MATH 3620
Fluid Dynamics II

Cédric Beaume (School of Mathematics (1.07)

c.m.l.beaume@leeds.ac.uk

www.cbeaume.com

Monday 10am LT10

Tuesday 11am LT01

"Tutorials": Tuesdays of W3, 5, 7, 9, 11

Chapter I : Mathematical tools

Fluid: substance that flows and takes the shape of its container.

Dynamics: study of the forces and torques when there is motion.

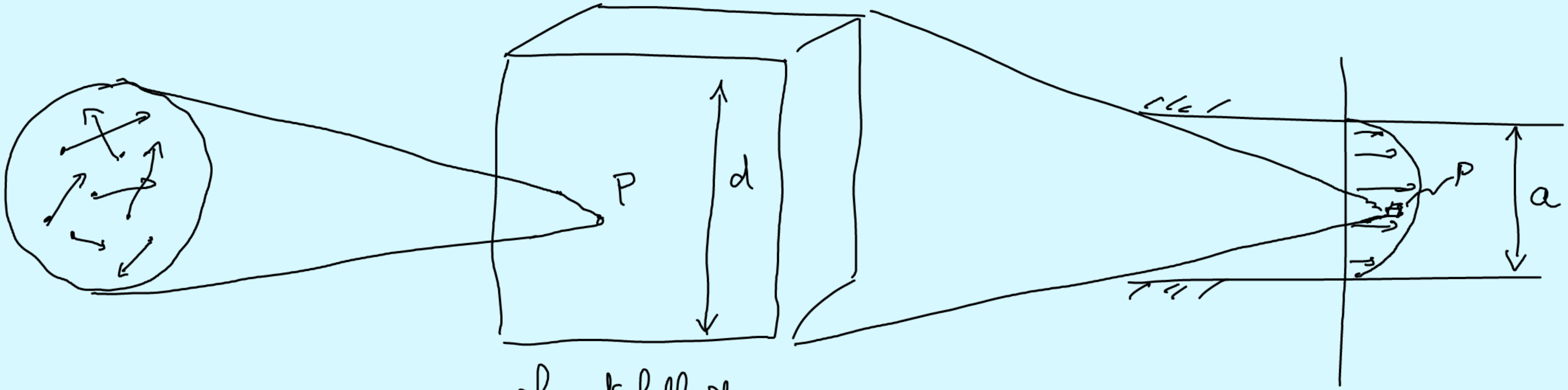
Why: aerodynamics (make things move in fluids more efficiently)
medicine (understanding blood flows)
meteorology (weather forecast)
...

Continuum hypothesis

$1\ cm^3$ of water : $\sim 10^{23}$ particles

particle size : $1\ \text{Å} = 10^{-10}\ m$

mean free path : $\ell_F \sim 10^{-8}\ m$ (average distance travelled before collision)



element of fluid
fluid "particle"
"blob" of fluid

MICRO

MESO

MACRO

CONTINUUM HYPOTHESIS

$d \gg \ell$, $a \gg d$

$$\vec{u}(P,t) = \frac{\sum_{\text{particles in } \delta V} \vec{u}_{\text{particle}}}{\sum_{\text{particles in } \delta V} 1}$$

δV : element of fluid centered on P

RHO, density

mass

$$\rho(P,t) = \frac{\sum_{\text{particles in } \delta V} m_{\text{particles}}}{\delta V}$$

Time-derivative

- $\dfrac{\partial f}{\partial t}(\vec{x},t)$: rate of change of f with time at $\vec{x}$ EULER

- $\dfrac{df}{dt}(\vec{x}(t),t) = \dfrac{\partial f}{\partial t} + \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial t}$

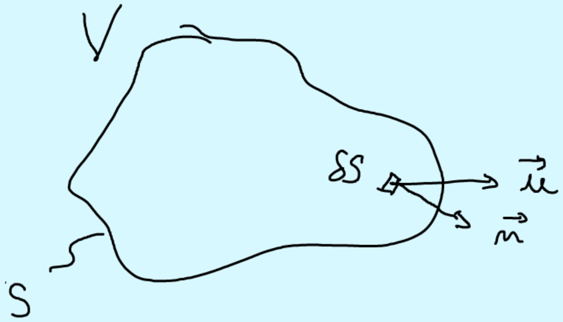
$$= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

$$= \frac{\partial f}{\partial t} + \underbrace{(\vec{u}\cdot\vec{\nabla})}_{\text{transport}} f$$

rate of change in time of f for a particle passing through $\vec{x}$ at time t. LAGRANGE

watch <u>INDEX NOTATION</u>

Conservation of mass



V: fixed volume of fluid

$\vec{n}$: unit outward pointing normal

Mass of fluid within V: $M_V = \int_V \rho \, dV$

The only cause for changes in M_V is the flow of fluid through S.

$$\frac{dM_V}{dt} = -\int_S \rho \, \underline{\vec{u} \cdot \vec{n}} \, dS$$

$$\Rightarrow \frac{d}{dt} \int_V \rho \, dV = -\int_S \rho \, \vec{u} \cdot \vec{n} \, dS$$

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t} \, dV = -\int_S \rho \, \underbrace{\vec{u} \cdot \vec{n} \, dS}_{\cdot \vec{n} \, dS}$$

→ divergence theorem

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t} \, dV = -\int_V \vec{\nabla} \cdot (\rho \vec{u}) \, dV$$

$$\Rightarrow \int_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u})\right] dV = 0$$

Because we did not prescribe a size for V, we can take V to be as small as the "blob" of fluid, therefore the integrand has to vanish pointwise.

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0}$$ – mass conservation continuity equation

$$\Rightarrow \underbrace{\frac{\partial \rho}{\partial t} + \rho\,\vec{\nabla}\cdot\vec{u} + (\vec{u}\cdot\vec{\nabla})\rho}\, = 0$$

$$\Rightarrow \boxed{\frac{D\rho}{Dt} + \rho\,\vec{\nabla}\cdot\vec{u} = 0}$$

<u>Incompressibility condition</u>: an incompressible fluid is a fluid whose density remains constant.

$$\frac{D\rho}{Dt} = 0 \Rightarrow \rho\,\vec{\nabla}\cdot\vec{u} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{u} = 0}$$

incompressibility condition

$$\vec{\nabla}\vec{u} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \\ - & - & \end{pmatrix}$$

The incompressibility condition imposes a strong constraint on the velocity field:

$$\vec{\nabla} \cdot (\underbrace{\vec{\nabla} \times \vec{\psi}}) = 0 \quad \text{always}$$

$$\vec{u} = \vec{\nabla} \times \vec{\psi} \longrightarrow \vec{\psi} : \underline{\text{streamfunction}}$$

For a 2D incompressible flow: $\vec{u} = u(x,y)\vec{e_x} + v(x,y)\vec{e_y}$

$$\Rightarrow \vec{\psi} = \psi(x,y)\vec{e_z}$$

$$u = \frac{\partial \psi}{\partial y} \;, \quad v = -\frac{\partial \psi}{\partial x}$$

Streamlines: lines where the streamfunction remains constant

$d\psi = 0$ on the streamline $\Rightarrow \dfrac{\partial \psi}{\partial x} dx + \dfrac{\partial \psi}{\partial y} dy = 0$

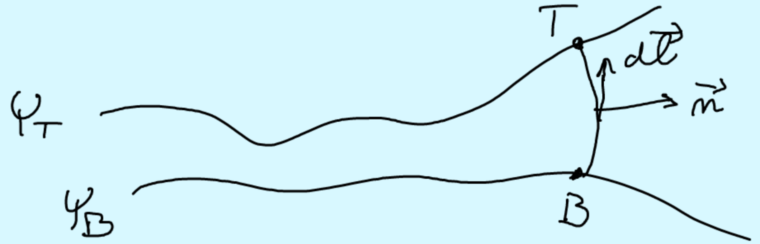
$$\Rightarrow -v\,dx + u\,dy = 0$$

$\Rightarrow \vec{u} \times \vec{dl} = \vec{0}$ where $\vec{dl} = (dx, dy)$ element of displacement along the streamline

↳ the streamline is everywhere tangent to the velocity.

other way: $\vec{u} \cdot \vec{\nabla} \psi = 0$

flux between two streamlines :



$\vec{n}$ unit outward pointing normal

$$\vec{dl} = dx\, \vec{e_x} + dy\, \vec{e_y}$$

$$ds\, \vec{n} = dy\, \vec{e_x} - dx\, \vec{e_y} = ds\left(\frac{\partial y}{\partial s}\vec{e_x} - \frac{\partial x}{\partial s}\vec{e_y}\right)$$

$$Q = \int_B^T \vec{u}\cdot\vec{n}\, ds$$

$$= \int_B^T \left[u\frac{\partial y}{\partial s} - v\frac{\partial x}{\partial s}\right] ds$$

$$= \int_B^T \left[\frac{\partial \psi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial s}\right] ds$$

$$= \int_B^T \frac{d\psi}{ds}\, ds$$

$$= \int_{\psi_B}^{\psi_T} d\psi$$

$$= \psi_T - \psi_B$$

$$\|\vec{u}\| = \sqrt{u^2 + v^2}$$

$$= \sqrt{\left(\frac{\partial \psi}{\partial y}\right)^2 + \left(-\frac{\partial \psi}{\partial x}\right)^2}$$

$$= \sqrt{\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2}$$

$$= \|\vec{\nabla}\psi\|$$

<u>Strain rate and vorticity tensors</u>: velocity gradient tensor: $\bar{\bar{K}}$

$$K_{ij} = \frac{\partial u_i}{\partial x_j}$$

$$= \underset{\substack{\text{symmetric} \\ E_{ij} = E_{ji} \\ \text{strain rate tensor}}}{E_{ij}} + \underset{\substack{\text{anti-symmetric} \\ \Omega_{ij} = -\Omega_{ji} \\ \text{vorticity tensor}}}{\Omega_{ij}}$$

$$E_{ij} = \frac{1}{2}\left(K_{ij} + K_{ji}\right) \qquad \Omega_{ij} = \frac{1}{2}\left(K_{ij} - K_{ji}\right)$$

Example in 2D: incompressibility $\rightarrow$ $\partial_x u = -\partial_y v$

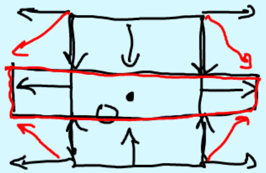
$$\bar{\bar{K}} = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$$

$$= \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & -\partial_x u \end{pmatrix}$$

3 unknowns

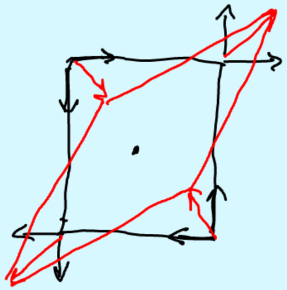
$$\bar{\bar{E}} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \qquad \bar{\bar{\Omega}} = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$$

- $\bar{\bar{E}}_a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\qquad \partial_x u = 1$, $\partial_y v = -1$



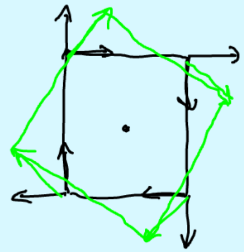
DEFORMATION

- $\bar{\bar{E}}_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad \partial_y u = 1$, $\partial_x v = 1$



DEFORMATION

- $\bar{\bar{\Omega}}_c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\qquad \partial_y u = 1$, $\partial_x v = -1$



SOLID BODY ROTATION

velocity gradient tensor $\bar{\bar{K}}$

- $\bar{\bar{E}}$ symmetric, strain rate tensor $\longrightarrow$ deformations (forces)
- $\bar{\bar{\Omega}}$ anti-symmetric, vorticity tensor $\longrightarrow$ solid body rotation (torques)

<u>Vorticity</u>: $\vec{\omega} = \vec{\nabla} \times \vec{u}$

$$\omega_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} u_k \quad \Longrightarrow \quad \varepsilon_{ilm}\, \omega_i = \varepsilon_{ilm}\, \varepsilon_{ijk} \frac{\partial}{\partial x_j} u_k$$

$$= \left(\delta_{lj}\, \delta_{mk} - \delta_{lk}\, \delta_{jm} \right) \left[\frac{\partial}{\partial x_j} u_k \right]$$

$$= \delta_{lj}\, \delta_{mk} \frac{\partial}{\partial x_j} u_k - \delta_{lk}\, \delta_{jm} \frac{\partial}{\partial x_j} u_k$$

$$= \frac{\partial}{\partial x_l} u_m - \frac{\partial}{\partial x_m} u_l$$

$$= 2\, \Omega_{ml}$$

$$\Longrightarrow \quad \varepsilon_{ilm}\, \omega_i = 2\, \Omega_{ml}$$

$$\Longrightarrow \quad \Omega_{ml} = \frac{1}{2}\, \varepsilon_{ilm}\, \omega_i$$

$$= \frac{1}{2} \varepsilon_{klm} \omega_k$$

$$\Rightarrow \Omega_{ij} = \frac{1}{2} \varepsilon_{kji} \omega_k$$

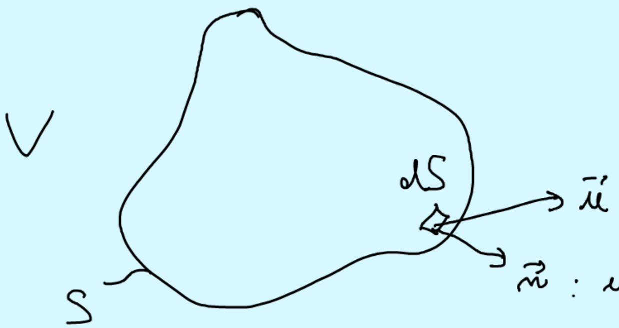
$$\Rightarrow \boxed{\Omega_{ij} = -\frac{1}{2} \varepsilon_{ijk} \omega_k}$$

$$\bar{\bar{\Omega}} = \begin{pmatrix} & \omega ? & \\ & & \\ & & \end{pmatrix}$$

4

Chapter 2: The Navier-Stokes equation

Equation of motion



V fixed

$\vec{n}$: unit outward-pointing normal vector momentum density : $\vec{q} = \rho \vec{u}$

momentum in V : $\displaystyle\int_V \vec{q}\, dV = \int_V \rho \vec{u}\, dV$

rate of change with time of the momentum in V : $\displaystyle\frac{d}{dt}\int_V \rho \vec{u}\, dV = \int_V \frac{\partial(\rho\vec{u})}{\partial t}\, dV$

$$\underbrace{\begin{pmatrix}\text{rate of change of}\\ \text{momentum in } V\end{pmatrix}}_{\vec{\mathrm{I}}} = \underbrace{\begin{pmatrix}\text{flow of momentum}\\ \text{through } S\end{pmatrix}}_{\vec{\mathrm{II}}} + \underbrace{\begin{pmatrix}\text{net forces}\\ \text{acting on } V\end{pmatrix}}_{\vec{\mathrm{III}}}$$

$$\vec{\mathrm{I}} = \int_V \frac{\partial(\rho\vec{u})}{\partial t}\, dV \qquad \mathrm{I}_i = \int_V \frac{\partial}{\partial t}(\rho u_i)\, dV$$

$$\vec{\mathrm{II}} = -\int_S \vec{q}\, \underbrace{\vec{u} \cdot \vec{n}}\, dS$$

$$= -\int_S \rho \vec{u}\, \vec{u} \cdot \vec{n}\, dS$$

flow of mass : $\displaystyle -\int_S \rho\, \underbrace{\vec{u} \cdot \vec{n}}\, dS$

$$\mathrm{II}_i = -\int_S q_i \, u_j \, n_j \, dS$$

$$= -\int_S \rho u_i \, u_j \, n_j \, dS$$

repeated index → divergence theorem

$$= -\int_V \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) dV$$

$$\vec{\mathrm{III}} = \int_V \vec{F}_{body} \, dV + \int_S \vec{F}_{local} \, dS$$

force densities

$$\vec{F}_{body} = \rho \vec{g} + \dots \quad \Rightarrow \quad \int_V \rho \vec{g} \, dV$$

$\vec{F}_{local} = \, ?$ in index notation: $\displaystyle\int_S F_{local_i} \, dS$

total stress tensor

I really want $\displaystyle\int_S \tau_{ji} \, n_j \, dS$

$$\Rightarrow F_{local_i} = \tau_{ji} \, n_j$$

$$\Rightarrow \int_S F_{\text{local}_i}\, dS = \int_S \tau_{ji} n_j \, dS$$

divergence theorem

$$= \int_V \frac{\partial}{\partial x_j}(\tau_{ji})\, dV$$

$$\Rightarrow \int_V \frac{\partial(\rho u_i)}{\partial t}\, dV = -\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)\, dV + \int_V \rho g_i \, dV + \int_V \frac{\partial}{\partial x_j}(\tau_{ji})\, dV$$

