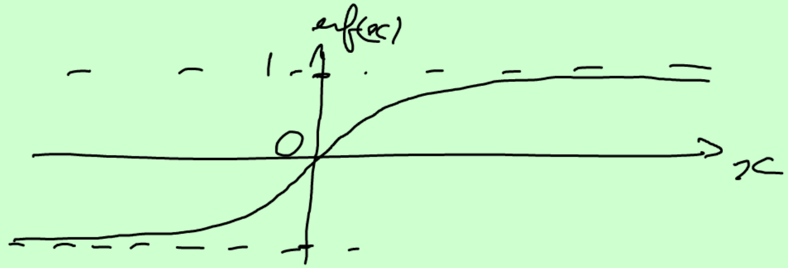
$$\Rightarrow \eta = y\, t^{-1/2} = \frac{y}{\sqrt{t}}$$

$$f'' = -\frac{\eta}{2\nu} f'$$

$$\Rightarrow f' = k_1 \exp\left(-\frac{\eta^2}{4\nu}\right)$$

$$\Rightarrow f = k_1 \underbrace{\int_0^{\eta} \exp\left(-\frac{\xi^2}{4\nu}\right) d\xi} + k_2$$

Definition of the error function: $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^{x} \exp(-y^2)\, dy$



$$y = \frac{\xi}{2\sqrt{\nu}} \quad \Rightarrow \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{2\sqrt{\nu}x} \exp\left(-\frac{\xi^2}{4\nu}\right) \frac{d\xi}{2\sqrt{\nu}}$$

$$= \frac{1}{\sqrt{\nu\pi}} \int_0^{2\sqrt{\nu}x} \exp\left(-\frac{\xi^2}{4\nu}\right) d\xi$$

$$\eta = 2\sqrt{\nu}\, x \quad \Rightarrow \quad \operatorname{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) = \frac{1}{\sqrt{\nu\pi}} \int_0^{\eta} \operatorname{erf}\left(-\frac{\xi^2}{4\nu}\right) d\xi$$

$$\Rightarrow f = k_1 \sqrt{\nu\pi}\, \operatorname{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) + k_2$$

$$\Rightarrow \quad f = k_3 \, \mathrm{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) + k_2$$

$$\Rightarrow \quad u(y,t) = k_3 \, \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) + k_2$$

BC: $y = 0 \longrightarrow u = U \quad \Rightarrow k_2 = U$

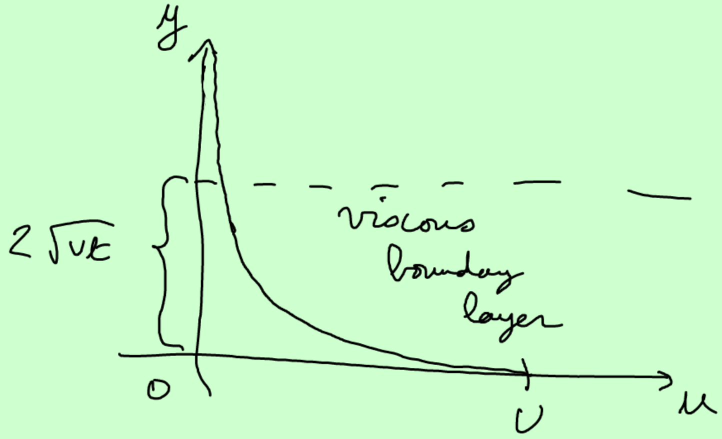
$y \to \infty \longrightarrow u \to 0 \quad \Rightarrow k_3 + k_2 \longrightarrow 0$

$\Rightarrow k_3 \to -U$

$$\Rightarrow \quad u(y,t) = U\left[1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right]$$

IC: $t \to 0 \quad u \to 0$ this is compatible with the above expression

$\Rightarrow$ the similarity condition is valid!

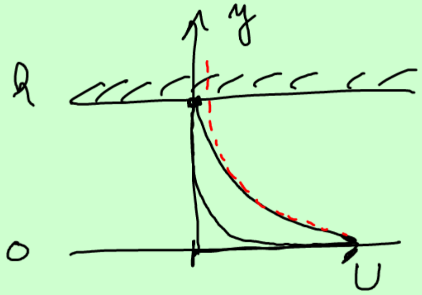


$u \ll U \quad \Rightarrow \frac{y}{2\sqrt{\nu t}} \gg 1$

$\Rightarrow y \gg 2\sqrt{\nu t}$

$\mathrm{erf}(2) \simeq 0.99 \quad \Rightarrow \quad u \lesssim 0.01\,U$ for $y \geq \underbrace{4\sqrt{\nu t}}_{\delta_{99}}$

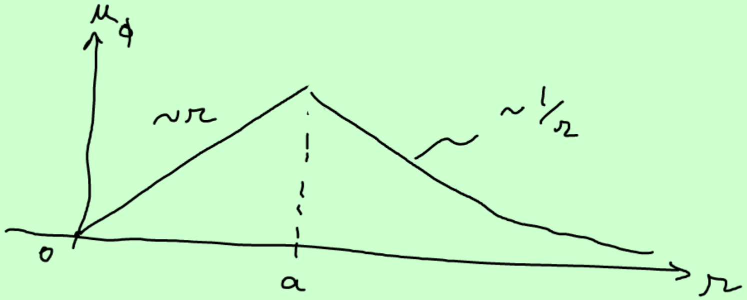
boundary layer thickness after the 99% criterion.



Chapter 5: Vorticity

$\vec{\omega} = \vec{\nabla} \times \vec{u}$

The Rankine vortex



$\vec{u} = k r \, \vec{e_\phi}$ for $r \leq a$

$\vec{u} = \frac{k a^2}{r} \vec{e_\phi}$ for $r > a$

vorticity: $\vec{\omega} = \vec{\nabla} \times \vec{u}$

$$= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \partial_r (\underbrace{r u_\phi}) \end{pmatrix}$$

$$= \begin{cases} 2k \, \vec{e_z} & \text{for } r \leq a \\ \vec{0} & \text{for } r > a \end{cases}$$

$\omega_0 = 2k$

8

$$\vec{\omega} = \vec{0}$$

Rankine vortex: $\vec{u} = \begin{cases} \dfrac{\omega_0 r}{2}\,\vec{e}_\phi & \text{for } r \le a \\[2mm] \dfrac{\omega_0 a^2}{2r}\,\vec{e}_\phi & \text{for } r > a \end{cases}$

Interesting calculation: calculate the pressure in the Rankine vortex

$$\vec{NS}\cdot\vec{e}_r : \quad \rho\,\frac{u_\phi^2}{r} = \frac{\partial p}{\partial r}$$

<u>Line/point vortex:</u>

Around the Rankine vortex: $\displaystyle\int_S \vec{\omega}\cdot\vec{n}\,dS = \pi a^2 \omega_0$

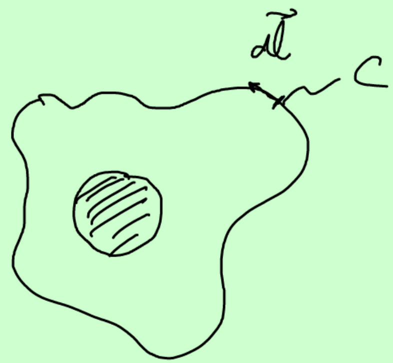
S – surface large enough to contain the Rankine vortex

By Stokes theorem: $\displaystyle\int_S \vec{\omega}\cdot\vec{n}\,dS = \oint_C \vec{u}\cdot d\vec{l}$

C closed contour corresponding to S

$$\Rightarrow \quad \pi a^2 \omega_0 = \oint_C \vec{u}\cdot d\vec{l} = \Gamma$$

circulation



Taking α to 0 while maintaining Γ constant (increasing ω_0 in proportion) leads to the creation of a point vortex (line vortex in 3D)

$\hookrightarrow$ potential flow theory

The vorticity equation

$$\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\,\vec{u}\right] = -\vec{\nabla}p + \mu\vec{\nabla}^2\vec{u} + \vec{F}$$

$$\begin{aligned}
\vec{u}\times(\vec{\nabla}\times\vec{u})\Big|_i &= \varepsilon_{ijk}\, u_j\, (\vec{\nabla}\times\vec{u})\Big|_k \\
&= \varepsilon_{ijk}\, u_j\, \varepsilon_{klm}\, \frac{\partial}{\partial x_l} u_m \\
&= \varepsilon_{ijk}\, \varepsilon_{klm}\, u_j \frac{\partial}{\partial x_l} u_m \\
&= \varepsilon_{kij}\, \varepsilon_{klm}\, u_j \frac{\partial}{\partial x_l} u_m \\
&= (\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl})\, u_j \frac{\partial}{\partial x_l} u_m \\
&= u_j \frac{\partial}{\partial x_i} u_j - u_j \frac{\partial}{\partial x_j} u_i
\end{aligned}$$

$$= \frac{1}{2} \frac{\partial}{\partial x_i} u_j^2 - u_j \frac{\partial}{\partial x_j} u_i$$

$$\Rightarrow \quad \vec{u} \times (\vec{\nabla} \times \vec{u}) = \frac{1}{2} \vec{\nabla} |\vec{u}|^2 - (\vec{u} \cdot \vec{\nabla}) \vec{u}$$

$$\hookrightarrow \rho \left[\frac{\partial \vec{u}}{\partial t} + \frac{1}{2} \vec{\nabla} |\vec{u}|^2 - \vec{u} \times \vec{\omega} \right] = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u} + \vec{F}$$

$$\Rightarrow \rho \left[\frac{\partial \vec{u}}{\partial t} - \vec{u} \times \vec{\omega} \right] = -\vec{\nabla} \left(p + \frac{1}{2} \rho \vec{u}^2 \right) + \mu \vec{\nabla}^2 \vec{u} + \vec{F}$$