The control volume V can be taken as small as the "blob" of fluid (cF continuum hypothesis).
$\Rightarrow$ The integrand has to vanish pointwise.

$$\hookrightarrow \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho g_i + \frac{\partial}{\partial x_j}\tau_{ji}$$

$$\Rightarrow \rho \frac{\partial u_i}{\partial t} + \underbrace{u_i \frac{\partial \rho}{\partial t} + u_i \frac{\partial}{\partial x_j}(\rho u_j)}_{0 \text{ by mass conservation}} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho g_i + \frac{\partial}{\partial x_j}\tau_{ji}$$

$$\Rightarrow \rho \left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right] = \rho g_i + \frac{\partial}{\partial x_j}\tau_{ji}$$

$$\boxed{\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = \rho\vec{g} + \vec{\nabla}\cdot\bar{\bar{\tau}}}$$ equation of motion

---

<u>Constitutive equations</u>: (5)

* Ideal fluid: $\vec{F}_{local} = -P\vec{n}$

total pressure

$$\Rightarrow \tau_{ji}\, n_j = -P n_i$$

$$\Rightarrow \underline{\tau_{ij} = -P\,\delta_{ij}}$$

$$\bar{\bar{\tau}} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

$$\Rightarrow \rho\left[\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right] = \rho g_i + \frac{\partial}{\partial x_j}\left(-P\,\delta_{ji}\right)$$

$$= \rho g_i - \frac{\partial P}{\partial x_i}$$

$$\Rightarrow \boxed{\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = \rho\vec{g} - \vec{\nabla}P}$$ Euler equation

+ mass conservation / incompressibility

* Newtonian fluids: isotropic fluid whose stress and strain are linear functions of one another.

$$\Rightarrow \underbrace{\tau_{ji}}_{\text{total stress tensor}} = -P\delta_{ji} + \underbrace{\sigma_{ji}}_{\text{viscous stress tensor}}$$

$$\sigma_{ji} = 2\mu\left[\epsilon_{ji} - \frac{1}{3}\delta_{ij}\cancel{\frac{\partial u_k}{\partial x_k}}\right] + \ldots$$

0 by incompressibility

$$= 2\mu\,\epsilon_{ji}$$

$$= \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\Rightarrow \tau_{ji} = -P\delta_{ji} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\hookrightarrow \rho\left[\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right] = \rho g_i + \frac{\partial}{\partial x_j}\left[-P\delta_{ji} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$$

$$= \rho g_i - \frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2} + \mu\cancel{\frac{\partial^2 u_j}{\partial x_i \partial x_j}}$$

0 by incompressibility

$$\Rightarrow \boxed{\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\,\vec{u}\right] = \rho\vec{g} - \vec{\nabla}P + \mu\vec{\nabla}^2\vec{u}}$$

Navier–Stokes equation

$\boxed{+ \text{ incompressibility}}$

rate of change of momentum with time

advection of momentum (transport of momentum with the flow)

weight

total pressure gradient (residual force to ensure incompressibility)

diffusion / viscous effects
dissipation

<u>The different kinds of pressure:</u>

no flow, no force (external): $\rho\vec{g} - \vec{\nabla}P_H = \vec{0}$ — hydrostatic pressure

$$\Rightarrow P_H = \rho\vec{g}\cdot\vec{x} + P_0$$

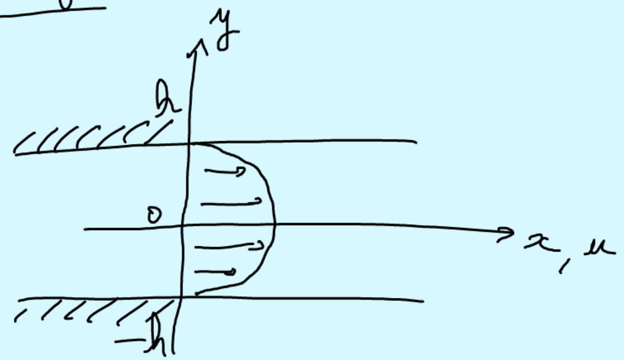
P_0 ~ reference pressure $(\vec{x} = \vec{0})$

$$\underbrace{P}_{\text{total}} = \underbrace{P_H}_{\text{hydrostatic}} + \underbrace{p}_{\text{dynamic}} \Rightarrow \boxed{\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\,\vec{u}\right] = -\vec{\nabla}p + \mu\vec{\nabla}^2\vec{u}}$$

Poiseuille flow

POISEUILLE

(0)



$\leftarrow \vec{\nabla} p = -G\,\vec{e}_x \qquad G > 0$

(1) 2D Navier–Stokes

$$\begin{cases} \rho \cancel{\dfrac{\partial u}{\partial t}} + \rho u \dfrac{\partial u}{\partial x} + \cancel{\rho v \dfrac{\partial u}{\partial y}} = -\dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u}{\partial x^2} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ \rho \cancel{\dfrac{\partial v}{\partial t}} + \cancel{\rho u \dfrac{\partial v}{\partial x}} + \cancel{\rho v \dfrac{\partial v}{\partial y}} = -\dfrac{\partial p}{\partial y} + \cancel{\mu \dfrac{\partial^2 v}{\partial x^2}} + \cancel{\mu \dfrac{\partial^2 v}{\partial y^2}} \end{cases}$$

Incompressibility condition : $\dfrac{\partial u}{\partial x} + \cancel{\dfrac{\partial v}{\partial y}} = 0$

(2) Boundary conditions : $y = -h$, $u = 0$ (no-slip) , $v = 0$ (impenetrability condition)

$y = +h$, $u = 0$, $v = 0$

(3) Hypotheses : • steady : $\dfrac{\partial}{\partial t} = 0$

• 1D flow : $v \equiv 0$

Example Sheet 1

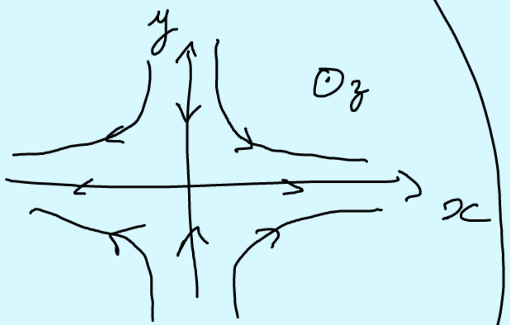
1(c)

$$\bar{\bar{E}} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\bar{\bar{E}}_{xy} = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$$

$$\frac{\partial u}{\partial x} = 4$$

$$\frac{\partial v}{\partial y} = -4$$

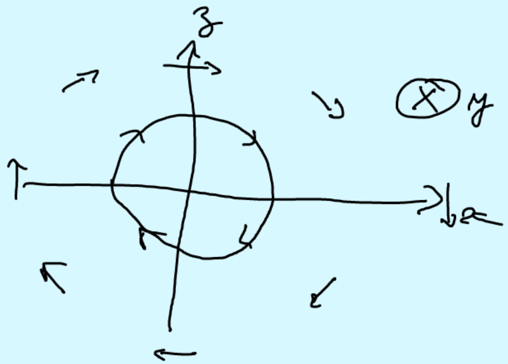


$$\bar{\bar{\Omega}} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

$$\bar{\bar{\Omega}}_{xz} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

$$\frac{\partial u}{\partial z} = 2$$

$$\frac{\partial w}{\partial x} = -2$$



② $\vec{u} = (\alpha x, -\alpha y, 0)$

no $f(z)$
no $f(t)$

$$\begin{cases} \cancel{\rho \frac{\partial u}{\partial t}} + \rho u \frac{\partial u}{\partial x} + \cancel{\rho v \frac{\partial u}{\partial y}} + \cancel{\rho w \frac{\partial u}{\partial z}} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \cancel{\mu \frac{\partial^2 u}{\partial y^2}} + \cancel{\mu \frac{\partial^2 u}{\partial z^2}} \\ \cancel{\rho \frac{\partial v}{\partial t}} + \cancel{\rho u \frac{\partial v}{\partial x}} + \rho v \frac{\partial v}{\partial y} + \cancel{\rho w \frac{\partial v}{\partial z}} = -\frac{\partial p}{\partial y} + \cancel{\mu \frac{\partial^2 v}{\partial x^2}} + \mu \frac{\partial^2 v}{\partial y^2} + \cancel{\mu \frac{\partial^2 v}{\partial z^2}} \\ \cancel{\rho \frac{\partial w}{\partial t}} + \cancel{\rho u \frac{\partial w}{\partial x}} + \cancel{\rho v \frac{\partial w}{\partial y}} + \cancel{\rho w \frac{\partial w}{\partial z}} = -\frac{\partial p}{\partial z} + \cancel{\mu \frac{\partial^2 w}{\partial x^2}} + \cancel{\mu \frac{\partial^2 w}{\partial y^2}} + \cancel{\mu \frac{\partial^2 w}{\partial z^2}} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0 \end{cases}$$

$$\begin{cases} \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad \rho \alpha^2 x = -\frac{\partial p}{\partial x} \\ \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial y^2} \quad \Rightarrow \quad \rho \alpha^2 y = -\frac{\partial p}{\partial y} \\ 0 = -\frac{\partial p}{\partial z} \quad \Rightarrow \quad \frac{\partial p}{\partial z} = 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \longrightarrow \quad \alpha - \alpha = 0 \quad \Rightarrow \text{flow is incompressible} \end{cases}$$

$$\frac{\partial p}{\partial x} = -\rho \alpha^2 x \quad \Rightarrow \quad p = -\rho \frac{\alpha^2 x^2}{2} + k_1(y, z)$$

$$\frac{\partial p}{\partial y} = -\rho \alpha^2 y \quad \Rightarrow \quad p = -\rho \frac{\alpha^2 y^2}{2} + k_2(x, z)$$

$$\frac{\partial p}{\partial z} = 0 \quad \Rightarrow \quad p = k_3(x, y)$$

$$k_4 = p(x=0, y=0)$$

$$\Rightarrow \boxed{p = -\frac{\rho \alpha^2}{2}(x^2 + y^2) + k_4}$$

3 $$(\vec{u} \cdot \vec{\nabla})\vec{u} = \frac{1}{2}\vec{\nabla}|\vec{u}|^2 - \vec{u} \times (\vec{\nabla} \times \vec{u})$$

$$\underline{\vec{u} \times (\vec{\nabla} \times \vec{u})}\Big|_i = \varepsilon_{ijk}\, u_j\, (\vec{\nabla} \times \vec{u})\Big|_k$$

$$= \varepsilon_{ijk}\, u_j\, \varepsilon_{klm} \frac{\partial}{\partial x_l}(u_m)$$

$$= \varepsilon_{ijk}\, \varepsilon_{klm}\, u_j \frac{\partial}{\partial x_l}(u_m)$$

$$= \varepsilon_{kij}\, \varepsilon_{klm}\, u_j \frac{\partial}{\partial x_l}(u_m)$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, u_j \frac{\partial}{\partial x_l}(u_m)$$

$$= u_j \frac{\partial}{\partial x_i}(u_j) - u_j \frac{\partial}{\partial x_j}(u_i)$$

$$\vec{u} \times (\vec{\nabla} \times \vec{u})\Big|_i = \frac{1}{2} \frac{\partial}{\partial x_i}(u_j^2) - u_j \frac{\partial}{\partial x_j}(u_i)$$

$$\Rightarrow \vec{u} \times (\vec{\nabla} \times \vec{u}) = \frac{1}{2} \vec{\nabla} \|\vec{u}\|^2 - (\vec{u} \cdot \vec{\nabla}) \vec{u}$$

$$\Rightarrow (\vec{u} \cdot \vec{\nabla}) \vec{u} = \frac{1}{2} \vec{\nabla} \|\vec{u}\|^2 - \vec{u} \times (\vec{\nabla} \times \vec{u})$$

$$\vec{\nabla}^2 \vec{u} = \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{u})$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{u})\Big|_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} \frac{\partial}{\partial x_l}(u_m)$$

$$= \varepsilon_{kij} \varepsilon_{klm} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l}(u_m)$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l}(u_m)$$

$$= \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i}(u_j) - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j}(u_i)$$

$$= \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}(u_j) - \frac{\partial^2}{\partial x_j^2}(u_i)$$

$$\Longrightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \vec{\nabla}^2 \vec{u}$$

$$\Longrightarrow \vec{\nabla}^2 \vec{u} = \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{u})$$

---

(4) Solution: 2

$$\begin{cases} \rho \cancel{u \dfrac{\partial u}{\partial x}} = -\dfrac{\partial p}{\partial x} + \cancel{\mu \dfrac{\partial^2 u}{\partial x^2}} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ 0 = -\dfrac{\partial p}{\partial y} \\ \dfrac{\partial u}{\partial x} = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} 0 = -\underbrace{\dfrac{\partial p}{\partial x}}_{p(x)} + \mu \underbrace{\dfrac{\partial^2 u}{\partial y^2}}_{u(y)} \\ 0 = -\dfrac{\partial p}{\partial y} \end{cases} \Longrightarrow 0 = -\dfrac{\partial^2 p}{\partial x^2} + \cancel{\dfrac{\partial}{\partial x}\left(\mu \dfrac{\partial^2 u}{\partial y^2}\right)}^{\,0}$$

$$\Rightarrow \frac{\partial p}{\partial x} = cst$$

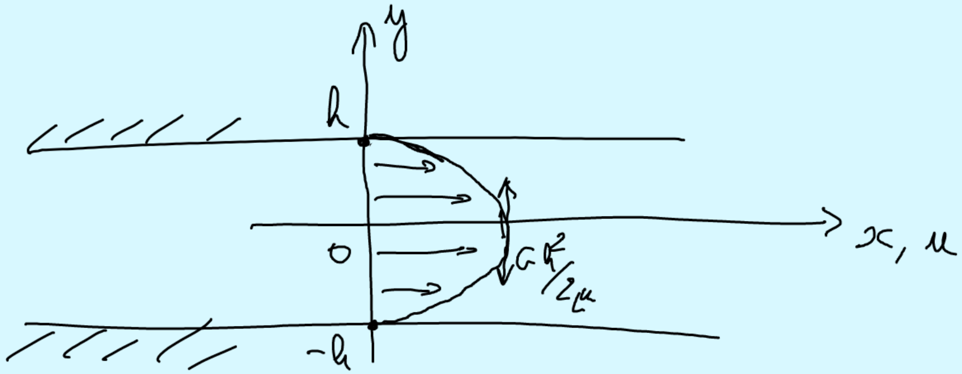
$$\Longrightarrow 0 = G + \mu \frac{\partial^2 u}{\partial y^2}$$

$$\Longrightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{G}{\mu}$$

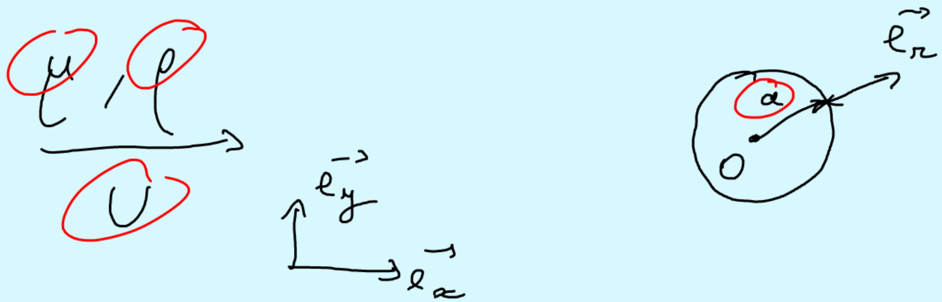
$$\Rightarrow \quad u = -\frac{G}{2\mu} y^2 + k_1 y + k_2$$

BC: $y = \pm h$, $u = 0 \quad \Rightarrow \quad u = \frac{G}{2\mu}\left(h^2 - y^2\right)$

(5) Sketching the flow:



The Reynolds number



4 parameters

Equations : $\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u}$

$\vec{\nabla}\cdot\vec{u} = 0$

4 parameters

BC : $r = a$: $\vec{u} = \vec{0}$

$r \to \infty$: $\vec{u} \to U \vec{e_x}$

Non-dimensionalization : $\vec{x} = a\,\vec{x}^*$
(similarity theory)

$[x] = L$

$[a] = L$

$\Rightarrow [x^*] = \frac{[x]}{[a]} = 1$, x^* is dimensionless
NO unit

$a = 5\text{ cm} = 5.10^{-2}\text{ m}$
$x = 1\text{ m}$
$\Rightarrow x^* = 20$

$\vec{u} = U\,\vec{u}^*$

$[u] = L.T^{-1}$

$[U] = L.T^{-1}$

$\Rightarrow [u^*] = 1$

$$t = \frac{a}{U}\, t^*$$

$$\hookrightarrow \frac{[a]}{[U]} = T \implies [t^*] = 1$$

$$p = P\, p^*$$

NOT the total pressure but a prescribed pressure scale

Into the equations: $\vec{\nabla}\cdot\vec{u} = 0 \implies \frac{1}{a}\vec{\nabla}^*\cdot\left(U\vec{u}^*\right) = 0$

$$\implies \frac{U}{a}\vec{\nabla}^*\cdot u^* = 0$$

$$\implies \vec{\nabla}^*\cdot\vec{u}^* = 0$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e_x} + \frac{\partial}{\partial y}\vec{e_y} + \dots$$

$$= \frac{\partial}{\partial(a x^*)}\vec{e_x} + \frac{\partial}{\partial(a y^*)}\vec{e_y} + \dots$$

$$= \frac{1}{a}\left[\frac{\partial}{\partial x^*}\vec{e_x} + \frac{\partial}{\partial y^*}\vec{e_y} + \dots\right]$$

$$\rho\left[\frac{\partial(U\vec{u}^*)}{\partial\left(\frac{a}{U}t^*\right)} + \left[(U\vec{u}^*)\cdot\frac{1}{a}\vec{\nabla}^*\right](U\vec{u}^*)\right] = -\frac{1}{a}\vec{\nabla}^*(P p^*) + \mu\frac{1}{a^2}\vec{\nabla}^{*2}(U\vec{u}^*)$$

$$\implies \frac{\rho U^2}{a}\left[\frac{\partial\vec{u}^*}{\partial t^*} + (\vec{u}^*\cdot\vec{\nabla}^*)\vec{u}^*\right] = -\frac{P}{a}\vec{\nabla}^* p^* + \frac{\mu U}{a^2}\vec{\nabla}^{*2}\vec{u}^*$$

8

$$\Rightarrow \quad \frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^* \cdot \vec{\nabla}^*)\vec{u}^* = -\frac{P}{\rho U^2}\vec{\nabla}^* p^* + \underbrace{\frac{\mu}{\rho U a}}_{\to \frac{1}{Re}} \vec{\nabla}^{*2}\vec{u}^*$$

$Re = \frac{\rho U a}{\mu}$ is the $\underbrace{\text{Reynolds number}}_{\text{dimensionless}}$

$= \frac{\text{inertial effects}}{\text{viscous effects}}$

2 choices for P:

- $P = \rho U^2 \leftarrow$ inertial pressure scale

$$\frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^* \cdot \vec{\nabla}^*)\vec{u}^* = -\vec{\nabla}^* p^* + \frac{1}{Re}\vec{\nabla}^{*2}\vec{u}^*$$

Useful for inertia driven flows ($Re \gg 1$), be careful, you cannot get rid of the rightmost term (singular perturbation)

- $P = \frac{\mu U}{a} \leftarrow$ viscous pressure scale

$$Re\left[\frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^* \cdot \vec{\nabla}^*)\vec{u}^*\right] = -\vec{\nabla}^* p^* + \vec{\nabla}^{*2}\vec{u}^*$$

Useful for viscous flows ($Re \ll 1$), dominated by the RHS

$\Rightarrow$ Stokes flow ($\vec{\nabla}^* p^* = \vec{\nabla}^{*2}\vec{u}^*$)

Incompressibility : $\vec{\nabla}^* \cdot \vec{u}^* = 0$

BC : $r = a$, $\vec{u} = \vec{0}$ $\longrightarrow$ $r^* = 1$, $\vec{u}^* = \vec{0}$

$r \to \infty$, $\vec{u} \to U \vec{e}_x$ $\longrightarrow$ $r^* \to \infty$, $\vec{u}^* \to \vec{e}_x$

1 parameter

$Re = \frac{\rho U a}{\mu}$

Chapter 3: Low Re flows

$$Re = \frac{\rho U a}{\mu} \ll 1$$

viscous scaling for the pressure: $P = \frac{\mu U}{a}$

$$\Rightarrow \quad Re \left[\frac{\partial \vec{u}^*}{\partial t^*} + \left(\vec{u}^* \cdot \vec{\nabla}^*\right) \vec{u}^* \right] = -\vec{\nabla}^* p^* + \vec{\nabla}^{*2} \vec{u}^*$$

$$\Rightarrow \quad \vec{0} = -\vec{\nabla}^* p^* + \vec{\nabla}^{*2} \vec{u}^*$$

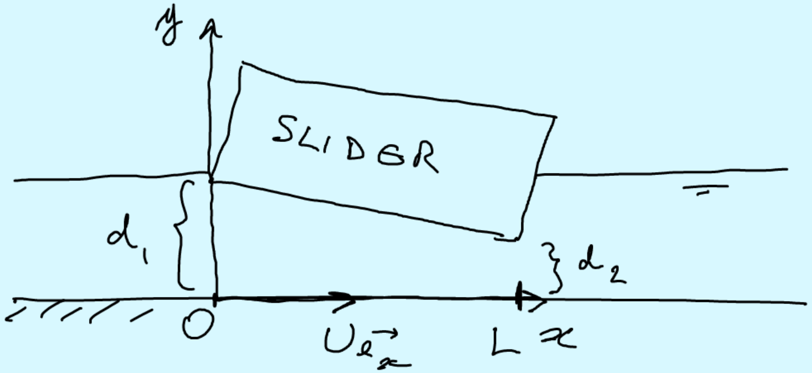
Revert the non-dimensionalization : $\vec{0} = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u}$

$$\Rightarrow \boxed{\vec{\nabla} p = \mu \vec{\nabla}^2 \vec{u}} \quad \text{Stokes equation}$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

The slider bearing flow

(0)



$$h(x) = d_1 + \underbrace{\frac{d_2 - d_1}{L}}_{h' = \frac{dh}{dx}} x$$

$$\bar{h} = \frac{d_1 + d_2}{2}$$

(3) Hypothesis :
- Low Re flow
- $|h'| \ll 1$

(1) Equations : Stokes flow

$$\begin{cases} \dfrac{\partial p}{\partial x} = \mu \left[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} \right] \\[2ex] \dfrac{\partial p}{\partial y} = \mu \left[\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} \right] \\[2ex] \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

(2) BC : at $y = 0$: $u = U$, $v = 0$

$y = h$: $u = 0$, $v = 0$

(4) Non-dimensionalization :

$$x = \bar{h}/h' \, x^*$$
$$y = \bar{h} \, y^*$$
$$u = U u^*$$
$$v = V v^*$$ not known
$$p = P p^*$$

(5) Determination of V and P :

$|h'| \ll 1$

- Incompressibility :

$$\frac{U h'}{\bar{h}} \frac{\partial u^*}{\partial x^*} + \frac{V}{\bar{h}} \frac{\partial v^*}{\partial y^*} = 0$$

$$\Rightarrow \quad U h' \frac{\partial u^*}{\partial x^*} + V \frac{\partial v^*}{\partial y^*} = 0$$

$\Rightarrow \quad V = U h'$ to allow for two-dimensional solutions.

- Stokes equation :

$$\frac{P h'}{\bar{h}} \frac{\partial p^*}{\partial x^*} = \mu \left(\frac{U h'^2}{\bar{h}^2} \frac{\partial^2 u^*}{\partial x^{*2}} + \frac{U}{\bar{h}^2} \frac{\partial^2 u^*}{\partial y^{*2}} \right)$$

$$\Rightarrow P \frac{\partial p^*}{\partial x^*} = \mu U \left(\cancel{\frac{h'}{\bar{h}} \frac{\partial^2 u^*}{\partial x^{*2}}} + \frac{1}{\bar{h} h'} \frac{\partial^2 u^*}{\partial y^{*2}} \right)$$

smaller compared to the other diffusion term

$$\Rightarrow P \frac{\partial p^*}{\partial x^*} = \frac{\mu U}{\bar{h} h'} \frac{\partial^2 u^*}{\partial y^{*2}} \longrightarrow P = \frac{\mu U}{\bar{h} h'}$$

$$\frac{P}{\bar{h}}\frac{\partial p^*}{\partial y^*} = \mu\left(\frac{U h'^3}{\bar{h}^2}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{U h'}{\bar{h}^2}\frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

$$\Rightarrow P\frac{\partial p^*}{\partial y^*} = \mu U\left(\underbrace{\cancel{\frac{h'^3}{\bar{h}}\frac{\partial^2 v^*}{\partial x^{*2}}}}_{\text{same reason}} + \frac{h'}{\bar{h}}\frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

$$\Rightarrow P\frac{\partial p^*}{\partial y^*} = \frac{\mu U h'}{\bar{h}}\frac{\partial^2 v^*}{\partial y^{*2}} \longrightarrow P = \frac{\mu U h'}{\bar{h}}$$

We take $P = \dfrac{\mu U}{\bar{h} h'}$, the larger of the two scales.

(6) Summary:

$$x = \frac{\bar{h}}{h'}x^*$$

$$y = \bar{h} y^*$$

$$u = U u^* + O(h'^2)$$

$$v = U h' v^* + O(h'^2)$$

$$p = \frac{\mu U}{\bar{h} h'} p^* + \frac{\mu U}{\bar{h}} p_0^* + O(h')$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{\partial p^*}{\partial x^*} = \frac{\partial^2 u^*}{\partial y^{*2}}$$

$$\frac{\partial p^*}{\partial y^*} = 0$$

$$y^* = 0 \begin{cases} u^* = 1 \\ v^* = 0 \end{cases}$$

$$y^* = h^* \begin{cases} u^* = 0 \\ v^* = 0 \end{cases}$$

$$h = \bar{h} h^*$$

Problem 4

$\hookrightarrow \quad \vec{u} = w(x,y)\,\vec{e_z}$

(a)
$$\begin{cases} \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = -\dfrac{G}{\mu} \\ w = 0 \quad \text{on } \delta S \end{cases}$$

(b)
$$w(x,y) = A\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

δS: ellipse of equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

If we inject the ellipse equation into the velocity expression, we get $w = 0$

$$\frac{\partial^2 w}{\partial x^2} = \frac{2A}{a^2} \quad , \quad \frac{\partial^2 w}{\partial y^2} = \frac{2A}{b^2}$$

Into NS: $\dfrac{2A}{a^2} + \dfrac{2A}{b^2} = -\dfrac{G}{\mu} \Rightarrow A\left(\dfrac{b^2+a^2}{a^2 b^2}\right) = -\dfrac{G}{2\mu}$

$$\Rightarrow A = -\frac{G a^2 b^2}{2\mu(a^2+b^2)}$$

$$\hookrightarrow \quad w = -\frac{G a^2 b^2}{2\mu (a^2+b^2)}\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right]$$

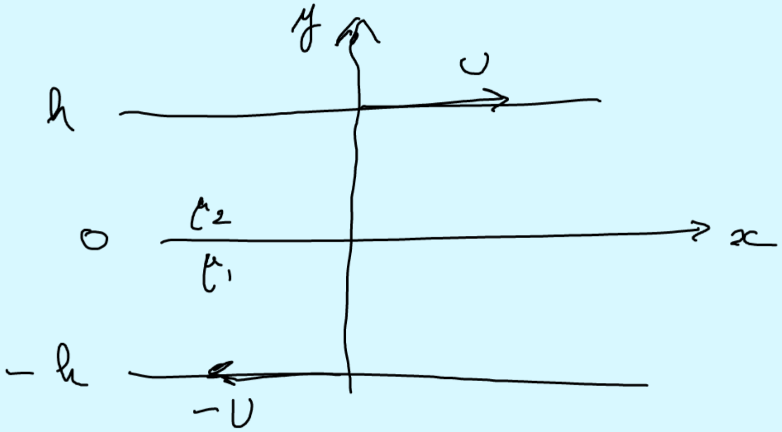
$$= \frac{G a^2 b^2}{2\mu (a^2+b^2)}\left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]$$

$$a = b \Rightarrow \quad w = \frac{G a^2}{4\mu}\left[1 - \frac{1}{a^2}\left(x^2+y^2\right)\right]$$

$$= \frac{G}{4\mu}\left[a^2 - \left(x^2+y^2\right)\right]$$

Going to cylindrical coordinates, we get: $w = \dfrac{G}{4\mu}\left(a^2 - r^2\right)$

(2)



$$\bar{\bar{\tau}} = -P\bar{\bar{I}} + \bar{\bar{\sigma}}$$

$$\tau_{ij} = -P\delta_{ij} + \sigma_{ij}$$

$$\vec{u} = u(y)\vec{e}_x$$

NS: $\dfrac{\partial u}{\partial x} + \cancel{\dfrac{\partial v}{\partial y}} = 0$

$$\boxed{0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{G}{\mu}$$

$$u = -\frac{G}{2\mu}y^2 + k_1 y + k_2$$

(a) BC at $y = 0$: continuity of the velocities: $u_1(y=0) = u_2(y=0) = u_0$

continuity of the stresses: $\tau_{xy_1}(y=0) = \tau_{xy_2}(y=0)$

$$\Rightarrow \mu_1 \left.\frac{\partial u_1}{\partial y}\right|_{y=0} = \mu_2 \left.\frac{\partial u_2}{\partial y}\right|_{y=0}$$

b) no pressure gradient: $\frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \begin{cases} u_1 = k_1 y + k_2 \\ u_2 = k_3 y + k_4 \end{cases}$

$u_1(-h) = -U$ (i)

$u_2(h) = U$ (ii)

~~$u_1(0) = u_2(0) = u_0$~~ (iii)

$\left(\mu_1 \left.\frac{\partial u_1}{\partial y}\right|_{y=0} = \mu_2 \left.\frac{\partial u_2}{\partial y}\right|_{y=0} \right.$ (iv)

$$\text{(iii)} \Rightarrow \begin{cases} u_1 = k_1 y + u_0 \\ u_2 = k_3 y + u_0 \end{cases}$$

$$\text{(i)} \Rightarrow -k_1 h + u_0 = -U \Rightarrow k_1 = \frac{u_0 + U}{h}$$

$$\text{(ii)} \Rightarrow k_3 h + u_0 = U \Rightarrow k_3 = \frac{U - u_0}{h}$$

LINEAR $\Rightarrow u_1 = \frac{u_0 + U}{h} y + u_0$ $\quad u_2 = \frac{U - u_0}{h} y + u_0$

$$u_1 = u_0\left(1 + \frac{y}{h}\right) + U\frac{y}{h} \qquad u_2 = u_0\left(1 - \frac{y}{h}\right) + U\frac{y}{h}$$

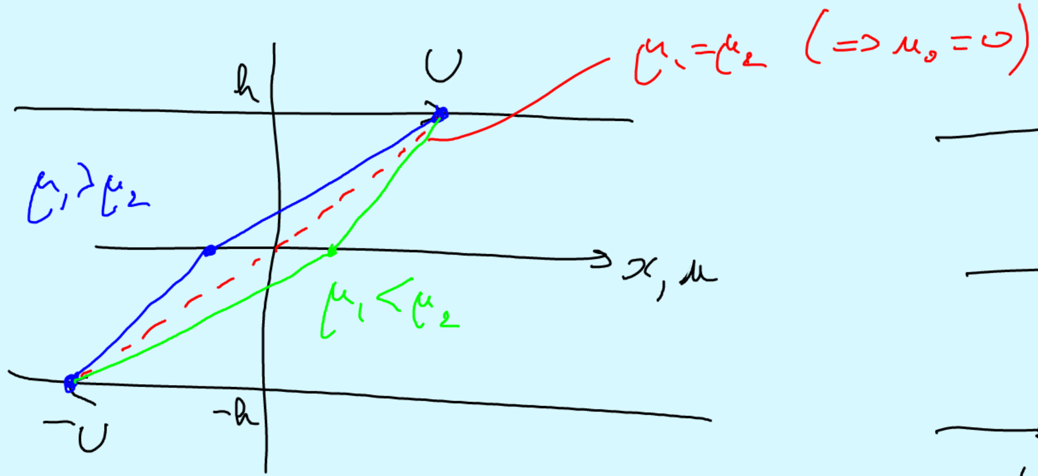
$$\mu_1 \frac{\partial u_1}{\partial y}\Big|_{y=0} = \mu_2 \frac{\partial u_2}{\partial y}\Big|_{y=0} \quad \Rightarrow \quad \mu_1 \frac{u_0+U}{h} = \mu_2 \frac{U-u_0}{h}$$