We take the curl:

$$\Rightarrow \rho \left[\frac{\partial \vec{\omega}}{\partial t} - \vec{\nabla} \times (\vec{u} \times \vec{\omega}) \right] = \vec{0} + \mu \vec{\nabla}^2 \vec{\omega} + \vec{0}$$

for $\vec{F} = \vec{\nabla} V$ force deriving from a potential

$$\vec{\nabla} \times (\vec{u} \times \vec{\omega}) \Big|_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\varepsilon_{klm} u_l \omega_m \right)$$

$$= \varepsilon_{kij} \varepsilon_{klm} \frac{\partial}{\partial x_j} (u_l \omega_m)$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial}{\partial x_j} (u_l \omega_m)$$

$$= \frac{\partial}{\partial x_j}\left(u_i \omega_j\right) - \frac{\partial}{\partial x_j}\left(u_j \omega_i\right)$$

$$= u_i \cancel{\frac{\partial \omega_j}{\partial x_j}} + \omega_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial \omega_i}{\partial x_j} - \omega_i \cancel{\frac{\partial u_j}{\partial x_j}}$$

0 incompressible

0 incompressible

$$= \omega_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial \omega_i}{\partial x_j}$$

$$\Rightarrow \quad \vec{\nabla} \times (\vec{u} \times \vec{\omega}) = (\vec{\omega} \cdot \vec{\nabla})\vec{u} - (\vec{u} \cdot \vec{\nabla})\vec{\omega}$$

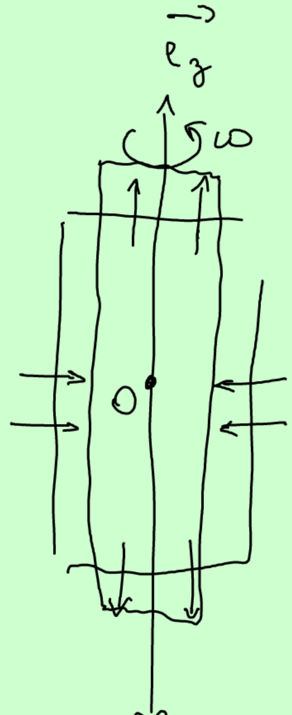
$$\hookrightarrow \quad \rho\left[\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{\omega} - (\vec{\omega} \cdot \vec{\nabla})\vec{u}\right] = \mu \vec{\nabla}^2 \vec{\omega}$$

$$\Rightarrow \boxed{\underbrace{\frac{\partial \vec{\omega}}{\partial t}}_{\text{rate of change}} + \underbrace{(\vec{u} \cdot \vec{\nabla})\vec{\omega}}_{\text{advection of vorticity}} = \underbrace{(\vec{\omega} \cdot \vec{\nabla})\vec{u}}_{\text{vortex stretching}} + \underbrace{\nu \vec{\nabla}^2 \vec{\omega}}_{\text{diffusion}}}$$

Vorticity equation

To understand $(\vec{\omega}\cdot\vec{\nabla})\vec{u}$, we isolate it: $\dfrac{\partial\vec{\omega}}{\partial t} = (\vec{\omega}\cdot\vec{\nabla})\vec{u}$

Align the z-axis with $\vec{\omega}$: $\vec{\omega} = \omega\,\vec{e_z}$



$$\Rightarrow \quad \frac{\partial\omega}{\partial t} = (\vec{\omega}\cdot\vec{\nabla})\,u_z$$

$$= \omega\,\frac{\partial u_z}{\partial z}$$

$$\Rightarrow \quad \frac{1}{\omega}\frac{\partial\omega}{\partial t} = \frac{\partial u_z}{\partial z}$$

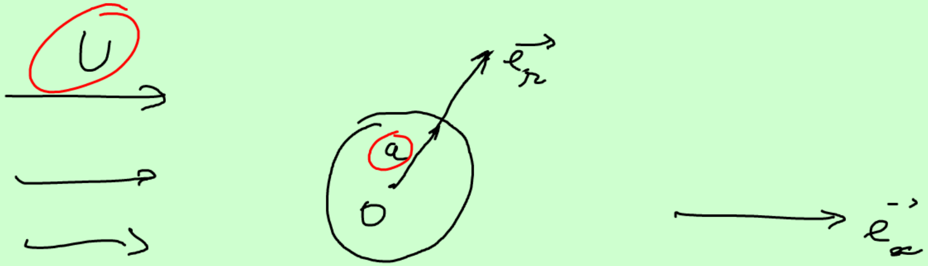
if $\dfrac{\partial u_z}{\partial z} > 0$, $\dfrac{\partial u_r}{\partial r} < 0$

we are sucking in fluid radially
to compensate for the z-expansion

$\Rightarrow$ stretching of the flow in z.

$\dfrac{1}{\omega}\dfrac{\partial\omega}{\partial t} > 0 \Rightarrow |\omega| \nearrow$

Chapter 6 : Dynamic similarity



μ : dynamic viscosity

ρ : density

Navier - Stokes : $\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\,\vec{u}\right] = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u}$

incompressibility : $\vec{\nabla}\cdot\vec{u} = 0$

BC ; no - slip : $\begin{cases} \vec{u}(r=a) = \vec{0} \\ \lim\limits_{r\to\infty} \vec{u} = U\,\vec{e_x} \end{cases}$

IC or look for a steady state ($\Leftarrow$ a strong hypothesis ?)

$\Longrightarrow$ 4 parameters : μ, ρ, a, U

Non-dimensionalization : $\vec{x} = a\,\vec{x}^*$

$\hookrightarrow [x] = L$ $\qquad x = 3.2\,m$

$[a] = L$ $\qquad a = 2\,m$

$[x^*] = \frac{[x]}{[a]} = \frac{L}{L} = 1$ $\qquad x^* = 1.6$

$\vec{u} = U\,\vec{u}^*$

$\hookrightarrow [u] = L.T^{-1}$ $\qquad u = 700\,m.s^{-1}$

$[U] = L.T^{-1}$ $\qquad U = 1000\,m.s^{-1}$

$[u^*] = 1$ $\qquad u^* = 0.7$

$t = \frac{a}{U}\,t^*$ $\qquad \frac{[a]}{[U]} = T$

$p = P\,p^*$ $\qquad [p^*] = 1$

not the total pressure but a pressure scale

Incompressibility : $\vec{\nabla}\cdot\vec{u} = 0 \Rightarrow \frac{1}{a}\vec{\nabla}^*\cdot\left(U\,\vec{u}^*\right) = 0$

$\Rightarrow \frac{U}{a}\,\vec{\nabla}^*\cdot\vec{u}^* = 0$

$$\Rightarrow \quad \vec{\nabla}^* \cdot \vec{u}^* = 0$$

Navier - Stokes :
$$\rho\left[\frac{U}{a}\frac{\partial}{\partial t^*}(U\vec{u}^*) + \frac{U^2}{a}(\vec{u}^*\cdot\vec{\nabla}^*)\vec{u}^*\right] = -\frac{P}{a}\vec{\nabla}^* p^* + \mu\frac{U}{a^2}\vec{\nabla}^{*2}\vec{u}^*$$

$$\Rightarrow \quad \rho\frac{U^2}{a}\left[\frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^*\cdot\vec{\nabla}^*)\vec{u}^*\right] = -\frac{P}{a}\vec{\nabla}^* p^* + \frac{\mu U}{a^2}\vec{\nabla}^{*2}\vec{u}^*$$

$$\Rightarrow \quad \frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^*\cdot\vec{\nabla}^*)\vec{u}^* = -\frac{P}{\rho U^2}\vec{\nabla}^* p^* + \frac{\mu}{\rho U a}\vec{\nabla}^{*2}\vec{u}^*$$

$$[P] = M.L^{-1}.T^{-2}$$

$$\left.\begin{array}{l}[\rho] = M.L^{-3}\\ [U^2] = L^2.T^{-2}\end{array}\right\} [\rho U^2] = M.L^{-1}.T^{-2}$$

$Re = \dfrac{\rho U a}{\mu}$: non-dimensional number quantifying the ratio between inertial and viscous effects.

$$\Rightarrow \quad \frac{\partial u^*}{\partial t^*} + (u^*\cdot\vec{\nabla}^*)\vec{u}^* = -\frac{P}{\rho U^2}\vec{\nabla}^* p^* + \frac{1}{Re}\vec{\nabla}^{*2}\vec{u}^*$$

Two choices for P : • $P = \rho U^2$ inertial scale

$$\hookrightarrow \quad \frac{\partial u^*}{\partial t^*} + (\vec{u}^* \cdot \vec{\nabla}^*)\,\vec{u}^* = -\vec{\nabla}^* p^* + \frac{1}{Re}\,\vec{\nabla}^{*2}\vec{u}^*$$

• $P = \frac{\mu U}{a}$ viscous scale

$$Re\left[\frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^* \cdot \vec{\nabla}^*)\,\vec{u}^*\right] = -\vec{\nabla}^* p^* + \vec{\nabla}^{*2}\vec{u}^*$$

BC: $r^* \to \infty$, $\vec{u}^* = \vec{e}_x$

$r^* = 1$, $\vec{u}^* = \vec{0}$

$Re = \rho \frac{Ua}{\mu} = 100$

say : $Ua = \frac{Re\,\mu}{\rho}$

Chapter 7: Potential flows

Two dimensional flow, irrotational, incompressible

$\vec{u}(x,y) = u(x,y)\,\vec{e_x} + v(x,y)\,\vec{e_y}$

irrotational: $\vec{\nabla} \times \vec{u} = \vec{0}$

incompressible: $\vec{\nabla} \cdot \vec{u} = 0$

First route: (1) incompressible: $\vec{\nabla}\cdot\vec{u} = 0$

↳ streamfunction ψ : $u = \dfrac{\partial \psi}{\partial y}$, $v = -\dfrac{\partial \psi}{\partial x}$

$$\Rightarrow \vec{\nabla}\cdot\vec{u} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y}$$

$$= 0$$

(2) irrotational: $\vec{\nabla}\times\vec{u} = \left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right)\vec{e}_z$

$$= \left(-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right)\vec{e}_z$$

$$= \vec{0}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}^2 \psi = 0}$$

$$\vec{\nabla}\times\vec{u} = \vec{0}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\phi\right) = \vec{0}$$

$$\varepsilon_{ijk}\frac{\partial}{\partial x_j} u_k = 0$$

$$u_k = \frac{\partial}{\partial x_k}\phi$$

Second route: (1) Irrotational: $\vec{\nabla}\times\vec{u} = \vec{0}$

↳ velocity potential ϕ : $u = \dfrac{\partial \phi}{\partial x}$, $v = \dfrac{\partial \phi}{\partial y}$

$$\Rightarrow \vec{\nabla}\times\vec{u} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{e}_z$$

$$= \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y}\right)\vec{e}_z$$

$= \vec{0}$

(2) Incompressible: $\vec{\nabla}\cdot\vec{u} = 0 \Rightarrow \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}^2\phi = 0}$$

The velocity is: $\boxed{\begin{aligned} u &= \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \\ v &= \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \end{aligned}}$ Cauchy - Riemann equations

$$\left.\begin{aligned} \frac{\partial^2\phi}{\partial x^2} &= \frac{\partial^2\psi}{\partial x\partial y} \\ \frac{\partial^2\phi}{\partial y^2} &= -\frac{\partial^2\psi}{\partial x\partial y} \end{aligned}\right\} \Rightarrow \vec{\nabla}^2\phi = 0$$

$$\left.\begin{aligned} \frac{\partial^2\phi}{\partial x\partial y} &= \frac{\partial^2\psi}{\partial y^2} \\ \frac{\partial^2\phi}{\partial x\partial y} &= -\frac{\partial^2\psi}{\partial x^2} \end{aligned}\right\} \Rightarrow \vec{\nabla}^2\psi = 0$$

Complex analysis: $\dfrac{df}{dx} = \lim\limits_{\delta x\to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x}$

$$\frac{df}{dz} = \lim_{|\delta z|\to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$z = x + iy$

has to be the same no matter the direction of δz for f to be complex differentiable.

$$f : \mathbb{C} \longmapsto \mathbb{C}$$

$$f(z) = g(x,y) + i\,h(x,y)$$

$$g : \mathbb{R}^2 \longmapsto \mathbb{R}$$

$$h : \mathbb{R}^2 \longmapsto \mathbb{R}$$

In the δx direction : $\dfrac{df}{dz} = \dfrac{\partial g}{\partial x} + i\dfrac{\partial h}{\partial x}$

In the $i\delta y$ direction : $\dfrac{df}{dz} = \dfrac{\partial g}{i\partial y} + i\dfrac{\partial h}{i\partial y} = \dfrac{\partial h}{\partial y} - i\dfrac{\partial g}{\partial y}$

For f to be complex differentiable : $\dfrac{\partial g}{\partial x} + i\dfrac{\partial h}{\partial x} = \dfrac{\partial h}{\partial y} - i\dfrac{\partial g}{\partial y}$

$$\Rightarrow \begin{cases} \dfrac{\partial g}{\partial x} = \dfrac{\partial h}{\partial y} \\[2ex] \dfrac{\partial h}{\partial x} = -\dfrac{\partial g}{\partial y} \end{cases}$$

Cauchy – Riemann equations

$\left(\Rightarrow \vec{\nabla}^2 g = \vec{\nabla}^2 h = 0 \right)$

f is complex differentiable $\Rightarrow$ its real and imaginary parts are harmonic functions.

We can then introduce a complex potential: $w = \phi + i\psi$

complex potential

velocity potential

streamfunction

w is complex differentiable $\Rightarrow$ ϕ and ψ are harmonic functions

$$\left(\vec{\nabla}^2 \phi = \vec{\nabla}^2 \psi = 0 \right)$$

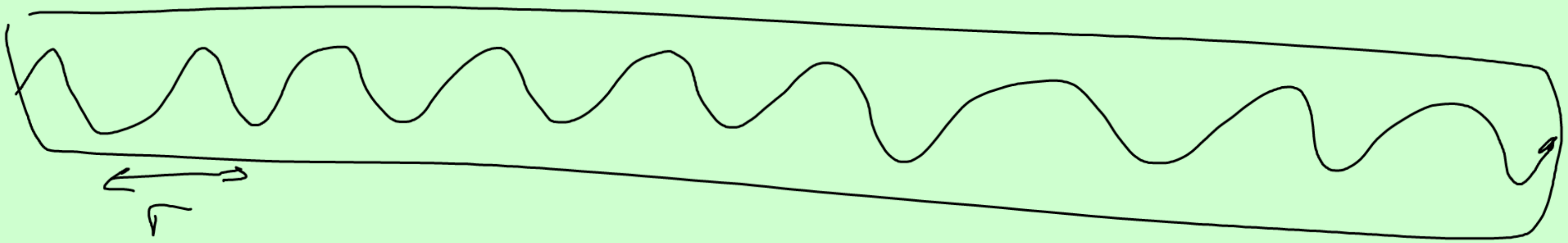
incompressible and irrotational

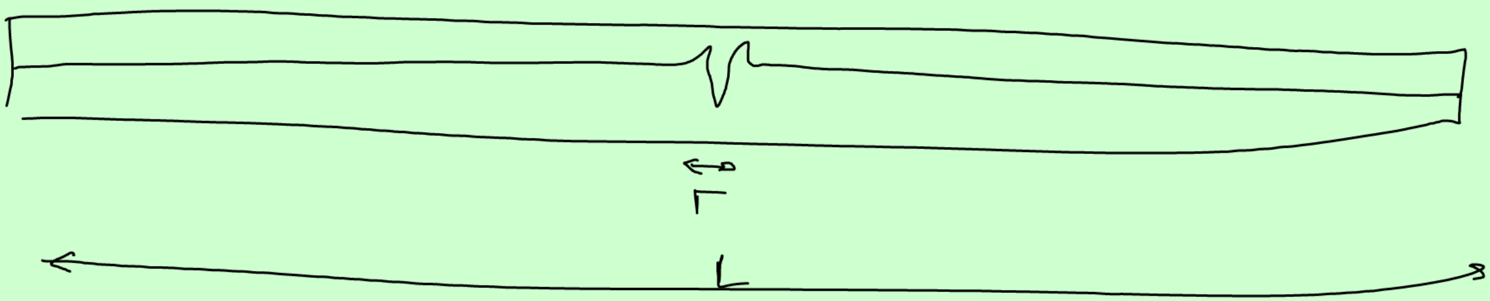
$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = u - iv$$

$$= \frac{\partial \phi}{i\partial y} + i \frac{\partial \psi}{i \partial y} = \frac{\partial \psi}{\partial y} - i \frac{\partial \phi}{\partial y} = u - iv$$

complex conjugate

$$u + iv = \overline{\frac{dw}{dz}}$$





$x = \varepsilon X, \quad \varepsilon \ll 1$

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X}$$

The Bernoulli equations

Inviscid flow $\longrightarrow$ Euler equation

$$\rho\left[\frac{\partial \vec{u}}{\partial t}+(\vec{u}\cdot\vec{\nabla})\vec{u}\right]=-\vec{\nabla}P-\rho g\vec{e}_z \qquad \uparrow \vec{e}_z \quad \downarrow \vec{g}$$

$$\Rightarrow \rho\left[\frac{\partial \vec{u}}{\partial t}-\vec{u}\times\vec{\omega}+\frac{1}{2}\vec{\nabla}|\vec{u}|^2\right]=-\vec{\nabla}P-\rho g\vec{e}_z$$

$$\Rightarrow \rho\left[\frac{\partial \vec{u}}{\partial t}-\vec{u}\times\vec{\omega}\right]=-\vec{\nabla}\left(P+\frac{1}{2}\rho|\vec{u}|^2\right)-\rho g\vec{e}_z$$

$$\Rightarrow \boxed{\rho\left[\frac{\partial \vec{u}}{\partial t}-\vec{u}\times\vec{\omega}\right]=-\vec{\nabla}\left(P+\frac{1}{2}\rho|\vec{u}|^2+\rho g z\right)}$$