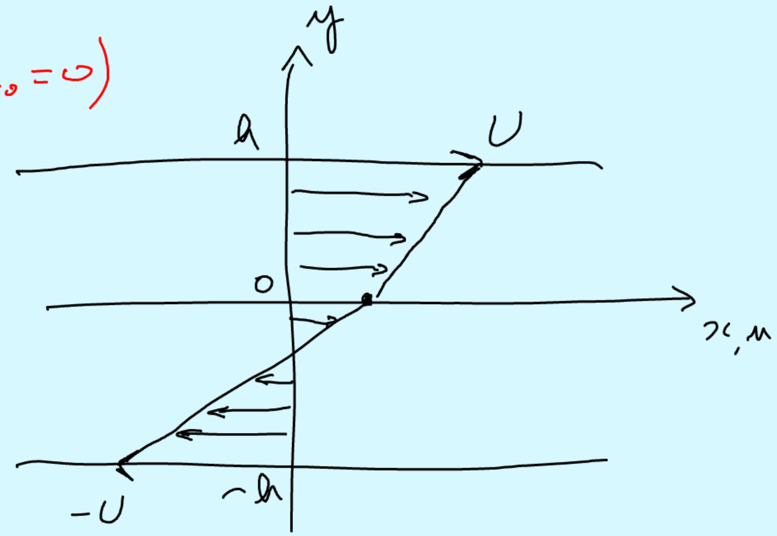
$$\Rightarrow \quad u_0(\mu_1+\mu_2) = U(\mu_2-\mu_1)$$

$$\Rightarrow \quad u_0 = U \frac{\mu_2-\mu_1}{\mu_1+\mu_2}$$

$$u_1 = U \frac{\mu_2-\mu_1}{\mu_1+\mu_2}\left(1+\frac{y}{h}\right) + U\frac{y}{h} \qquad u_2 = U \frac{\mu_2-\mu_1}{\mu_1+\mu_2}\left(1-\frac{y}{h}\right) + U\frac{y}{h}$$

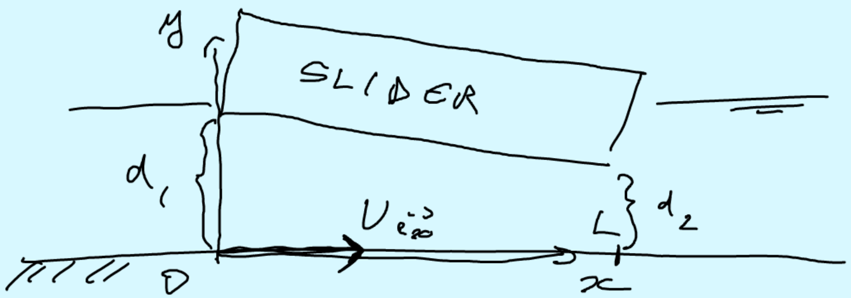
$$= U\left[\frac{\mu_2-\mu_1}{\mu_1+\mu_2}\left(1+\frac{y}{h}\right) + \frac{y}{h}\right] \qquad = U\left[\frac{\mu_2-\mu_1}{\mu_1+\mu_2}\left(1-\frac{y}{h}\right) + \frac{y}{h}\right]$$





$\mu_1 < \mu_2$

Slider bearing flow (continued):



$$h = d_1 + \underbrace{\frac{d_2 - d_1}{L}}_{h'} x \qquad |h'| \ll 1$$

$$\bar{h} = \frac{d_1 + d_2}{2}$$

$$x = \frac{\bar{h}}{h'} x^*$$

$$y = \bar{h} y^*$$

$$u = U u^*$$

$$v = h' U v^*$$

$$p = \frac{\mu U}{h' \bar{h}} p^*$$

$$\frac{\partial p^*}{\partial x^*} = \frac{\partial^2 u^*}{\partial y^{*2}} \quad (1)$$

$$\frac{\partial p^*}{\partial y^*} = 0 \quad (2)$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \quad (3)$$

$y^* = 0$ $\quad u^* = 1 \quad v^* = 0$

$y^* = h^*$ $\quad u^* = v^* = 0$

$h = \bar{h} h^*$

(6) Solution: $(2) \Rightarrow p^* = p^*(x^*)$

$$(1) \Rightarrow u^* = \frac{1}{2} \frac{\partial p^*}{\partial x^*} y^{*2} + k_1 y^* + k_2$$

BC: $y^* = 0 \rightarrow u^* = 1 \Rightarrow k_2 = 1$

$$y^* = h^*, \quad u^* = 0 \Rightarrow \frac{1}{2}\frac{\partial p^*}{\partial x^*} h^{*2} + k_1 h^* + 1 = 0$$

$$\Rightarrow k_1 = -\frac{1}{h^*} - \frac{1}{2}\frac{\partial p^*}{\partial x^*} h^*$$

$$\Rightarrow \boxed{u^* = \frac{1}{2}\frac{\partial p^*}{\partial x^*}\left(y^{*2} - h^* y^*\right) + 1 - \frac{y^*}{h^*}}$$

$$(3) \Rightarrow \frac{\partial v^*}{\partial y^*} = -\frac{\partial u^*}{\partial x^*} \Rightarrow \frac{\partial v^*}{\partial y^*} = -\frac{1}{2}\frac{\partial^2 p^*}{\partial x^{*2}}\left(y^{*2} - h^* y^*\right) + \frac{1}{2}\frac{\partial p^*}{\partial x^*}\frac{\partial h^*}{\partial x^*} y^* - \frac{y^*}{h^{*2}}\frac{\partial h^*}{\partial x^*}$$

$$\Rightarrow \int_0^{y^*} \frac{\partial v^*}{\partial \xi}\, d\xi = \int_0^{y^*}\left[-\frac{1}{2}\frac{\partial^2 p^*}{\partial x^{*2}}\left(\xi^2 - h^*\xi\right) + \frac{1}{2}\frac{\partial p^*}{\partial x^*}\frac{\partial h^*}{\partial x^*}\xi - \frac{\xi}{h^{*2}}\frac{\partial h^*}{\partial x^*}\right] d\xi$$

BC

$$\Rightarrow \boxed{v^*(y^*) = -\frac{1}{2}\frac{\partial^2 p^*}{\partial x^{*2}}\left(\frac{y^{*3}}{3} - h^*\frac{y^{*2}}{2}\right) + \frac{1}{4}\frac{\partial p^*}{\partial x^*}\frac{\partial h^*}{\partial x^*} y^{*2} - \frac{1}{h^{*2}}\frac{\partial h^*}{\partial x^*}\frac{y^{*2}}{2}}$$

$$\text{BC: } y^* = h^*, \quad v^* = 0 \Rightarrow 0 = -\frac{1}{2}\frac{\partial^2 p^*}{\partial x^{*2}}\left(\frac{h^{*3}}{3} - \frac{h^{*3}}{2}\right) + \frac{1}{4}\frac{\partial p^*}{\partial x^*}\frac{\partial h^*}{\partial x^*} h^{*2} - \frac{1}{2}\frac{\partial h^*}{\partial x^*}$$

$$\Rightarrow 0 = \frac{1}{12}\frac{\partial^2 p^*}{\partial x^{*2}} h^{*3} + \frac{1}{4}\frac{\partial p^*}{\partial x^*}\frac{\partial h^*}{\partial x^*} h^{*2} - \frac{1}{2}\frac{\partial h^*}{\partial x^*}$$

$$\Rightarrow \frac{\partial^2 p^*}{\partial x^{*2}} h^{*3} + 3\frac{\partial p^*}{\partial x^*}\frac{\partial h^*}{\partial x^*} h^{*2} = 6\frac{\partial h^*}{\partial x^*}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x^*}\left(\frac{\partial p^*}{\partial x^*} h^{*3}\right) = 6\frac{\partial h^*}{\partial x^*}}$$

Reynolds equation (equation linking the pressure gradient with the height of fluid in lubrication flows)

$$\Rightarrow \frac{\partial p^*}{\partial x^*} h^{*3} = 6h^* + k_3$$

$$\Rightarrow \frac{\partial p^*}{\partial x^*} = \frac{6}{h^{*2}} + \frac{k_3}{h^{*3}}$$

$$h = d_1 + h' x \rightarrow h^* = \frac{d_1}{\bar{h}} + \frac{h'}{\bar{h}}\frac{\bar{h}}{h'} x^*$$

$$= \frac{d_1}{\bar{h}} + x^*$$

$$\Rightarrow dh^* = dx^*$$

$$\Rightarrow \int_0^{L^*} \frac{\partial p^*}{\partial x^*} dx^* = \int_0^{L^*}\left(\frac{6}{h^{*2}} + \frac{k_3}{h^{*3}}\right) dx^*$$

$$L = \frac{\bar{h}}{h'} L^*$$

$$\Rightarrow 0 = \int_{d_1^*}^{d_2^*}\left(\frac{6}{h^{*2}} + \frac{k_3}{h^{*3}}\right) dh^*$$

because the pressure outside the slider has to be equal (and $\frac{\partial p^*}{\partial y^*} = 0$)

$$d_1 = \bar{h}\, d_1^*$$

$$d_2 = \bar{h}\, d_2^*$$

$$\Rightarrow 0 = \left(-\frac{6}{h^*} - \frac{k_3}{2h^{*2}}\right)_{d_1^*}^{d_2^*}$$

$$\Rightarrow \quad 0 = \frac{6}{d_1^*} - \frac{6}{d_2^*} + \frac{k_3}{2{d_1^*}^2} - \frac{k_3}{2{d_2^*}^2}$$

$$\Rightarrow \quad 0 = 12\left(\frac{d_2^* - d_1^*}{d_1^* d_2^*}\right) + k_3\left(\frac{{d_2^*}^2 - {d_1^*}^2}{{d_1^*}^2 {d_2^*}^2}\right)$$

$$\Rightarrow \quad k_3 = -12\,\frac{d_2^* - d_1^*}{d_1^* d_2^*}\;\frac{{d_1^*}^2 {d_2^*}^2}{{d_2^*}^2 - {d_1^*}^2}$$

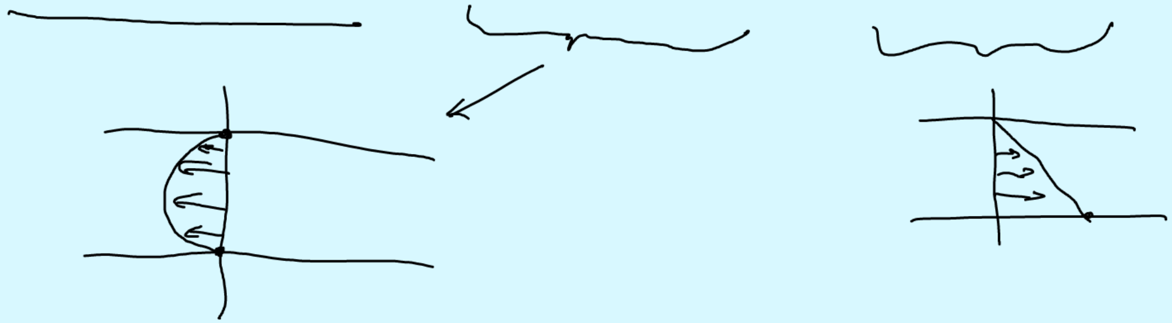
$$\Rightarrow \quad k_3 = -12\,\frac{d_1^* d_2^*}{d_1^* + d_2^*}$$

$$d_1^* + d_2^* = \frac{d_1}{h} + \frac{d_2}{h} = \frac{d_1}{\frac{d_1+d_2}{2}} + \frac{d_2}{\frac{d_1+d_2}{2}} = 2$$

$$\Rightarrow \quad k_3 = -6\, d_1^* d_2^*$$

$$\hookrightarrow \quad \frac{\partial p^*}{\partial x^*} = 6\left(\frac{1}{{h^*}^2} - \frac{d_1^* d_2^*}{{h^*}^3}\right)$$

$$\Rightarrow \quad \boxed{u^* = 3\left(\frac{1}{{h^*}^2} - \frac{d_1^* d_2^*}{{h^*}^3}\right)\left({y^*}^2 - h^* y^*\right) + 1 - \frac{y^*}{h^*}}$$

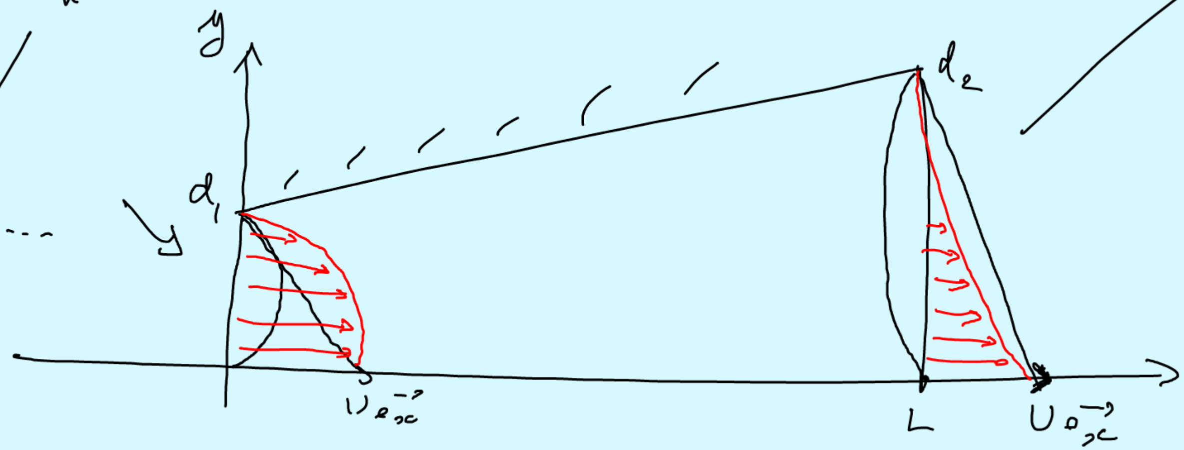


$$d_2^* = \frac{d_2}{h}$$

$$h^* = d_1^*$$

$$u^* = 3\left(\frac{1}{d_1^{*2}} - \frac{d_2^*}{d_1^{*2}}\right)\ldots$$

$$<0$$



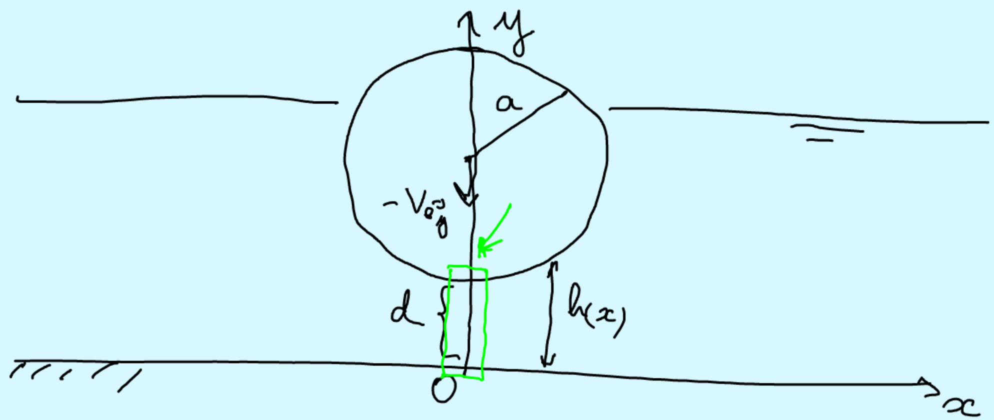
$$>0$$

$$u^* = 3\left(\frac{1}{d_2^{*2}} - \frac{d_1^*}{d_2^{*2}}\right)\ldots$$

$$d_1^* < d_2^*$$

Fluid expulsion between a cylinder and a wall

(a)



$d \ll a$

$\underline{x \ll a}$ → so that the slope of $h(x)$ is small

$$h(x) = d + a - \sqrt{a^2 - x^2}$$

$$= d + a - a\sqrt{1 - \frac{x^2}{a^2}}$$

$$\simeq d + a - a\left(1 - \frac{x^2}{2a^2}\right)$$

$$\simeq d + \frac{x^2}{2a} \longrightarrow d\,h^* = d + \frac{x^2}{2a} \Rightarrow \underline{h^* = 1 + \frac{x^2}{2ad}} \longrightarrow x \sim \sqrt{2ad} \sim \sqrt{ad} \sim d\sqrt{\frac{a}{d}}$$

$$\boxed{x \sim d\,\varepsilon^{-1} \quad \varepsilon = \sqrt{\frac{d}{a}} \ll 1}$$

$$h^* = 1 + \frac{x^{*2}}{2}$$

(1) Equations:

$$\begin{cases} \dfrac{\partial p}{\partial x} = \mu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ \dfrac{\partial p}{\partial y} = \mu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

(2) BC: $y = 0, \quad u = 0, \quad v = 0$

$y = h(x), \quad u = 0, \quad v = -V$

(3) Hypotheses: • low Re ($\Rightarrow$ Stokes equations)

• $d \ll a$, $x \ll a$

asymptotic, lubrication flow

(4) Non-dimensionalization:

$$x = d\,\varepsilon^{-1} x^*$$

$$y = d\,y^*$$

$$u = U u^* = V\varepsilon^{-1} u^*$$

$$v = V v^*$$

$$p = P p^*$$

(5) Determination of U and P: $\quad \dfrac{U}{d\varepsilon^{-1}} \dfrac{\partial u^*}{\partial x^*} + \dfrac{V}{d} \dfrac{\partial v^*}{\partial y^*} = 0$