Case 1: steady, irrotational

$$\vec{\nabla}\left(P+\frac{1}{2}\rho|\vec{u}|^2+\rho g z\right)=\vec{0}$$

$$\Rightarrow P+\frac{1}{2}\rho|\vec{u}|^2+\rho g z=k_1(t)$$

$$\Rightarrow \boxed{P+\frac{1}{2}\rho|\vec{u}|^2+\rho g z=k_2}$$ because the flow is steady

constant everywhere and at every time

Case 2: steady

$$\rho\, \vec{u} \times \vec{\omega} = \vec{\nabla}\left(P + \frac{1}{2}\rho|\vec{u}|^2 + \rho g z\right)$$

$$\Rightarrow \rho\, \vec{u} \cdot (\vec{u} \times \vec{\omega}) = \vec{u} \cdot \vec{\nabla}\left(P + \frac{1}{2}\rho|\vec{u}|^2 + \rho g z\right)$$

$$\Rightarrow 0 = \vec{u} \cdot \vec{\nabla}\left(P + \frac{1}{2}\rho|\vec{u}|^2 + \rho g z\right)$$

$$\Rightarrow \boxed{P + \frac{1}{2}\rho|\vec{u}|^2 + \rho g z = k_3(\psi)}$$

constant along a streamline
(varies with the streamfunction value)

Case 3: irrotational

$$\rho \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla}\left(P + \frac{1}{2}\rho|\vec{u}|^2 + \rho g z\right)$$

$$\Rightarrow \rho \frac{\partial \vec{\nabla}\phi}{\partial t} = -\vec{\nabla}\left(P + \frac{1}{2}\rho|\vec{u}|^2 + \rho g z\right)$$

$$\Rightarrow \vec{\nabla}\left(\rho \frac{\partial \phi}{\partial t} + P + \frac{1}{2}\rho|\vec{u}|^2 + \rho g z\right) = \vec{0}$$

$$\Rightarrow \boxed{\rho \frac{\partial \phi}{\partial t} + P + \frac{1}{2}\rho|\vec{u}|^2 + \rho g z = k_4(t)}$$

constant that can vary with time

Lecture 19: High Re flows

9W

$$\frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^* \cdot \vec{\nabla}^*)\vec{u}^* = -\vec{\nabla}^* p^* + \underbrace{\frac{1}{Re}\vec{\nabla}^{*2}\vec{u}^*}$$

do not neglect!
singular perturbation.

Boundary layer will develop near structures.
Away from the boundary layer, inertia dominates

Inside the boundary layer: $\rho(\vec{u}\cdot\vec{\nabla})\vec{u} \sim \mu \vec{\nabla}^2 \vec{u}$

$$\Rightarrow \rho \frac{U^2}{L} \sim \mu \frac{U}{\delta^2}$$

L: inertial scale / advective

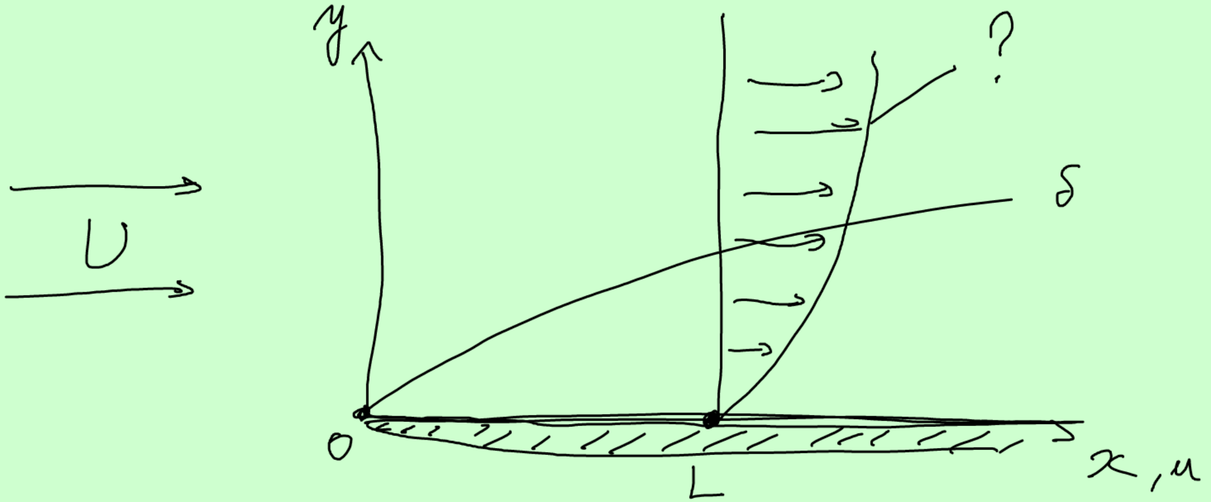
δ: boundary layer thickness

$$\Rightarrow \delta \sim \sqrt{\frac{\mu L}{\rho U}}$$

$$\Rightarrow \frac{\delta}{L} \sim Re_L^{-1/2} \qquad Re_L = \rho\frac{UL}{\mu}$$

The Blasius boundary layer

PRANDTL & BLASIUS



Hypotheses : • steady
• 2D
• large Re
• thin boundary layer $\varepsilon = \frac{\delta}{L} \ll 1$

Equations :
$$\begin{cases} \rho u \partial_x u + \rho v \partial_y u = -\partial_x p + \mu \partial_x^2 u + \mu \partial_y^2 u \\ \rho u \partial_x v + \rho v \partial_y v = -\partial_y p + \mu \partial_x^2 v + \mu \partial_y^2 v \\ \partial_x u + \partial_y v = 0 \end{cases}$$

3 unknowns : u, v, p

2D : x, y

BC: $y = 0 \;,\; u = 0 \;,\; v = 0$

$\; y \to \infty \;,\; u \to U \;,\; v \to 0$

Non-dimensionalization:

$$\left[\begin{aligned} x &= L x^* \\ y &= \delta y^* = \varepsilon L y^* \\ u &= U u^* \end{aligned}\right.$$

$$v = \varepsilon U v^*$$

$$p = \rho U^2 p^*$$

$$\begin{cases} \dfrac{\rho U^2}{L}\left[u^* \partial_{x^*} u^* + v^* \partial_{y^*} u^* \right] = -\dfrac{\rho U^2}{L} \partial_{x^*} p^* + \mu \left[\dfrac{U}{L^2} \partial^2_{x^*} u^* + \dfrac{U}{\varepsilon^2 L^2} \partial^2_{y^*} u^* \right] \\[2ex] \dfrac{\rho \varepsilon U^2}{L}\left[u^* \partial_{x^*} v^* + v^* \partial_{y^*} v^* \right] = -\dfrac{\rho U^2}{\varepsilon L} \partial_{y^*} p^* + \mu \left[\dfrac{U\varepsilon}{L^2} \partial^2_{x^*} v^* + \dfrac{U}{\varepsilon L^2} \partial^2_{y^*} v^* \right] \\[2ex] \partial_{x^*} u^* + \partial_{y^*} v^* = 0 \end{cases}$$

smaller than

$$\Rightarrow \begin{cases} u^* \partial_{x^*} u^* + v^* \partial_{y^*} u^* = -\partial_{x^*} p^* + \cancel{\dfrac{\mu}{\rho U L} \partial^2_{x^*} u^*} + \dfrac{\mu \varepsilon^{-2}}{\rho U L} \partial^2_{y^*} u^* \\[2ex] \cancel{u^* \partial_{x^*} v^* + v^* \partial_{y^*} v^*} = -\varepsilon^{-2} \partial_{y^*} p^* + \cancel{\dfrac{\mu}{\rho U L} \partial^2_{x^*} v^*} + \dfrac{\mu \varepsilon^{-2}}{\rho U L} \partial^2_{y^*} v^* \end{cases}$$

$\leftarrow O(\varepsilon^{-2})$

$$\partial_{x^*} u^* + \partial_{y^*} v^* = 0$$

smaller than

$$\Rightarrow \begin{cases} u^* \partial_{x^*} u^* + v^* \partial_{y^*} u^* = -\partial_{x^*} p^* + \dfrac{\varepsilon^{-2}}{Re_L} \partial_{y^*}^2 u^* \\ 0 = -\varepsilon^{-2} \partial_{y^*} p^* + \cancel{\dfrac{\varepsilon^{-2}}{Re_L} \partial_{y^*}^2 v^*} \\ \partial_{x^*} u^* + \partial_{y^*} v^* = 0 \end{cases}$$

$$Re_L = \frac{\rho U L}{\mu}$$

small because $Re_L \gg 1$ compared to the $\partial_y p$ term

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M 10

To preserve the balance between advection (inertia) and diffusion in the boundary layer, we have to set $Re_L = \varepsilon^{-2}$

$$\left(\Rightarrow \frac{\rho U L}{\mu} = \varepsilon^{-2} = \frac{L^2}{\delta^2} \right.$$

$$\left. \Rightarrow \delta = \sqrt{\frac{L\mu}{\rho U}} \right)$$

$$\Rightarrow \begin{cases} \rho u \partial_x u + \rho v \partial_y u = -\partial_x p + \mu \partial_y^2 u \\ \partial_y p = 0 \\ \partial_x u + \partial_y v = 0 \end{cases}$$

Since $\partial_y p = 0$, we can "read" the pressure inside the boundary layer by taking its value infinitely far away from it but at the same value of x.

On a $y \to \infty$ streamline : $p + \frac{1}{2}\rho \|\vec{u}\|^2 = cst \Rightarrow p + \frac{1}{2}\rho U^2 = cst$

$\Rightarrow p = cst$

$\Rightarrow \partial_x p = 0$

$$\Rightarrow \begin{cases} \rho u \partial_x u + \rho v \partial_y u = \mu \partial_y^2 u \\ \partial_x u + \partial_y v = 0 \end{cases}$$

2 unknowns : u, v

2D : x, y

Because the flow is 2D and incompressible, we can introduce the streamfunction ψ :

$$u = \partial_y \psi \qquad v = -\partial_x \psi$$

$$\Rightarrow \partial_y \psi \, \partial_{xy} \psi - \partial_x \psi \, \partial_{yy} \psi = \nu \, \partial_{yyy} \psi$$

, $\nu = \frac{\mu}{\rho}$ the kinematic viscosity

$$\boxed{\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}}$$

1 unknown : ψ

2D : x, y

Blasius: the boundary layer is self-similar:

• $\eta = y\left(\dfrac{U}{\nu x}\right)^{1/2}$

$= \dfrac{y}{g(x)}$ $\qquad g(x) = \left(\dfrac{\nu x}{U}\right)^{1/2}$

• $u = U f'$ $\qquad f' = \dfrac{df}{d\eta}$

$u = \dfrac{\partial \psi}{\partial y} \Rightarrow U f' = \dfrac{\partial \psi}{\partial \eta}\dfrac{\partial \eta}{\partial y}$

$= \dfrac{\partial \psi}{\partial \eta}\dfrac{1}{g}$

$\Rightarrow \dfrac{\partial \psi}{\partial \eta} = U g f'$

$\Rightarrow \psi = U g f$

$$\boxed{\frac{\partial \eta}{\partial x} = -\frac{y g'}{g^2}}$$

$\partial_x \psi = U g' f + U g \dfrac{\partial f}{\partial \eta}\dfrac{\partial \eta}{\partial x}$

$= U g' f - \dfrac{U y f' g'}{g}$ $\qquad = -v$

$\partial_y \psi = U g f' \dfrac{\partial \eta}{\partial y}$

$= U f'$ $\qquad = u$

$\partial_{yy}\psi = \partial_y (U f')$

$= U f'' \dfrac{\partial \eta}{\partial y}$

$= \dfrac{U f''}{g}$

$\partial_{yyy}\psi = \dfrac{U f'''}{g^2}$

$$\partial_{xy}\psi = \partial_x\left(U f'\right)$$

$$= U f'' \frac{\partial \eta}{\partial x}$$

$$= - \frac{U f'' y \delta'}{\delta^2}$$

$$\Rightarrow U f' \left[- \frac{U y \delta'}{\delta^2} f'' \right] - \left(U \delta' f - \frac{U y \delta'}{\delta} f' \right) \frac{U f''}{\delta} = \nu \frac{U f'''}{\delta^2}$$

$$\Rightarrow - \frac{U^2 y \delta'}{\delta^2} f' f'' - \frac{U^2 \delta'}{\delta} f f'' + \frac{U^2 y \delta'}{\delta^2} f' f'' = \frac{\nu}{\delta^2} U f'''$$

$$\Rightarrow \frac{\nu U}{\delta^2} f''' + \frac{U^2 \delta'}{\delta} f f'' = 0$$

$$\Rightarrow f''' + \frac{U \delta \delta'}{\nu} f f'' = 0$$

Note: $\delta = \left(\frac{\nu x}{U}\right)^{1/2} \Rightarrow \delta' = \frac{1}{2}\left(\frac{\nu}{U x}\right)^{1/2}$

$$\Rightarrow \delta\delta' = \left(\frac{\nu x}{U}\right)^{1/2} \frac{1}{2} \left(\frac{\nu}{U x}\right)^{1/2}$$

$$\Rightarrow \delta\delta' = \frac{1}{2} \frac{\nu}{U}$$

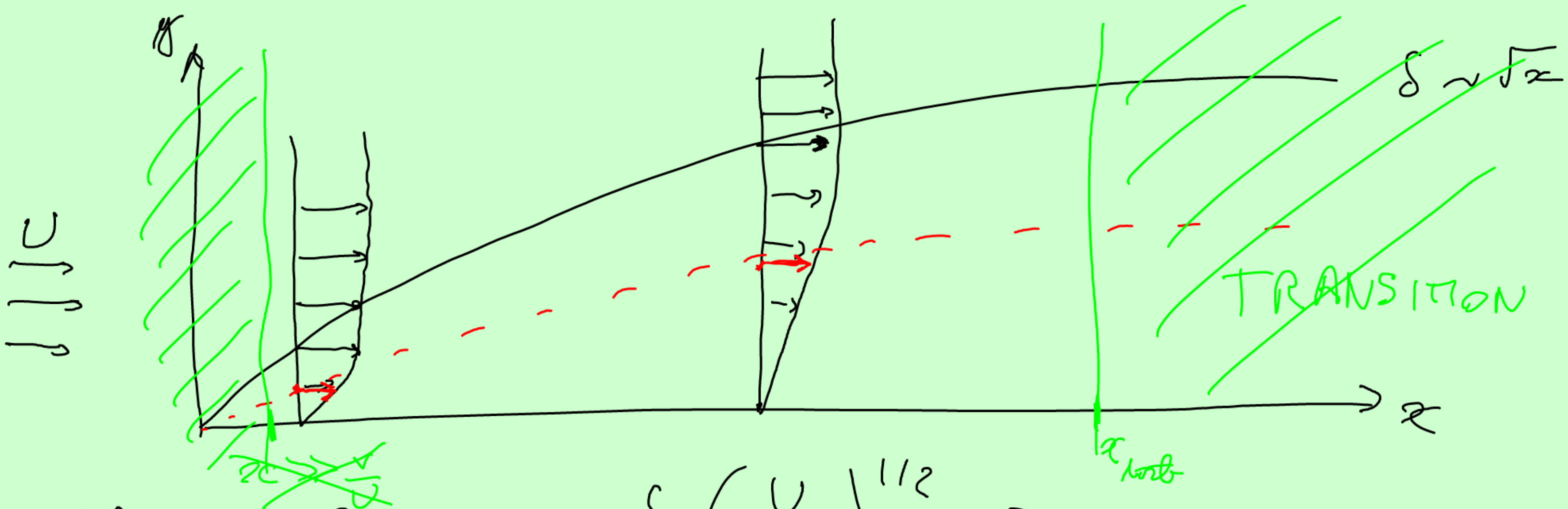
$$\Rightarrow \boxed{f''' + \frac{1}{2} f f'' = 0}$$

1 unknown : f

1D : η

BC : $y = 0$: $u = 0$, $v = 0$; $\eta = 0$: $f = 0$, $f' = 0$

$y \to \infty$: $u \to U$, $v \to 0$; $\eta \to \infty$: $f' \to 1$



$\frac{\delta}{x} = \varepsilon \ll 1$

$\sqrt{x} \ll x$

$Re_x = \frac{Ux}{\nu} \gg 1$

$x \gg \frac{\nu}{U}$

BL below : $\eta \cong 5 \Rightarrow \delta_{99} \left(\frac{U}{\nu x}\right)^{1/2} \cong 5$

$$\Rightarrow \delta_{99} \simeq 5 \left(\frac{\nu x}{U}\right)^{1/2}$$

$$\Rightarrow \quad \frac{\delta_{99}}{x} \simeq 5\, Re_x^{-1/2} \qquad Re_x = \frac{Ux}{\nu}$$

aspect ratio

Lecture 20: from laminar to turbulence

We need the help of dynamical systems theory!