$\Rightarrow \quad U = V\varepsilon^{-1}$ to keep the two-dimensionality of the flow.

$$\frac{P}{d\varepsilon^{-1}} \frac{\partial p^*}{\partial x^*} = \mu \left(\cancel{\frac{V\varepsilon^{-1}}{d^2 \varepsilon^{-2}} \frac{\partial^2 u^*}{\partial x^{*2}}} + \frac{V\varepsilon^{-1}}{d^2} \frac{\partial^2 u^*}{\partial y^{*2}} \right)$$

ε^1

small compared to y-diffusion

$$\Rightarrow \quad P \frac{\partial p^*}{\partial x^*} = \frac{\mu V \varepsilon^{-2}}{d} \frac{\partial^2 u^*}{\partial y^{*2}} \quad \longrightarrow \quad P = \frac{\mu V \varepsilon^{-2}}{d}$$

this is the largest scale for the pressure, so we use it.

$$\frac{P}{d} \frac{\partial p^*}{\partial y^*} = \mu \left(\cancel{\frac{V}{d^2 \varepsilon^{-2}} \frac{\partial^2 v^*}{\partial x^{*2}}} + \frac{V}{d^2} \frac{\partial^2 v^*}{\partial y^{*2}} \right)$$

ε^2

$$\Rightarrow \quad P \frac{\partial p^*}{\partial y^*} = \frac{\mu V}{d} \frac{\partial^2 v^*}{\partial y^{*2}} \quad \longrightarrow \quad \cancel{P = \frac{\mu V}{d}}$$

(6) Recap:

$$x = d\varepsilon^{-1}x^*$$
$$y = d\,y^*$$
$$u = V\varepsilon^{-1}u^*$$
$$v = V v^*$$
$$p = \frac{\mu V \varepsilon^{-2}}{d}\,p^*$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \quad (i) \checkmark$$

$$\frac{\partial p^*}{\partial x^*} = \frac{\partial^2 u^*}{\partial y^{*2}} \quad (ii) \checkmark$$

$$\frac{\partial p^*}{\partial y^*} = 0 \quad (iii) \checkmark$$

$y^* = 0$, $u^* = v^* = 0$ ✓ ✓

$y^* = h^*$, $u^* = 0$ ✓, $v^* = -1$

$d h^* = h$

---

(7) Solution: (ii) $\Rightarrow$ $u^* = \frac{1}{2}\frac{\partial p^*}{\partial x^*}\,y^{*2} + k_1 y^* + k_2$

13

BC: $y^* = 0$, $u^* = 0$ $\Rightarrow$ $k_2 = 0$

$y^* = h^*$, $u^* = 0$ $\Rightarrow$ $0 = \frac{1}{2}\frac{\partial p^*}{\partial x^*}h^{*2} + k_1 h^*$

$\Rightarrow$ $k_1 = -\frac{1}{2}\frac{\partial p^*}{\partial x^*}h^*$

$$\Rightarrow \quad u^* = \frac{1}{2}\frac{\partial p^*}{\partial x^*}\left(y^{*2} - h^* y^*\right)$$

(i) $\Rightarrow$ $\frac{\partial v^*}{\partial y^*} = -\frac{\partial u^*}{\partial x^*}$

$$= -\frac{1}{2}\frac{\partial^2 p^*}{\partial x^{*2}}\left(y^{*2} - h^* y^*\right) + \frac{1}{2}\frac{\partial p^*}{\partial x^*}\,y^*\frac{\partial h^*}{\partial x^*}$$

$$\Rightarrow \int_0^{y^*} \frac{\partial v^*}{\partial \xi}\, d\xi = \int_0^{y^*} \left[-\frac{1}{2} \frac{\partial^2 p^*}{\partial x^{*2}} \left(\xi^2 - h^* \xi \right) + \frac{1}{2} \frac{\partial p^*}{\partial x^*}\, \xi\, \frac{\partial h^*}{\partial x^*} \right] d\xi$$

$$\Rightarrow v^*(y^*) - v^*(0) = -\frac{1}{2} \frac{\partial^2 p^*}{\partial x^{*2}} \left(\frac{y^{*3}}{3} - \frac{h^* y^{*2}}{2} \right) + \frac{1}{4} \frac{\partial p^*}{\partial x^*}\, y^{*2} \frac{\partial h^*}{\partial x^*}$$

$$\Rightarrow v^*(y^*) = -\frac{1}{12} \frac{\partial^2 p^*}{\partial x^{*2}} \left(2 y^{*3} - 3 h^* y^{*2} \right) + \frac{1}{4} \frac{\partial p^*}{\partial x^*}\, y^{*2} \frac{\partial h^*}{\partial x^*} \quad \leftarrow$$

$$v^*(h^*) = -1 \quad \Rightarrow \quad -1 = -\frac{1}{12} \frac{\partial^2 p^*}{\partial x^{*2}} \left(-h^{*3} \right) + \frac{1}{4} \frac{\partial p^*}{\partial x^*} \frac{\partial h^*}{\partial x^*}\, h^{*2}$$

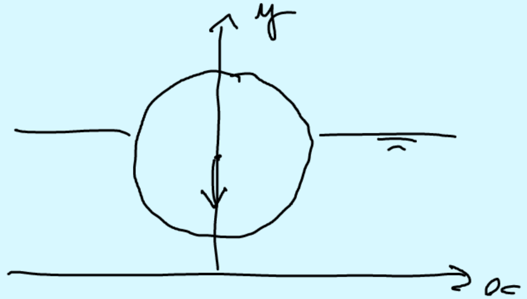
$$\Rightarrow \frac{\partial^2 p^*}{\partial x^{*2}}\, h^{*3} + 3 \frac{\partial p^*}{\partial x^*} \frac{\partial h^*}{\partial x^*}\, h^{*2} = -12$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x^*} \left[\frac{\partial p^*}{\partial x^*}\, h^{*3} \right] = -12}$$

Reynolds equation

$$\Rightarrow \frac{\partial p^*}{\partial x^*}\, h^{*3} = -12 x^* + k_3$$

$$\Rightarrow \quad \frac{\partial p^*}{\partial x^*} = -\frac{12x^*}{h^{*3}} + \frac{k_3}{h^{*3}}$$

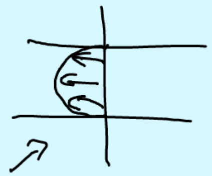


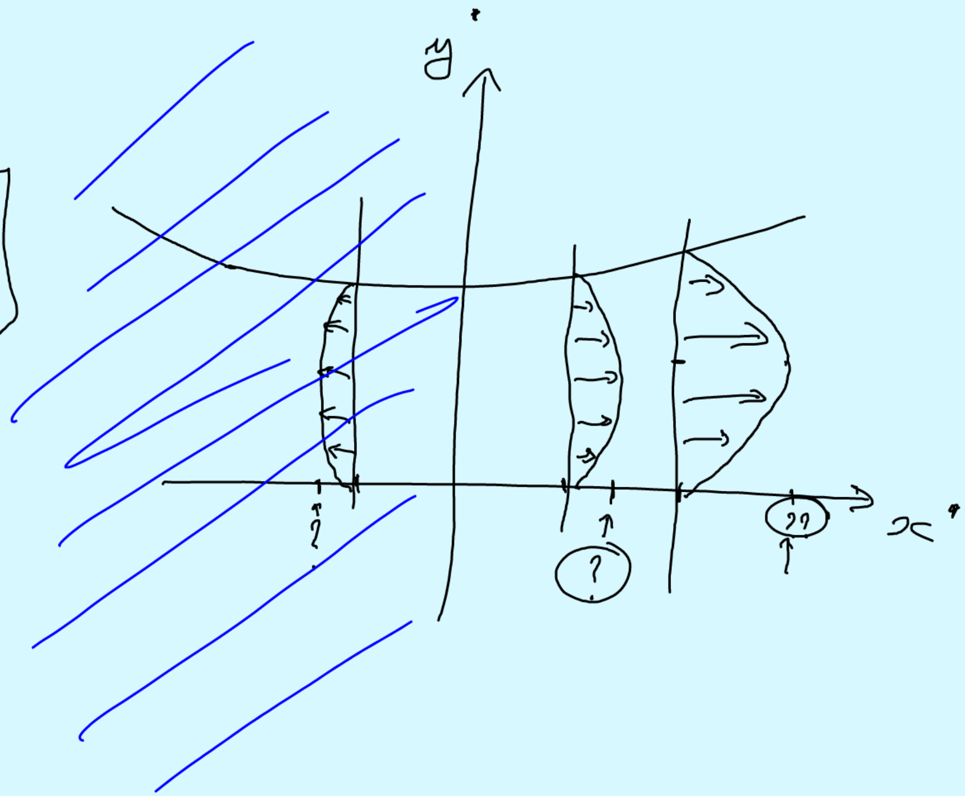
The configuration is symmetric around $x=0$, so we need to impose $\left.\frac{\partial p^*}{\partial x^*}\right|_{x^*=0} = 0$ (no force due to pressure gradient on the axis of symmetry)

$$\Rightarrow k_3 = 0$$

$$\hookrightarrow \frac{\partial p^*}{\partial x^*} = -\frac{12x^*}{h^{*3}}$$

$$\Rightarrow \boxed{u^* = -6\frac{x^*}{h^{*3}}\left(y^{*2} - h^* y^*\right)}$$

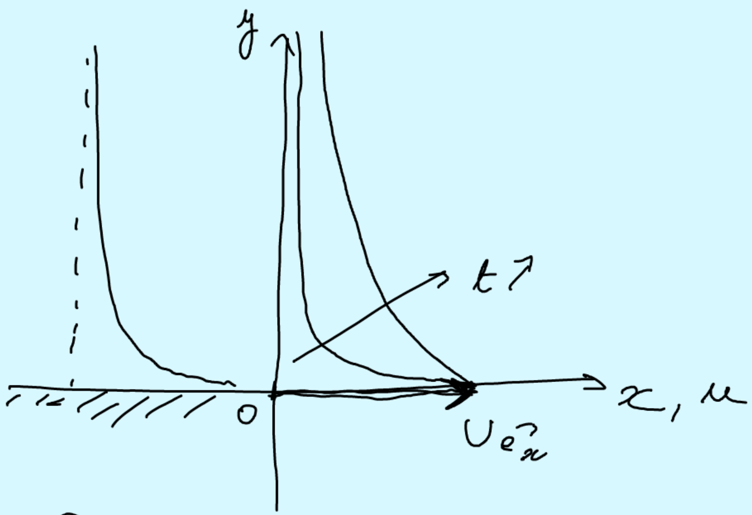




Chapter 4: Boundary layers

Stokes first problem

(0) Sketch:



(1) Equations:

$$\begin{cases} \rho\left(\partial_t u + u\partial_x u + v\partial_y u\right) = -\partial_x p + \mu\partial_x^2 u + \mu\partial_y^2 u \\ \rho\left(\partial_t v + u\partial_x v + v\partial_y v\right) = -\partial_y p + \mu\partial_x^2 v + \mu\partial_y^2 v \\ \partial_x u + \partial_y v = 0 \end{cases}$$

(2) BC: initial condition: $t=0, \quad u=0, \quad v=0$

$y=0$: $u=U, \quad v=0$

$y\to\infty$: $u\to 0, \quad v\to 0$

(3) Hypotheses : 1D flow : $\vec{u} = u\,\vec{e_x}$

Example Sheet 3

③

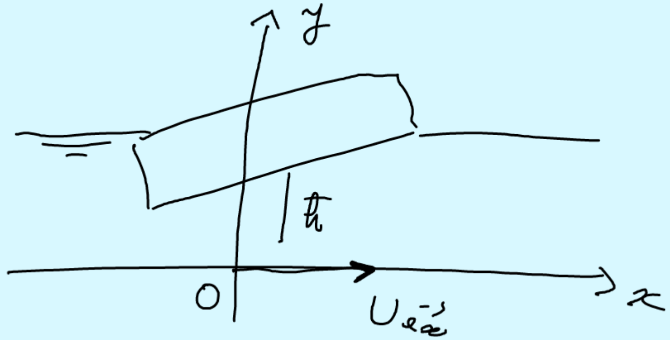
$$x = \frac{h}{h'}\, x^*$$

$$y = h\, y^*$$

$$u = U\, u^*$$

$$v = U h'\, v^*$$

$$p = \frac{\mu U}{h h'}\, p^*$$



$|h'| \ll 1$

$$\rho \frac{D\vec{u}}{Dt} = \begin{cases} \cancel{\rho \frac{\partial u}{\partial t}} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \rho U u^* \frac{\partial (U u^*)}{\partial \left(\frac{h}{h'} x^*\right)} + \dots = \rho \frac{U^2 h'}{h} u^* \frac{\partial u^*}{\partial x^*} + \dots \\ \cancel{\rho \frac{\partial v}{\partial t}} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} \end{cases}$$

$$\left.\begin{aligned} \rho u \frac{\partial u}{\partial x} &\sim \rho \frac{U^2 |h'|}{h} \\ \rho v \frac{\partial u}{\partial y} &\sim \rho \frac{U^2 |h'|}{h} \end{aligned}\right\} \Rightarrow \rho \frac{Du}{Dt} \sim \rho \frac{U^2 |h'|}{h}$$

$$\Rightarrow \rho \frac{D\vec{u}}{Dt} \sim \rho \frac{U^2 |h'|}{h}$$

$$\left.\begin{array}{l} \rho u \dfrac{\partial v}{\partial x} \sim \rho \dfrac{U^2 |h'|^2}{h} \\ \rho v \dfrac{\partial v}{\partial y} \sim \rho \dfrac{U^2 |h'|^2}{h} \end{array}\right\} \Rightarrow \rho \frac{Dv}{Dt} \sim \rho \frac{U^2 |h'|^2}{h}$$

Viscous terms:

$$\left.\begin{array}{l} \mu \dfrac{\partial^2 u}{\partial x^2} \sim \mu \dfrac{U |h'|^2}{h^2} \\ \mu \dfrac{\partial^2 u}{\partial y^2} \sim \mu \dfrac{U}{h^2} \end{array}\right\} \Rightarrow \mu \vec{\nabla}^2 u \sim \frac{\mu U}{h^2}$$

$$\left.\begin{array}{l} \mu \dfrac{\partial^2 v}{\partial x^2} \sim \mu \dfrac{U |h'|^3}{h^2} \\ \mu \dfrac{\partial^2 v}{\partial y^2} \sim \mu \dfrac{U |h'|}{h^2} \end{array}\right\} \Rightarrow \mu \vec{\nabla}^2 v \sim \mu \frac{U |h'|}{h^2}$$

$$\Rightarrow \mu \vec{\nabla}^2 \vec{u} \sim \frac{\mu U}{h^2}$$

Inertia can be neglected if: $\rho \dfrac{U^2 |h'|}{h} \ll \dfrac{\mu U}{h^2}$ and $\rho \dfrac{U^2 |h'|^2}{h} \ll \mu \dfrac{U |h'|}{h^2}$

$$\Rightarrow \rho \frac{U^2 |h'|}{h} \ll \frac{\mu U}{h^2}$$

$$\Rightarrow \quad U \ll \frac{\mu}{\rho \bar{h} |h'|}$$

$$\Rightarrow \quad U \ll \frac{2 \mu L}{\rho (d_1 + d_2) |d_2 - d_1|}$$

$$\Rightarrow \quad U \ll \frac{2 \mu L}{\rho |d_1^2 - d_2^2|}$$

$$Re = \frac{\rho U (d_1 + d_2)}{2\mu} \quad \Rightarrow \quad Re \ll \frac{\cancel{\rho}\, \cancel{2\mu} L \,(d_1 + d_2)}{\cancel{\rho}\, |d_2^2 - d_1^2|\, \cancel{2\mu}}$$

$$\Rightarrow \quad Re \ll \frac{L}{|d_2 - d_1|}$$

$$\Rightarrow \quad Re \ll \frac{1}{|h'|}$$

$$\Rightarrow \quad Re\, |h'| \ll 1$$

The Reynolds number can be large provided $\frac{1}{|h'|}$ is much larger

Continue on Stokes first problem:

(4) Solution:

$$\begin{cases} \rho\left[\partial_t u + u\partial_x u + v\partial_y u\right] = -\partial_x p + \mu\left(\partial_x^2 u + \partial_y^2 u\right) \\ \rho\left[\partial_t v + u\partial_x v + v\partial_y v\right] = -\partial_y p + \mu\left(\partial_x^2 v + \partial_y^2 v\right) \\ \partial_x u + \partial_y v = 0 \end{cases}$$

$$\vec{u} = u\,\vec{e}_x \Rightarrow \begin{cases} \rho\left[\partial_t u + u\partial_x u\right] = -\partial_x p + \mu\left(\partial_x^2 u + \partial_y^2 u\right) \\ \partial_y p = 0 \\ \partial_x u = 0 \longrightarrow u = u(y,t) \end{cases}$$

$$\Rightarrow \begin{cases} \rho\,\partial_t u = -\partial_x p + \mu\,\partial_y^2 u \\ \partial_y p = 0 \end{cases}$$

$$\longrightarrow \partial_x p = cst = 0$$ to match the BC at infinity.