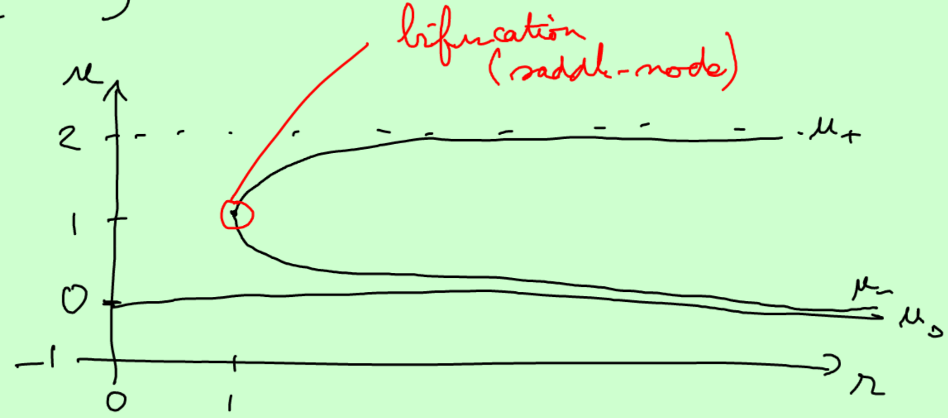
Toy problem: $$d_t u = u\left[1 - \frac{1}{r} - (u-1)^2\right], \quad r \geq 0$$

state variable scalar

parameter

Steady states:
$$\begin{cases} u_0 = 0 \\ u_+ = 1 + \sqrt{1 - \frac{1}{r}} \\ u_- = 1 - \sqrt{1 - \frac{1}{r}} \end{cases} \quad \left.\right\} r \geq 1$$

bifurcation diagram:



Stability: $u = \underset{\substack{\text{base state} \\ (u_0 \text{ or } u_+ \text{ or } u_-)}}{U} + \underset{\text{perturbation}}{\tilde{u}}$

$$\Rightarrow \underbrace{d_t U}_{=0} + d_t \tilde{u} = (U + \tilde{u}) \left[1 - \frac{1}{r} - (U + \tilde{u} - 1)^2 \right]$$

$$\Rightarrow d_t \tilde{u} = \underbrace{U \left[1 - \frac{1}{r} - (U-1)^2 \right]}_{=0} + U \left[-\tilde{u}^2 - 2\tilde{u}(U-1) \right]$$
$$+ \tilde{u} \left[1 - \frac{1}{r} - (U + \tilde{u} - 1)^2 \right]$$

$$\Rightarrow d_t \tilde{u} = -U\tilde{u}^2 - 2U^2\tilde{u} + 2U\tilde{u} + \tilde{u} - \frac{\tilde{u}}{r} - U^2\tilde{u} - \tilde{u}^3 - \tilde{u}$$
$$-2U\tilde{u}^2 + 2U\tilde{u} + 2\tilde{u}^2$$

$$\Rightarrow d_t \tilde{u} = \tilde{u} \left[-2U^2 + 2U + 1 - \frac{1}{r} - U^2 - 1 + 2U \right]$$
$$+ \tilde{u}^2 \left[-U - 2U + 2 \right] + \tilde{u}^3 (-1)$$

$$\Rightarrow d_t \tilde{u} = \tilde{u} \left[4U - 3U^2 - \frac{1}{r} \right] + \tilde{u}^2 \left[2 - 3U \right] - \tilde{u}^3$$

We look at small perturbations: $\tilde{u} = \varepsilon \hat{u}$, $\varepsilon \ll 1$, $\hat{u} = O(1)$

$$\Rightarrow \quad \varepsilon\, d_t \hat{u} = \varepsilon \hat{u}\left[4U - 3U^2 - \frac{1}{r}\right] + \varepsilon^2 \hat{u}^2 \left[2 - 3U\right] - \varepsilon^3 \hat{u}^3$$

$$\Rightarrow \quad \underbrace{d_t \hat{u} = \hat{u}\left[4U - 3U^2 - \frac{1}{r}\right]} + O(\varepsilon)$$

We have, at leading order in ε, a first order linear ODE, so the solutions read:

$$\hat{u} = \underbrace{k}_{\text{amplitude}}\, e^{\lambda t}$$

λt — temporal growth rates

$\lambda > 0 \longrightarrow$ the perturbation grows with time

$\Rightarrow U$ is unstable

$\lambda < 0 \longrightarrow$ the perturbation decays

$\Rightarrow U$ is stable

$$\Rightarrow \quad \lambda k e^{\lambda t} = k e^{\lambda t}\left[4U - 3U^2 - \frac{1}{r}\right]$$

$$\Rightarrow \quad \boxed{\lambda = 4U - 3U^2 - \frac{1}{r}}$$

for $U = u_0$: $\lambda = -\frac{1}{r} < 0$

$\Rightarrow u_0$ is stable

for $U = u_+$: $\lambda = 4\left(1 + \sqrt{1 - \frac{1}{r}}\right) - 3\left(1 + \sqrt{1 - \frac{1}{r}}\right)^2 - \frac{1}{r}$

$$= 4 + 4\sqrt{1-\frac{1}{r}} - 3 - 6\sqrt{1-\frac{1}{r}} - 3\left(1-\frac{1}{r}\right) - \frac{1}{r}$$

$$= 1 - 2\sqrt{1-\frac{1}{r}} - 3 + \frac{3}{r} - \frac{1}{r}$$

$$= -2 + \frac{2}{r} - 2\sqrt{1-\frac{1}{r}}$$

$$= -2\left[1 - \frac{1}{r} + \sqrt{1-\frac{1}{r}}\right]$$

$\Rightarrow \lambda < 0$ except at $r = 1$ where $\lambda = 0$

$\Rightarrow u_+$ is stable

for $U = u_-$: $\lambda = 4\left(1 - \sqrt{1-\frac{1}{r}}\right) - 3\left(1 - \sqrt{1-\frac{1}{r}}\right)^2 - \frac{1}{r}$

$$= 4 - 4\sqrt{1-\frac{1}{r}} - 3 + 6\sqrt{1-\frac{1}{r}} - 3\left(1-\frac{1}{r}\right) - \frac{1}{r}$$

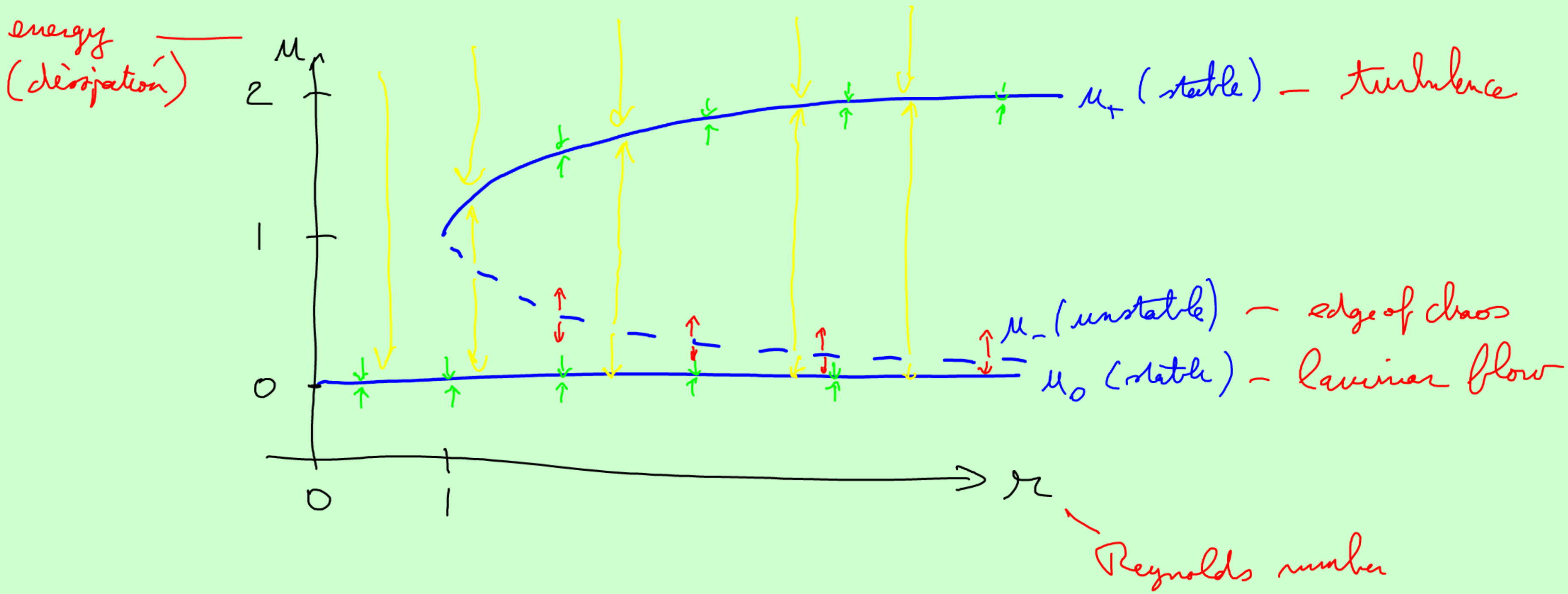
$$= 1 + 2\sqrt{1-\frac{1}{r}} - 3 + \frac{3}{r} - \frac{1}{r}$$

$$= -2 + 2\sqrt{1-\frac{1}{r}} + \frac{2}{r}$$

$$= -2\left[1 - \frac{1}{r} - \sqrt{1-\frac{1}{r}}\right]$$

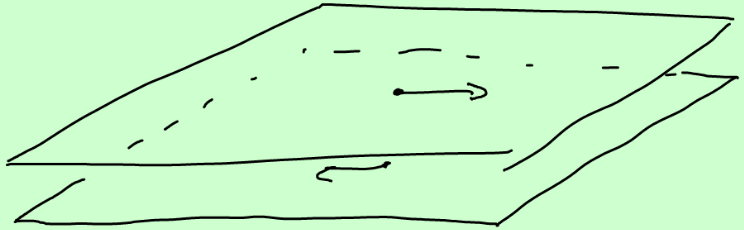
$\Rightarrow \lambda > 0$ except at $r = 1$ for which $\lambda = 0$

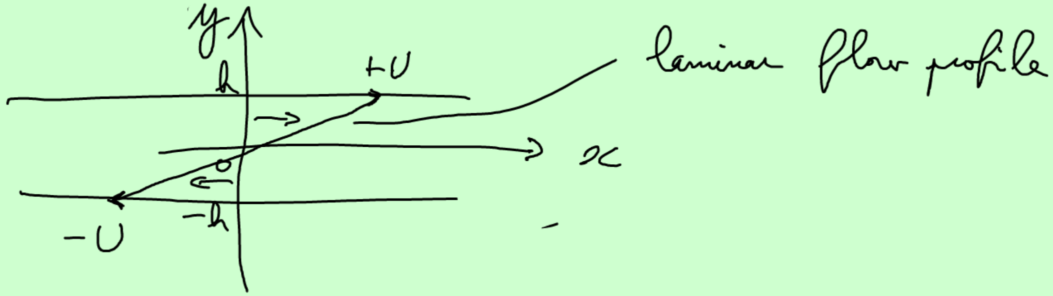
$\Rightarrow u_-$ is unstable



W10

Plane Couette flow:



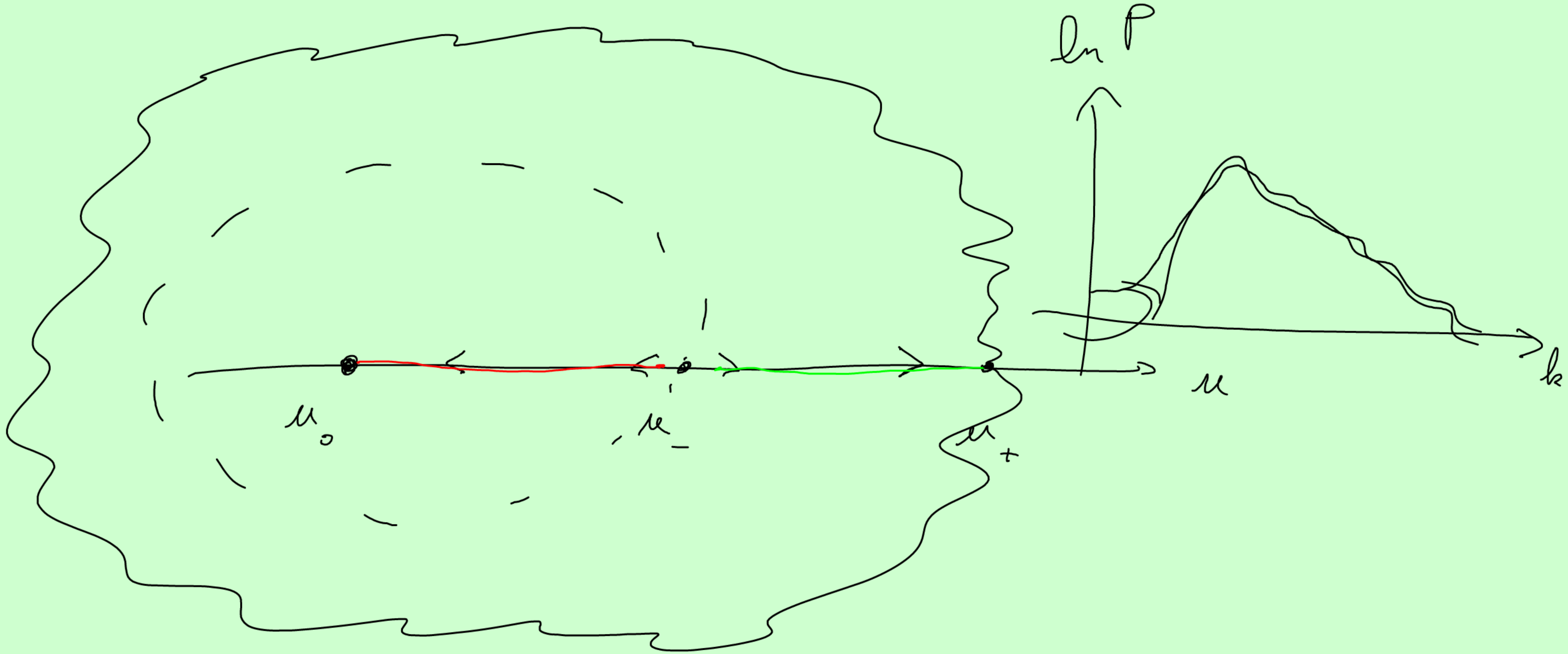


Navier – Stokes:
$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\,\vec{u} = -\vec{\nabla} p + \frac{1}{Re}\,\vec{\nabla}^2 \vec{u}$$
$$\vec{\nabla}\cdot\vec{u} = 0$$

BC : $y = \pm 1$, $\vec{u} = \pm \vec{e}_x$ (wall normal direction)

x periodic direction (streamwise direction)

z periodic direction (spanwise direction)



Lecture 21: High Reynolds number flows

Kolmogorov (1941)

$$\partial_t \vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla}p + \nu \vec{\nabla}^2 \vec{u}$$

ensure incompressibility

look at u^2 nonlinearity

$u \sim e^{ikx} \Rightarrow u^2 \sim e^{i2kx}$

redistribution of the energy among wavenumbers

brings energy to larger wavenumbers

$\partial_t u = \nu \partial_x^2 u$

$\partial_t \hat{u} = -\nu k^2 \hat{u}$

$\hat{u} = \hat{u}_0 e^{-\nu k^2 t}$

dissipates energy

smaller structures dissipate faster

* Large scale flow : $Re = \dfrac{UL}{\nu} \gg 1$

$$\left.\begin{array}{l} \text{Kinetic energy: } E_k \sim U^2 \\ \text{Eddy turnover time: } \tau \sim \dfrac{L}{U} \end{array}\right\}$$

Power input at large scale:

$$P = \frac{E_k}{\tau} = \frac{U^3}{L}$$

$L^2 \cdot T^{-3}$

* At the small scales, dissipation and inertia are in balance:

$$Re_\ell \sim 1 \quad \Rightarrow \quad \frac{u\,\ell}{\nu} \sim 1$$

Dissipation: $\varepsilon = \nu \left(\dfrac{u}{\ell}\right)^2$

ε: $L^2 . T^{-3}$

Using the expression of Re_ℓ: $\varepsilon = \dfrac{\nu^3}{\ell^4}$ $\qquad \varepsilon = \dfrac{u^4}{\nu}$

For an established flow: $P = \varepsilon$

(1) $$\frac{U^3}{L} = \frac{\nu^3}{\ell^4} \qquad\qquad \frac{U^3}{L} = \frac{u^4}{\nu}$$

$$\Rightarrow \ell^4 = L\,\frac{\nu^3}{U^3} \qquad\qquad \Rightarrow u^4 = U^3\,\frac{\nu}{L}$$

$$\Rightarrow \ell^4 = L^4\,\frac{\nu^3}{U^3 L^3} \qquad\qquad \Rightarrow u^4 = U^4\,\frac{\nu}{U L}$$

$$\Rightarrow \boxed{\ell = L\ Re^{-3/4}} \qquad\qquad \Rightarrow \boxed{u = U\ Re^{-1/4}}$$

Kolmogorov scale

Kolmogorov laws

* Intermediate scales, turbulence is self-similar: it will dissipate energy in proportion to the available energy at that scale.

$$\underset{L^2 \cdot T^{-3}}{\varepsilon} \sim \underset{L^2 \cdot T^{-2}}{E_k}$$

NO!

$E_k \sim v^2$ — intermediate scale

$\varepsilon \sim v^2 \dfrac{v}{r}$ — intermediate length scale

$$\Rightarrow \quad \varepsilon \sim \frac{v^3}{r}$$

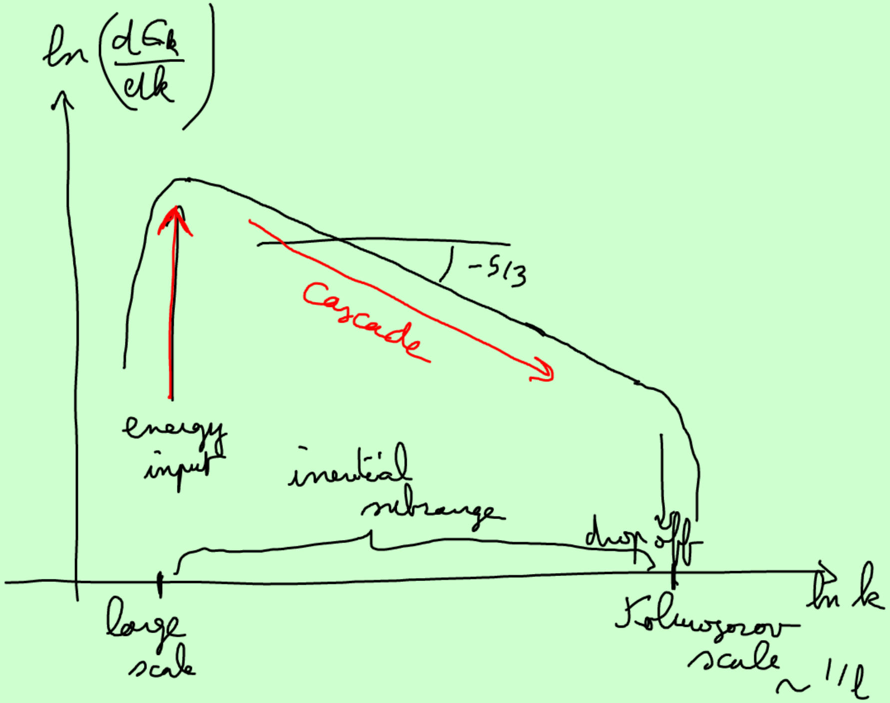
$$\Rightarrow \quad v^3 \sim \varepsilon r$$

$$\Rightarrow \quad E_k \sim \varepsilon^{2/3} r^{2/3}$$

In terms of wavenumber: $E_k \sim \varepsilon^{2/3} k^{-2/3}$

The results are generally presented using the density of kinetic energy per wavenumber:

$$\boxed{\frac{dE_k}{dk} \sim \varepsilon^{2/3} k^{-5/3}} \quad \text{Kolmogorov } -5/3 \text{ law}$$



$$\partial_t u = NLT + \text{dissipation}$$

$$\frac{u^+ - u^-}{\Delta t} = NLT^- + \text{dissipation}^-$$

$$\Rightarrow \quad u^+ = u^- + \Delta t\left(NLT^- + \text{dissipation}^-\right)$$