$$\Rightarrow \boxed{\partial_t u = \nu\,\partial_y^2 u} \qquad \nu = \frac{\mu}{\rho}$$ kinematic viscosity

diffusion equation (heat)

2 methods : • separation of variables : $u(y,t) = Y(y)T(t)$

• similarity variable : $\eta = y t^{\alpha}$, $f(\eta) = u(y,t)$ ✓

Note : try both methods to understand from failure

$$\left.\begin{aligned} \partial_t &= \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = y a t^{a-1}\frac{\partial}{\partial \eta} \\ \partial_y &= \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y} = t^{a}\frac{\partial}{\partial \eta} \\ \partial_y^2 &= t^{2a}\frac{\partial^2}{\partial \eta^2} \end{aligned}\right\} \Rightarrow y a t^{a-1} f' = \nu t^{2a} f''$$

$$\Rightarrow f'' = \underbrace{\frac{y a t^{-a-1}}{\nu}}\, f'$$

$$y t^{-a-1} = y t^{a} \; (= \eta)$$

$$\Rightarrow a = -a-1$$

$$\Rightarrow a = -\frac{1}{2}$$

$$\eta = y t^{-1/2} = \frac{y}{\sqrt{t}} \quad \text{and} \quad \boxed{f'' = -\frac{\eta}{2\nu} f'}$$

$$\Rightarrow \quad f' = k_1 \exp\left(-\frac{\eta^2}{4\nu}\right)$$

$$\Rightarrow f(\eta) = k_1 \int_0^{\eta} \exp\left(-\frac{\xi^2}{4\nu}\right) d\xi + \overset{f(0)}{k_2}$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} \exp(-y^2)\, dy$$

$$y = \frac{\xi}{2\sqrt{\nu}} \quad \Rightarrow \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{2\sqrt{\nu}x} \exp\left(-\frac{\xi^2}{4\nu}\right) \frac{d\xi}{2\sqrt{\nu}}$$

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi\nu}} \int_0^{2\sqrt{\nu}x} \exp\left(-\frac{\xi^2}{4\nu}\right) d\xi$$

$$\eta = 2\sqrt{\nu}\, x \quad \Rightarrow \quad \operatorname{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) = \frac{1}{\sqrt{\pi\nu}} \underbrace{\int_0^{\eta} \exp\left(-\frac{\xi^2}{4\nu}\right) d\xi}$$

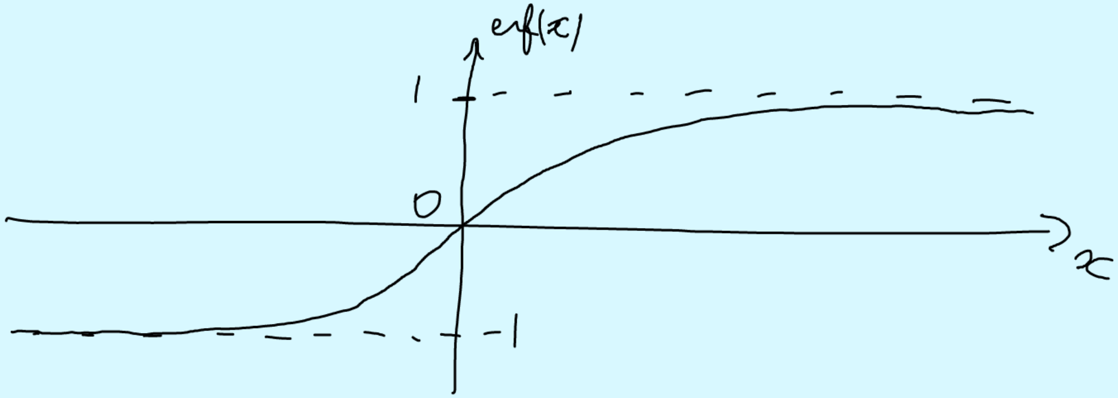
$$\Rightarrow f(\eta) = \underbrace{k_1 \sqrt{\pi\nu}}_{k_3} \operatorname{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) + k_2$$

$$\Rightarrow u(y,t) = k_3 \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) + k_2$$

BC : $y = 0 \quad u = U$

$y \to \infty \quad u \to 0$

IC : $t \to 0 \quad u \to 0$



$$y = 0 \;:\; u = k_2 = U$$

$$\Rightarrow \quad u(y,t) = k_3 \,\mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) + U$$

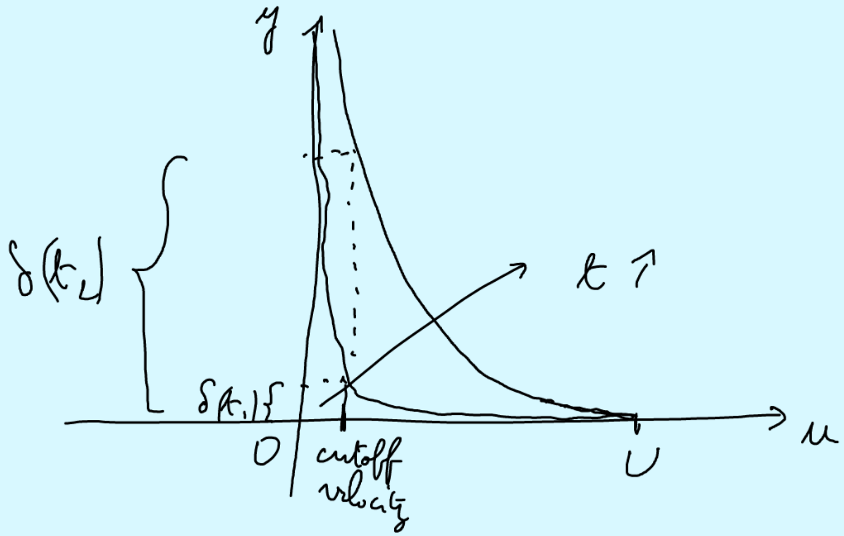
$$y \to \infty \;:\; u \to k_3 + U = 0$$

$$\Rightarrow \quad \boxed{u(y,t) = U\left[1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right]}$$

The initial condition is compatible with our solution, so it is validated

---

16



δ : viscous boundary layer thickness

$$\frac{u(\delta,t)}{U} = 0.01 \quad \Rightarrow \quad \mathrm{erf}\left(\frac{\delta_{99}}{2\sqrt{\nu t}}\right) = 0.99$$

99% criterion

$$\Rightarrow \quad \frac{\delta_{99}}{2\sqrt{\nu t}} \simeq 2$$

$$\Rightarrow \quad \delta_{99} = 4\sqrt{\nu t}$$

Vorticity equation

Navier – Stokes equation : $\rho\left[\partial_t \vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u}$ $\qquad \vec{\nabla}\cdot\vec{u} = 0$

Remember: $\vec{u}\times(\vec{\nabla}\times\vec{u}) = \frac{1}{2}\vec{\nabla}(\vec{u}^2) - (\vec{u}\cdot\vec{\nabla})\vec{u}$

$$\Rightarrow \rho\left[\partial_t \vec{u} + \frac{1}{2}\vec{\nabla}(\vec{u}^2) - \vec{u}\times(\vec{\nabla}\times\vec{u})\right] = -\vec{\nabla} p + \mu\vec{\nabla}^2\vec{u}$$

$$\Rightarrow \rho\left[\partial_t \vec{u} - \vec{u}\times(\vec{\nabla}\times\vec{u})\right] = -\vec{\nabla}\left(p + \frac{1}{2}\rho\vec{u}^2\right) + \mu\vec{\nabla}^2\vec{u}$$

$$\Rightarrow \rho\left[\partial_t \vec{u} - \vec{u}\times\vec{\omega}\right] = -\vec{\nabla}\left(p + \frac{1}{2}\rho\vec{u}^2\right) + \mu\vec{\nabla}^2\vec{u}$$

Taking the curl: $\rho\left[\partial_t\vec{\omega} - \vec{\nabla}\times(\vec{u}\times\vec{\omega})\right] = \vec{0} + \mu\vec{\nabla}^2\vec{\omega}$

$$\vec{\nabla}\times(\vec{u}\times\vec{\omega})\Big|_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{u}\times\vec{\omega})\Big|_k$$

$$= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\varepsilon_{klm}u_l\omega_m\right)$$

$$= \varepsilon_{ijk}\,\varepsilon_{klm}\,\frac{\partial}{\partial x_j}\left(u_l\,\omega_m\right)$$

$$= \varepsilon_{kij}\,\varepsilon_{klm}\,\frac{\partial}{\partial x_j}\left(u_l\,\omega_m\right)$$

$$= \left(\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}\right)\left[\frac{\partial}{\partial x_j}\left(u_l\,\omega_m\right)\right]$$

$$= \frac{\partial}{\partial x_j}\left(u_i\,\omega_j\right) - \frac{\partial}{\partial x_j}\left(u_j\,\omega_i\right)$$

$$= \underset{\text{incompressibility}}{\cancel{u_i\,\frac{\partial \omega_j}{\partial x_j}}} + \omega_j\,\frac{\partial u_i}{\partial x_j} - u_j\,\frac{\partial \omega_i}{\partial x_j} - \underset{\text{incompressibility}}{\cancel{\omega_i\,\frac{\partial u_j}{\partial x_j}}}$$

$$= \omega_j\,\frac{\partial u_i}{\partial x_j} - u_j\,\frac{\partial \omega_i}{\partial x_j}$$

$$= \left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{u}\,\Big|_i - \left(\vec{u}\cdot\vec{\nabla}\right)\vec{\omega}\,\Big|_i$$

$$\Rightarrow \quad \rho\left[\partial_t\vec{\omega} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{\omega} - \left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{u}\right] = \mu\,\vec{\nabla}^2\vec{\omega}$$

$$\Rightarrow \quad \boxed{\rho\Big[\underbrace{\underset{\text{rate of change}}{\partial_t\vec{\omega}} + \underset{\text{advection}}{\left(\vec{u}\cdot\vec{\nabla}\right)\vec{\omega}}}_{\text{inertia}}\Big] = \underset{\text{? vortex stretching}}{\rho\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{u}} + \underset{\text{diffusion}}{\mu\,\vec{\nabla}^2\vec{\omega}}}$$

Vorticity equation

$$\partial_t f = \partial_x^2 f$$

$$\partial_t \hat{f} = -k_x^2\,\hat{f}$$

$\hat{k} = \hat{k}_0 \, e^{-k_x t}$

Let us isolate $(\vec{\omega}\cdot\vec{\nabla})\vec{u}$: $\quad \rho\, \partial_t \vec{\omega} = \rho (\vec{\omega}\cdot\vec{\nabla})\vec{u}$

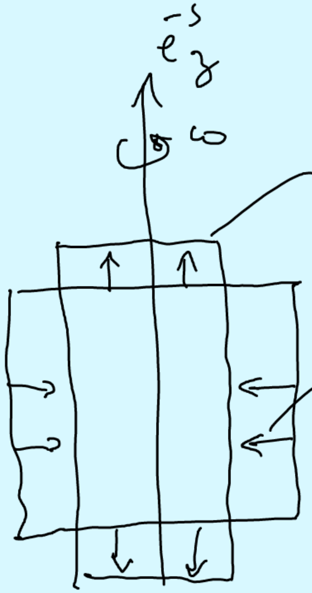
Aligning $\vec{\omega}$ with $\vec{e}_z$: $\quad \vec{\omega} = \omega\, \vec{e}_z \quad \rightarrow \quad \partial_t \omega = \omega \dfrac{\partial u_z}{\partial z}$

u_z velocity in $\vec{e}_z$

$$\Rightarrow \frac{1}{\omega}\partial_t \omega = \frac{\partial u_z}{\partial z}$$

relative rate of change : positive $\rightarrow$ to infinity

negative $\rightarrow$ to 0

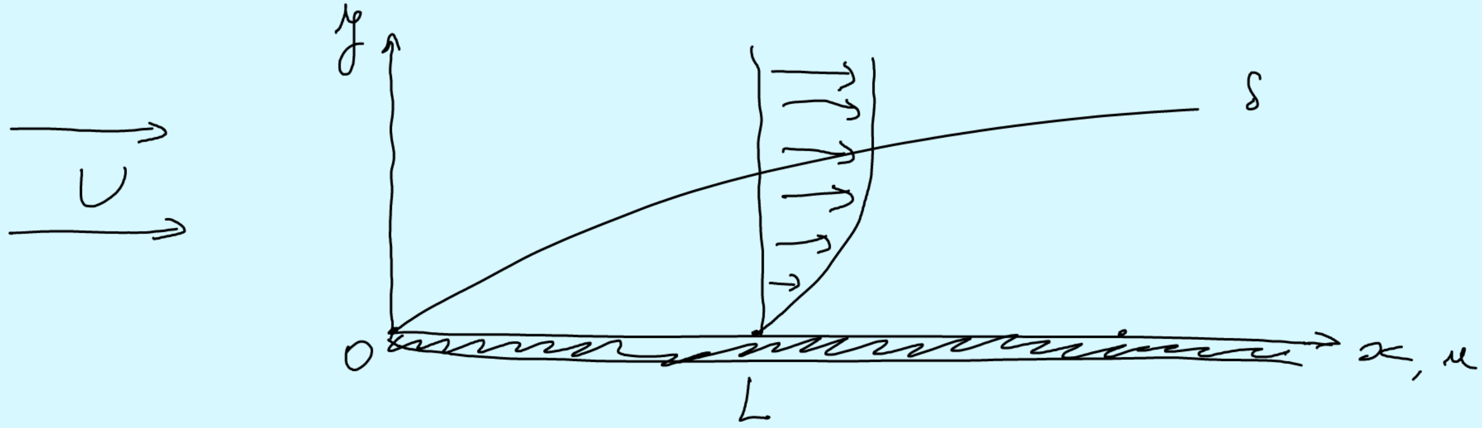


$\dfrac{\partial u_z}{\partial z} > 0 \;\Rightarrow$ acceleration of the fluid spin

incompressibility

The Blasius boundary layer

BLASIUS



Hypotheses : • 2D flow
• steady
• Large Re
• thin boundary layer } $\Rightarrow \varepsilon = \frac{\delta}{L} \ll 1$

Equations : $\begin{cases} \rho u \partial_x u + \rho v \partial_y u = -\partial_x p + \mu \partial_x^2 u + \mu \partial_y^2 u \\ \rho u \partial_x v + \rho v \partial_y v = -\partial_y p + \mu \partial_x^2 v + \mu \partial_y^2 v \\ \partial_x u + \partial_y v = 0 \end{cases}$

3 unknowns : u, v, p
2D : x, y

BC : $y = 0$: $u = 0$, $v = 0$
$y \to \infty$: $u \to U$, $v \to 0$

Nondimensionalization : $x = L x^*$
$y = \delta y^* = \varepsilon L y^*$
$u = U u^*$

$\delta = \delta(L)$

$v = \varepsilon U v^*$ ⟵ obtained via the incompressibility condition

$p = \rho U^2 p^*$

Into the equations:

$$\begin{cases} \rho \dfrac{U^2}{L}\left(u^* \dfrac{\partial u^*}{\partial x^*} + v^* \dfrac{\partial u^*}{\partial y^*}\right) = -\rho\dfrac{U^2}{L}\dfrac{\partial p^*}{\partial x^*} + \mu\left[\cancel{\dfrac{U}{L^2}\dfrac{\partial^2 u^*}{\partial x^{*2}}} + \dfrac{U}{\varepsilon^2 L^2}\dfrac{\partial^2 u^*}{\partial y^{*2}}\right] \\ \rho\dfrac{\varepsilon U^2}{L}\left(u^*\dfrac{\partial v^*}{\partial x^*} + v^*\dfrac{\partial v^*}{\partial y^*}\right) = -\rho\dfrac{U^2}{\varepsilon L}\dfrac{\partial p^*}{\partial y^*} + \mu\left[\cancel{\dfrac{\varepsilon U}{L^2}\dfrac{\partial^2 v^*}{\partial x^{*2}}} + \dfrac{U}{\varepsilon L^2}\dfrac{\partial^2 v^*}{\partial y^{*2}}\right] \\ \dfrac{\partial u^*}{\partial x^*} + \dfrac{\partial v^*}{\partial y^*} = 0 \end{cases}$$

because they are smaller than their y-counterpart

$$\Rightarrow \begin{cases} u^*\dfrac{\partial u^*}{\partial x^*} + v^*\dfrac{\partial u^*}{\partial y^*} = -\dfrac{\partial p^*}{\partial x^*} + \dfrac{1}{Re_L}\varepsilon^{-2}\dfrac{\partial^2 u^*}{\partial y^{*2}} \\ \cancel{u^*\dfrac{\partial v^*}{\partial x^*}} + \cancel{v^*\dfrac{\partial v^*}{\partial y^*}} = -\varepsilon^{-2}\dfrac{\partial p^*}{\partial y^*} + \cancel{\dfrac{1}{Re_L}\varepsilon^{-2}\dfrac{\partial^2 v^*}{\partial y^{*2}}} \\ \dfrac{\partial u^*}{\partial x^*} + \dfrac{\partial v^*}{\partial y^*} = 0 \end{cases}$$

$Re_L = \dfrac{\rho U L}{\mu} \gg 1$

all small compared to the pressure gradient

To preserve the balance between inertia and viscous effects, we need $Re_L = \varepsilon^{-2}$

$\|$

$$\hookrightarrow \begin{cases} \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial p}{\partial y} = 0 \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \end{cases}$$

$$\Downarrow$$

$$\rho\frac{UL}{\mu} = \frac{L^2}{\delta^2}$$

$$\Rightarrow \boxed{\delta = \sqrt{\frac{\mu L}{\rho U}}}$$

$$\frac{\delta}{L} = \sqrt{\frac{\mu}{\rho UL}} \Rightarrow \frac{\delta}{L} = Re_L^{-1/2}$$

Because $\frac{\partial p}{\partial y} = 0$, the pressure inside the boundary layer can be obtained along a streamline located infinitely far away:

$$p + \frac{1}{2}\rho|\vec{u}|^2 = cst \Rightarrow p = cst'$$

$$\Rightarrow \frac{\partial p}{\partial x} = 0$$

$$\Rightarrow \begin{cases} \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \end{cases}$$

~~2~~ unknowns : u, v

2D : x, y

Let us introduce the streamfunction : $u = \frac{\partial \psi}{\partial y}$ $\quad v = -\frac{\partial \psi}{\partial x}$

$$\hookrightarrow \boxed{\partial_y \psi \, \partial_{xy} \psi - \partial_x \psi \, \partial_{yy} \psi = \nu \, \partial_{yyy} \psi} \quad , \quad \nu = \frac{\mu}{\rho}$$

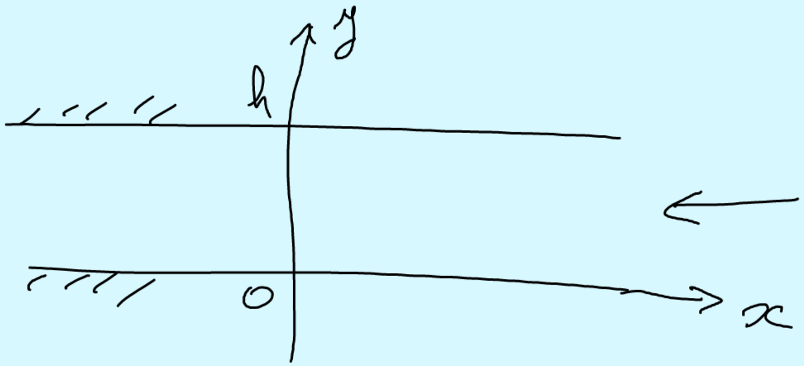
1 unknown : ψ

2D : x, y

18

Example Sheet 4

Problem 2



at $t=0$

$\vec{\nabla} p = -G \vec{e}_x \qquad G > 0$

(a)

The Navier-Stokes equations in 2D read :

$$\begin{cases} \rho \left[\dfrac{\partial u}{\partial t} + u \dfrac{\partial u}{\partial x} + v \dfrac{\partial u}{\partial y} \right] = - \dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u}{\partial x^2} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ \rho \left[\dfrac{\partial v}{\partial t} + u \dfrac{\partial v}{\partial x} + v \dfrac{\partial v}{\partial y} \right] = - \dfrac{\partial p}{\partial y} + \mu \dfrac{\partial^2 v}{\partial x^2} + \mu \dfrac{\partial^2 v}{\partial y^2} \end{cases}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Let us write $\vec{u} = u(y,t)\,\vec{e_x}$, the remaining terms are:

$$\begin{cases} \rho \dfrac{\partial u}{\partial t} = -\dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ 0 = -\dfrac{\partial p}{\partial y} \end{cases}$$

Because p does not depend on y: $\rho \dfrac{\partial u}{\partial t} - \mu \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial p}{\partial x} = G$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = \frac{G}{\rho} + \nu \frac{\partial^2 u}{\partial y^2}, \quad \nu = \frac{\mu}{\rho} \qquad (1)$$

(b) Let us consider $u_{ss}(y) = \dfrac{G\,y(h-y)}{2\rho\nu}$

Let us look at: $\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{G}{\rho\nu}$

In the right-hand-side of equation (1): $\dfrac{G}{\rho} + \nu \dfrac{\partial^2 u_{ss}}{\partial y^2} = \dfrac{G}{\rho} + \nu\left(-\dfrac{G}{\rho\nu}\right)$

$= 0$

Since u_{SS} does not depend on time, $\frac{\partial u_{SS}}{\partial t} = 0$ and we can conclude that u_{SS} satisfies equation (1).

(c) We write $u = u_{SS} + v$

$$= \frac{G\,y(h-y)}{2\rho\nu} + v$$

We inject this expression into equation (1):

$$\frac{\partial u_{SS}}{\partial t} + \frac{\partial v}{\partial t} = \frac{G}{\rho} + \nu \frac{\partial^2 u_{SS}}{\partial y^2} + \nu \frac{\partial^2 v}{\partial y^2}$$

[Based on the previous calculations: $\frac{\partial u_{SS}}{\partial t} = \frac{G}{\rho} + \nu \frac{\partial^2 u_{SS}}{\partial y^2} = 0$, and we get:]

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial y^2} \qquad (2)$$

The boundary conditions are: $u(y=0,t) = 0 \Rightarrow u_{SS}(y=0) + v(y=0,t) = 0$

$$\Rightarrow \left.\frac{G\,y(h-y)}{2\rho\nu}\right|_{y=0} + v(y=0,t) = 0$$

$$\Rightarrow v(y=0,t) = 0$$

$$u(y=h,t)=0 \implies u_{ss}(y=h)+v(y=h,t)=0$$

$$\implies \left.\frac{Gy(h-y)}{2\rho\nu}\right|_{y=h} + v(y=h,t)=0$$

$$\implies v(y=h,t)=0$$

(d) We use separation of variables: $v(y,t)=Y(y)\,T(t)$

In equation (2): $YT' = \nu Y''T \implies \frac{T'}{\nu T} = \frac{Y''}{Y} = -k_1 \quad , \; k_1 \in \mathbb{C}$

$$\implies \begin{cases} T' = -\nu k_1 T \\ Y'' = -k_1 Y \end{cases}$$

$$\implies \begin{cases} T = k_2 \exp(-\nu k_1 t) \\ Y = k_3 \cos(\sqrt{k_1}\, y) + k_4 \sin(\sqrt{k_1}\, y) \end{cases}$$

$$(k_2, k_3, k_4) \in \mathbb{C}^3$$

$$\Longrightarrow v(y,t) = \exp(-\nu k_1 t)\left[k_5 \cos(\sqrt{k_1}\, y) + k_6 \sin(\sqrt{k_1}\, y)\right]$$

$$k_5 = k_2 k_3 \ , \quad k_6 = k_2 k_4$$

We now apply the boundary conditions: • $v(y=0) = 0 \Rightarrow k_5 = 0$

$$\Rightarrow v(y,t) = k_6 \sin(\sqrt{k_1}\, y) \exp(-\nu k_1 t)$$

• $v(y=h) = 0 \Rightarrow \sqrt{k_{1_n}}\, h = n\pi$, n is an integer

$$\Rightarrow k_{1_n} = \frac{n^2\pi^2}{h^2}$$

$$\Rightarrow v_n(y,t) = k_{6_n} \sin\left(\frac{n\pi y}{h}\right) \exp\left(-\frac{\nu n^2\pi^2}{h^2} t\right)$$

$$\Rightarrow v(y,t) = \sum_{n=1}^{\infty} k_{6_n} \sin\left(\frac{n\pi y}{h}\right) \exp\left(-\frac{\nu n^2\pi^2}{h^2} t\right)$$

(e) at $t = 0$: $-\dfrac{G\, y(h-y)}{2\rho\nu} = \displaystyle\sum_n a_n \sin\left(\frac{n\pi y}{h}\right)$

$$\int_0^h \sin\left(\frac{n\pi y}{h}\right)\sin\left(\frac{q\pi y}{h}\right)dy = \int_0^h \frac{e^{\frac{in\pi y}{h}} - e^{-\frac{in\pi y}{h}}}{2i}\cdot\frac{e^{\frac{iq\pi y}{h}} - e^{-\frac{iq\pi y}{h}}}{2i}\,dy$$

n, q are positive integers

$$= \int_0^h \frac{e^{\frac{i(n+q)\pi y}{h}} + e^{-\frac{i(n+q)\pi y}{h}} - e^{\frac{i(n-q)\pi y}{h}} - e^{-\frac{i(n-q)\pi y}{h}}}{-4}\,dy$$

$$= \int_0^h \left[\underbrace{-\frac{1}{2}\cos\left[\frac{(n+q)\pi y}{h}\right]}_{=0} + \frac{1}{2}\cos\left[\frac{(n-q)\pi y}{h}\right]\right]dy$$

$$= \begin{cases} \frac{h}{2} & \text{if } n=q \\ 0 & \text{otherwise} \end{cases}$$

19

$$\partial_y\psi\,\partial_{xy}\psi - \partial_x\psi\,\partial_{yy}\psi = \nu\,\partial_{yyy}\psi$$

Blasius : the boundary layer is self-similar $\therefore$ $\eta = y\left(\frac{U}{\nu x}\right)^{1/2}$

$= y/g(x)$ $\qquad g(x) = \left(\frac{\nu x}{U}\right)^{1/2}$

$\bullet \; u = U f' \qquad f' = \dfrac{df}{d\eta}$

note: $u = \dfrac{\partial \psi}{\partial y} \Rightarrow U f' = \dfrac{\partial \psi}{\partial \eta} \dfrac{\partial \eta}{\partial y}$

$\Rightarrow \dfrac{\partial \psi}{\partial \eta} = U f' \delta$

$\Rightarrow \psi = U f \delta$

$\dfrac{\partial \eta}{\partial x} = -\dfrac{y \delta'}{\delta^2}$

$\dfrac{\partial \eta}{\partial y} = \dfrac{1}{\delta}$

We have: $\partial_x \psi = U f' \delta \dfrac{\partial \eta}{\partial x} + U f \delta'$

$= -\dfrac{U y \delta'}{\delta} f' + U f \delta' \quad = -v$

$\partial_y \psi = U \delta f' \dfrac{\partial \eta}{\partial y}$

$= U f' \quad = u$

$\partial_{yy} \psi = \partial_y (U f')$

$= U f'' \dfrac{\partial \eta}{\partial y}$

$= \dfrac{U f''}{\delta}$

$\partial_{yyy} \psi = \dfrac{U f'''}{\delta^2}$

$$\partial_{xy}\psi = \partial_x\left(U f'\right)$$

$$= U f'' \frac{\partial \eta}{\partial x}$$

$$= -\frac{U f'' y \delta'}{\delta^2}$$

$$U f'\left[-\frac{U y \delta'}{\delta^2} f''\right] - \left(U \delta' f - \frac{U y \delta'}{\delta} f'\right)\frac{U f''}{\delta} = \nu \frac{U f'''}{\delta^2}$$

$$\Rightarrow -\frac{U^2 y \delta'}{\delta^2} f' f'' - \underline{\frac{U^2 \delta'}{\delta}} f f'' + \frac{U^2 y \delta'}{\delta^2} f' f'' = \frac{\nu U}{\delta^2} f'''$$

$$\Rightarrow \frac{\nu U}{\delta^2} f''' + \frac{U^2 \delta'}{\delta} f f'' = 0$$

$$\Rightarrow f''' + \frac{U \delta \delta'}{\nu} f f'' = 0$$

Remember: $\delta = \left(\frac{\nu x}{U}\right)^{1/2} \Rightarrow \delta' = \frac{1}{2}\left(\frac{\nu}{U x}\right)^{1/2}$

$$\Rightarrow \delta \delta' = \frac{1}{2}\left(\frac{\nu}{U x}\right)^{1/2}\left(\frac{\nu x}{U}\right)^{1/2}$$

$$= \frac{1}{2}\frac{\nu}{U}$$

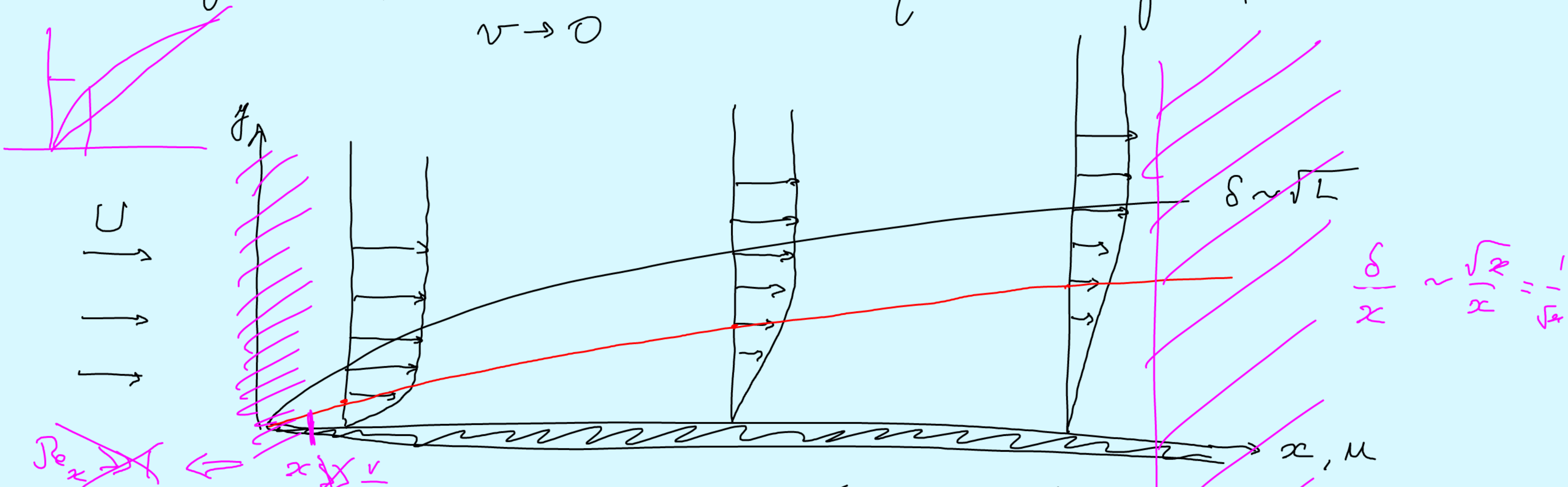
In the end, we get: $\boxed{f''' + \frac{1}{2} f f'' = 0}$

1 unknown: f

1D: η

BC: $y = 0$, $u = 0$, $v = 0$ $\eta = 0$ $f = 0$, $f' = 0$

$y \to \infty$, $u \to U$, $v \to 0$ $\eta \to \infty$ $f' \to 1$



The boundary layer is located in $\eta \leq 5$ (numerically)

$$\Rightarrow \delta_{99}\left(\frac{U}{\nu x}\right)^{1/2} = 5$$

$$\Rightarrow \delta_{99} = 5 \left(\frac{\nu x}{U}\right)^{1/2}$$

$$\Rightarrow \frac{\delta_{99}}{x} = 5 \left(\frac{\nu}{U x}\right)^{1/2} \qquad Re_x = \frac{U x}{\nu}$$

$$\Rightarrow \frac{\delta_{99}}{x} = 5 \, Re_x^{-1/2}$$

aspect ratio of the boundary layer

Chapter 5: Potential flows

20

Bernoulli equations: inviscid flow $\Rightarrow$ Euler equation

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} \right] = - \vec{\nabla} P - \rho g \vec{e}_z$$

see Example sheet 1:

$$(\vec{u} \cdot \vec{\nabla}) \vec{u} = \frac{1}{2} \vec{\nabla} \|\vec{u}\|^2 - \vec{u} \times \overbrace{\vec{\nabla} \times \vec{u}}^{\vec{\omega}}$$

$$\Rightarrow \quad \rho\left[\frac{\partial \vec{u}}{\partial t} - \vec{u}\times\vec{\omega}\right] = -\vec{\nabla}\left[P + \frac{1}{2}\rho\|\vec{u}\|^2\right] - \rho g\,\vec{e_z}$$

$$\Rightarrow \quad \boxed{\rho\left[\frac{\partial \vec{u}}{\partial t} - \vec{u}\times\vec{\omega}\right] = -\vec{\nabla}\left[P + \frac{1}{2}\rho\|\vec{u}\|^2 + \rho g z\right]}$$

First law : steady , irrotational

$$\vec{0} = -\vec{\nabla}\left[P + \frac{1}{2}\rho\|\vec{u}\|^2 + \rho g z\right]$$

$$\Rightarrow \quad P + \frac{1}{2}\rho\|\vec{u}\|^2 + \rho g z = cst(t)$$

$$\Rightarrow \quad \boxed{P + \frac{1}{2}\rho\|\vec{u}\|^2 + \rho g z = cst}$$

Second law : steady

$$-\rho\,\vec{u}\times\vec{\omega} = -\vec{\nabla}\left[P + \frac{1}{2}\rho\|\vec{u}\|^2 + \rho g z\right]$$

$$\Rightarrow \quad \rho\,\vec{u}\cdot\underbrace{(\vec{u}\times\vec{\omega})}_{\perp \vec{u}} = \vec{u}\cdot\vec{\nabla}\left[P + \frac{1}{2}\rho\|\vec{u}\|^2 + \rho g z\right]$$

$$\Rightarrow \quad 0 = \vec{u} \cdot \vec{\nabla}\left[P + \frac{1}{2}\rho\|\vec{u}\|^2 + \rho g z\right]$$

$$\Rightarrow \quad \vec{u} \cdot \vec{\nabla} f = 0 \qquad , \qquad f = P + \frac{1}{2}\rho\|\vec{u}\|^2 + \rho g z$$

$$\Rightarrow \quad \vec{u} \perp \vec{\nabla} f$$

$\Rightarrow$ f is constant in the direction tangent to $\vec{u}$

$\Rightarrow$ f is constant along a streamline

$$\boxed{P + \frac{1}{2}\rho\|\vec{u}\|^2 + \rho g z = \text{cst}(\psi)}$$

Planar, incompressible, irrotational flows:

$\vec{u} = u(x,y)\vec{e_x} + v(x,y)\vec{e_y}$ $\quad$ $\vec{\nabla}\cdot\vec{u} = 0$ $\quad$ $\vec{\nabla}\times\vec{u} = \vec{0}$

Do we need Navier–Stokes given the above assumptions?