Lecture 22: Computational models for fluid dynamics

W11

$$\left\{\begin{array}{l}\partial_t \vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}p + \frac{1}{Re}\vec{\nabla}^2\vec{u} \\ \vec{\nabla}\cdot\vec{u} = 0\end{array}\right\}$$ DNS: direct numerical simulation

RANS: Reynolds-averaged Navier-Stokes equation

$$\vec{u}(\vec{x},t) = \underbrace{\vec{\bar{u}}(\vec{x})}_{\text{time-averaged}} + \underbrace{\vec{u}'(\vec{x},t)}_{\text{fluctuation}}$$

$$\vec{\bar{u}}(\vec{x}) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} \vec{u}(\vec{x},t)\,dt$$

$$\int_{t_0}^{t_0+T} \vec{u}'(\vec{x},t)\,dt = \vec{0} \quad \longleftarrow$$

Navier-Stokes:

$$\left\{\begin{array}{l}\rho\frac{\partial\vec{\bar{u}}}{\partial t} + \rho\frac{\partial\vec{u}'}{\partial t} + \rho(\vec{\bar{u}}\cdot\vec{\nabla})\vec{\bar{u}} + \rho(\underline{\vec{\bar{u}}\cdot\vec{\nabla}})\vec{u}' + \rho(\underline{\vec{u}'\cdot\vec{\nabla}})\vec{\bar{u}} + \rho(\vec{u}'\cdot\vec{\nabla})\vec{u}' \\ \qquad = -\vec{\nabla}\bar{p} - \vec{\nabla}p' + \mu\vec{\nabla}^2\vec{\bar{u}} + \mu\vec{\nabla}^2\vec{u}' \\ \vec{\nabla}\cdot\vec{\bar{u}} + \vec{\nabla}\cdot\underline{\vec{u}'} = 0\end{array}\right.$$

Average :
$$\begin{cases} \rho \dfrac{\partial \bar{\vec{u}}}{\partial t} + \rho(\bar{\vec{u}}\cdot\vec{\nabla})\bar{\vec{u}} + \underbrace{\rho\overline{(\vec{u}'\cdot\vec{\nabla})\vec{u}'}} = -\vec{\nabla}\bar{p} + \mu\vec{\nabla}^2\bar{\vec{u}} \\ \vec{\nabla}\cdot\bar{\vec{u}} = 0 \end{cases}$$

$$(\Rightarrow \vec{\nabla}\cdot\vec{u}' = 0)$$

$$\rho u_i' \frac{\partial u_j'}{\partial x_i} = \rho \frac{\partial(u_i' u_j')}{\partial x_i} - \rho u_j' \underbrace{\frac{\partial u_i'}{\partial x_i}}_{=0}$$

$$= \frac{\partial(\rho u_i' u_j')}{\partial x_i}$$

$$= \vec{\nabla}\cdot(\rho\vec{u}'\vec{u}')\Big|_i$$

$$\Rightarrow \rho\frac{\partial\bar{\vec{u}}}{\partial t} + \rho(\bar{\vec{u}}\cdot\vec{\nabla})\bar{\vec{u}} = -\vec{\nabla}\bar{p} + \mu\vec{\nabla}^2\bar{\vec{u}} + \overline{\vec{\nabla}\cdot(-\rho\vec{u}'\vec{u}')}$$

$\bar{\bar{\tau}}^R = -\rho\vec{u}'\vec{u}'$: Reynolds stress

$$\Rightarrow \rho\frac{\partial\bar{\vec{u}}}{\partial t} + \rho(\bar{\vec{u}}\cdot\vec{\nabla})\bar{\vec{u}} = \vec{\nabla}\cdot(\bar{\bar{\tau}} + \overline{\bar{\bar{\tau}}^R})$$

We usually work :
$$\begin{cases} \rho\dfrac{\partial\bar{\vec{u}}}{\partial t} + \rho(\bar{\vec{u}}\cdot\vec{\nabla})\bar{\vec{u}} = -\vec{\nabla}\bar{p} + \mu\vec{\nabla}^2\bar{\vec{u}} + \vec{\nabla}\cdot(-\underline{\rho\overline{\vec{u}'\vec{u}'}}) \\ \vec{\nabla}\cdot\bar{\vec{u}} = 0 \end{cases}$$

Residual :
$$\begin{cases} \rho \dfrac{\partial \vec{u}'}{\partial t} + \rho\left(\vec{\bar{u}}\cdot\vec{\nabla}\right)\vec{u}' + \rho\left(\vec{u}'\cdot\vec{\nabla}\right)\vec{\bar{u}} + \rho\left(\vec{u}'\cdot\vec{\nabla}\right)\vec{u}' - \rho\,\overline{\left(\vec{u}'\cdot\vec{\nabla}\right)\vec{u}'} = -\vec{\nabla}p' \\ \qquad + \mu\vec{\nabla}^2\vec{u}' \\ \vec{\nabla}\cdot\vec{u}' = 0 \end{cases}$$

Index :
$$\rho\frac{\partial u_i'}{\partial t} + \rho\,\bar{u}_k\frac{\partial u_i'}{\partial x_k} + \rho\,u_k'\frac{\partial \bar{u}_i}{\partial x_k} + \rho\,u_k'\frac{\partial u_i'}{\partial x_k} - \rho\,\overline{u_k'\frac{\partial u_i'}{\partial x_k}} = -\frac{\partial p'}{\partial x_i} + \mu\,\partial_k^2 u_i'$$

Multiply by u_j' :
$$\rho\,u_j'\frac{\partial u_i'}{\partial t} + \rho\,\bar{u}_k u_j'\frac{\partial u_i'}{\partial x_k} + \rho\,u_k' u_j'\frac{\partial \bar{u}_i}{\partial x_k} + \rho\,u_k' u_j'\frac{\partial u_i'}{\partial x_k} - \rho\,u_j'\,\overline{u_k'\frac{\partial u_i'}{\partial x_k}}$$
$$= -u_j'\frac{\partial p'}{\partial x_i} + \mu\,u_j'\,\partial_k^2 u_i'$$

Swap j and i :
$$\rho\,u_i'\frac{\partial u_j'}{\partial t} + \rho\,\bar{u}_k u_i'\frac{\partial u_j'}{\partial x_k} + \rho\,u_k' u_i'\frac{\partial \bar{u}_j}{\partial x_k} + \rho\,u_k' u_i'\frac{\partial u_j'}{\partial x_k} - \rho\,u_i'\,\overline{u_k'\frac{\partial u_j'}{\partial x_k}}$$
$$= \ldots$$

Sum :
$$\rho\frac{\partial (u_i' u_j')}{\partial t} + \rho\,\bar{u}_k\left(u_j'\frac{\partial u_i'}{\partial x_k} + u_i'\frac{\partial u_j'}{\partial x_k}\right) + \frac{\partial \bar{u}_i}{\partial x_k}\rho\,u_k' u_j' + \frac{\partial \bar{u}_j}{\partial x_k}\rho\,u_k' u_i'$$
$$+ \rho\,u_k'\left(u_j'\frac{\partial u_i'}{\partial x_k} + u_i'\frac{\partial u_j'}{\partial x_k}\right) \ldots$$

$$\Rightarrow \frac{\partial(-\tau^R_{ij})}{\partial t} + \underbrace{\rho\,\overline{u_k}\left(u'_j \frac{\partial u'_i}{\partial x_k} + u'_i \frac{\partial u'_j}{\partial x_k}\right)}_{\rho \frac{\partial(\overline{u_k}\, u'_i u'_j)}{\partial x_k}} + \frac{\partial \overline{u_i}}{\partial x_k}\left(-\tau^R_{kj}\right) + \frac{\partial \overline{u_j}}{\partial x_k}\left(-\tau^R_{ki}\right)$$

$$\frac{\partial}{\partial x_k}\left(-\overline{u_k}\,\tau^R_{ij}\right)$$

$$+ \underbrace{\rho\, u'_k \frac{\partial (u'_i u'_j)}{\partial x_k}}_{\underbrace{\rho \frac{\partial (u'_i u'_j u'_k)}{\partial x_k}}_{3^{rd}\text{ order moment}} - \rho\, u'_i u'_j \cancel{\frac{\partial u'_k}{\partial x_k}} = 0} + \ldots$$

$\longrightarrow$ need a closure!