* Incompressible : $\vec{\nabla}\cdot\vec{u} = 0 \Rightarrow$ streamfunction ψ

$$u = \frac{\partial \psi}{\partial y} \quad , \quad v = -\frac{\partial \psi}{\partial x}$$

note: $\vec{\nabla}\cdot\vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$

$$= \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y}$$

$$= 0$$

$$(\vec{\nabla}\times\vec{u})\cdot\vec{e_z} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$= -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}$$

Irrotational $\Rightarrow$ $\boxed{\vec{\nabla}^2 \psi = 0}$

* Irrotational: $\vec{\nabla}\times\vec{u} = \vec{0}$ $\Rightarrow$ velocity potential ϕ

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \qquad (\vec{u} = \vec{\nabla}\phi)$$

note: $(\vec{\nabla}\times\vec{u})\cdot\vec{e_z} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

$$= \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}$$

$$= 0$$

Incompressibility: $\vec{\nabla}\cdot\vec{u} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\Rightarrow \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}^2 \Phi = 0}$$

ψ and Φ are solutions of homogeneous Laplace equations $\longrightarrow$ harmonic function theory

Complex calculus interlude: $\quad f : \mathbb{C} \mapsto \mathbb{C}$

f is complex differentiable if: $\displaystyle \frac{df}{dz} = \lim_{|\delta z| \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ returns the same value no matter the direction of δz.

Let us introduce $\quad f(z) = g(x,y) + i\, h(x,y) \qquad z = x + iy$

$$g : \mathbb{R}^2 \longmapsto \mathbb{R}$$

$$h : \mathbb{R}^2 \longmapsto \mathbb{R}$$

$\delta z = \delta x$: $\quad \displaystyle \frac{df}{dz} = \frac{\partial g}{\partial x} + i \frac{\partial h}{\partial x}$

$\delta z = i\delta y$: $\quad \displaystyle \frac{df}{dz} = \frac{\partial g}{i\partial y} + i \frac{\partial h}{i \partial y} = \frac{\partial h}{\partial y} - i \frac{\partial g}{\partial y}$

If f is complex differentiable : $\frac{\partial g}{\partial x} + i\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} - i\frac{\partial g}{\partial y}$

$$\Rightarrow \begin{cases} \frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} & (1) \\ \frac{\partial h}{\partial x} = -\frac{\partial g}{\partial y} & (2) \end{cases}$$ Cauchy – Riemann equations

$\frac{\partial(1)}{\partial x} + \frac{\partial(2)}{\partial y}$: $\frac{\partial^2 g}{\partial x^2} + \cancel{\frac{\partial^2 h}{\partial x \partial y}} = \cancel{\frac{\partial^2 h}{\partial x \partial y}} - \frac{\partial^2 g}{\partial y^2}$

$\Rightarrow \vec{\nabla}^2 g = 0$

$\frac{\partial(1)}{\partial y} - \frac{\partial(2)}{\partial x}$: $\vec{\nabla}^2 h = 0$

same conditions as ψ and ϕ for a planar, incompressible, irrotational flow

How about writing a complex potential : $w = \phi + i\psi$

velocity potential

streamfunction

w is complex differentiable $\Rightarrow \vec{\nabla}^2\phi = \vec{\nabla}^2\psi = 0$

$$\underbrace{w}_{\text{complex potential}} = \underbrace{\phi}_{\text{velocity potential}} + i\,\underbrace{\psi}_{\text{streamfunction}}$$

If w is complex differentiable, then it is associated with a planar, incompressible and irrotational flow.

We have: $\vec{\nabla}^2 \phi = \vec{\nabla}^2 \psi = 0$

Because these equations are linear and homogeneous, we can apply the principle of superposition: if ϕ_1 and ϕ_2 are solutions, then $\alpha_1 \phi_1 + \alpha_2 \phi_2$ is also a solution.
$(\alpha_1, \alpha_2) \in \mathbb{R}$

Building blocks of potential flow theory:

* uniform flow: $\boxed{w = \overline{U_0 e^{-i\alpha} z}}$

$$\overline{\frac{dw}{dz}} = U_0 e^{i\alpha}$$

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x}$$

$$= u - iv$$

$$\frac{dw}{dz} = \frac{\partial \phi}{i\partial y} + i \frac{\partial \psi}{i\partial y}$$

$$= U_0 (\cos\alpha + i \sin\alpha)$$

$$= -i \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y}$$

$$= -iv + u$$

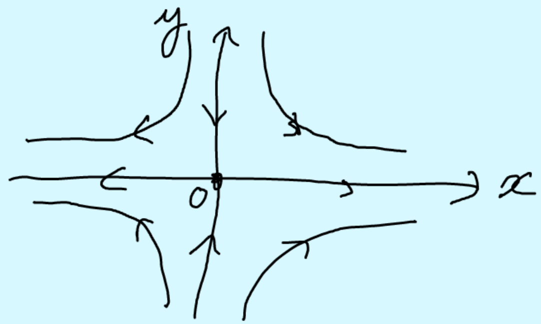
$$u + iv = \overline{\frac{dw}{dz}} \leftarrow \text{complex conjugated}$$

* saddle point flow: $w = Az^2, \quad A \in \mathbb{R}$

$$\frac{dw}{dz} = 2Az$$

$$= 2Ax + i\,2Ay$$

$$u + iv = 2Ax - i\,2Ay$$



$A > 0$

* sources/sinks: $w = A \ln z$

$$= A \ln(re^{i\theta})$$

$$= \underbrace{A \ln r}_{\phi} + i \underbrace{A\theta}_{\psi}$$

$$u_r = \frac{\partial \phi}{\partial r} = \frac{A}{r} \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

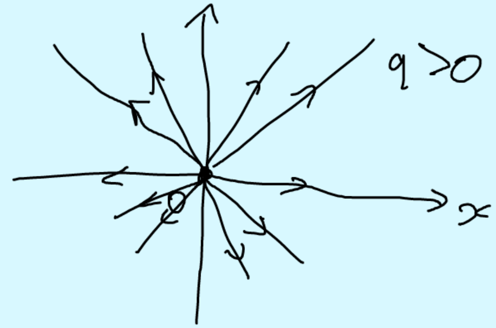
$A > 0$ source

$A < 0$ sink

inflow : $q = \int_0^{2\pi} u_r \Big|_{r=a} a \, d\theta$

$$= \int_0^{2\pi} A \, d\theta$$

$$= 2\pi A$$



$$\Rightarrow A = \frac{q}{2\pi}$$

The source/sink complex potential is: $\boxed{w = \frac{q}{2\pi} \ln z}$

* Vortex : $\boxed{w = \frac{-i\Gamma}{2\pi} \ln z}$

$$= -\frac{i\Gamma}{2\pi} \ln(r e^{i\theta})$$

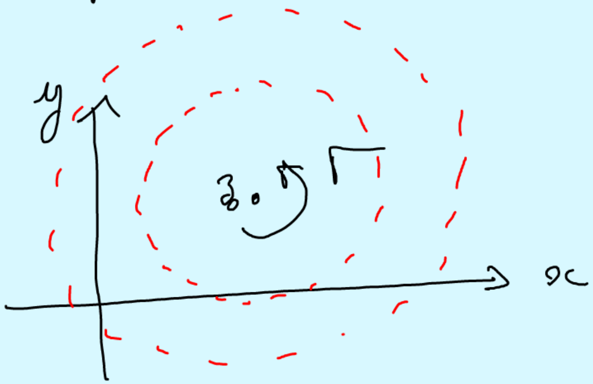
$$= \underbrace{-\frac{i\Gamma}{2\pi} \ln r}_{i\psi} + \underbrace{\frac{\Gamma\theta}{2\pi}}_{\phi}$$

$$u_r = \frac{\partial \phi}{\partial r} = 0 \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r} \rightarrow \text{rotation}$$

$$\int_0^{2\pi} u_\theta \Big|_{r=a} a\, d\theta = \int_0^{2\pi} \frac{\Gamma}{2\pi} d\theta = \Gamma \text{ — circulation}$$

What about boundary conditions ?

Method of images:



$$f(z) = -\frac{i\Gamma}{2\pi} \ln(z - z_0)$$

To create a wall at $y=0$, we apply : $w(z) = f(z) + \overline{f(\bar{z})}$

Let $z = x$, $w(x) = f(x) + \overline{f(\bar{x})}$

$$= f(x) + \overline{f(x)}$$

$$\in \mathbb{R}$$

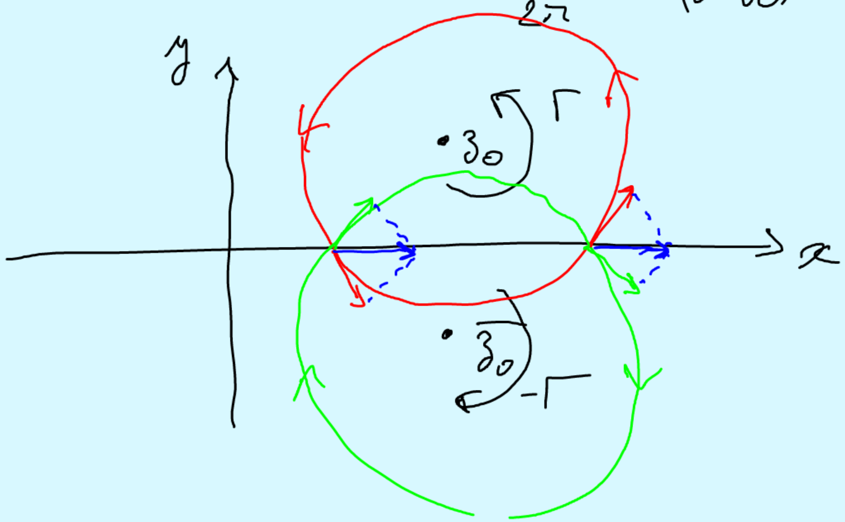
$$w = \phi + i \underbrace{\psi}_{=0}$$

On $z = x$, $\psi = 0$. Since the streamfunction is constant on the $z = x$ line, this line is a streamline.

In our example: $w(z) = f(z) + \overline{f(\bar{z})}$

$$= -\frac{i\Gamma}{2\pi} \ln(z - z_0) + \overline{-\frac{i\Gamma}{2\pi} \ln(\bar{z} - z_0)}$$

$$= -\frac{i\Gamma}{2\pi} \ln(z - z_0) + \frac{i\Gamma}{2\pi} \ln(z - \bar{z}_0)$$



To create a wall at $x=0$, we apply: $w(z) = f(z) + \overline{f(-\bar{z})}$

The Milne-Thompson circle theorem:

To create a boundary on $|z| = a$, we apply the Milne-Thompson circle theorem:

$$w(z) = f(z) + \overline{f\left(\frac{a^2}{\bar{z}}\right)}$$

$$z = a e^{i\theta} \rightarrow w(a e^{i\theta}) = f(a e^{i\theta}) + \overline{f\left(\frac{a^2}{a e^{-i\theta}}\right)}$$

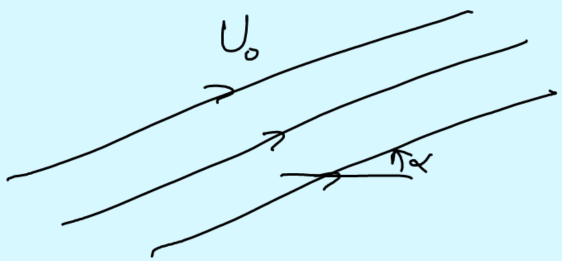
$$= f(a e^{i\theta}) + \overline{f(a e^{i\theta})}$$

$$\in \mathbb{R}$$

$\Rightarrow$ The circle $|z| = a$ is a streamline

Example: flow past a cylinder

(1) uniform flow: $f(z) = U_0 e^{-i\alpha} z$

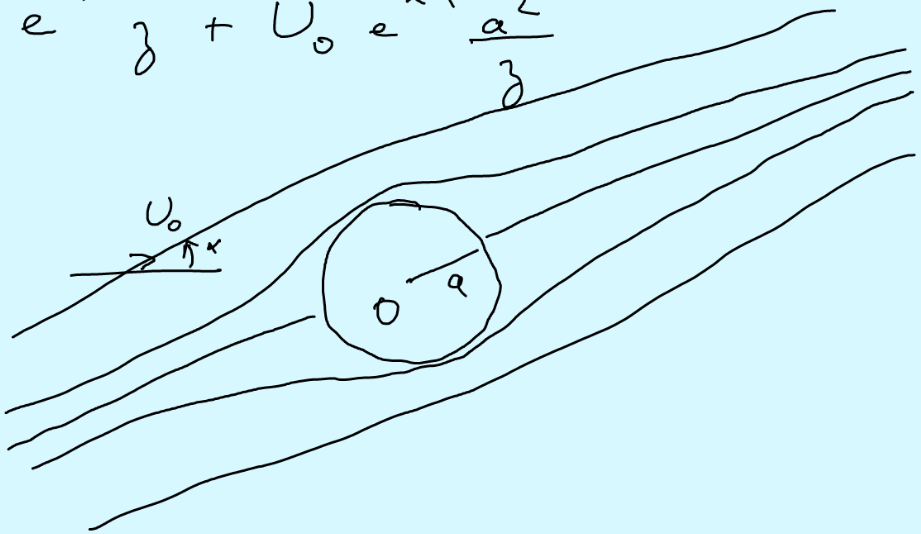


(2) Milne-Thompson circle theorem :

$$w(z) = f(z) + \overline{f\left(\frac{a^2}{\bar{z}}\right)}$$

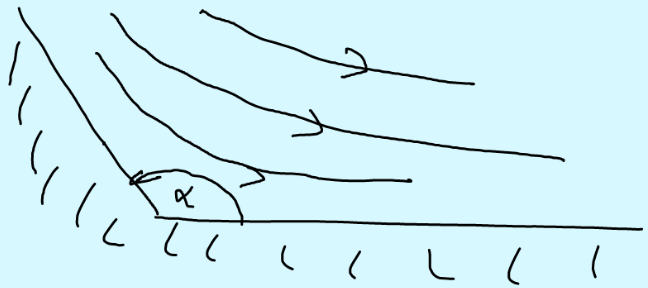
$$= U_0 e^{-i\alpha} z + \overline{U_0 e^{-i\alpha} \frac{a^2}{\bar{z}}}$$

$$= U_0 e^{-i\alpha} z + U_0 e^{i\alpha} \frac{a^2}{z}$$



(3) Add circulation : $w(z) = U_0 e^{-i\alpha} z + U_0 e^{i\alpha} \frac{a^2}{z} - \frac{i\Gamma}{2\pi} \ln z$

Conformal mapping :



(1) Generate a uniform flow : $w = U_0 z$

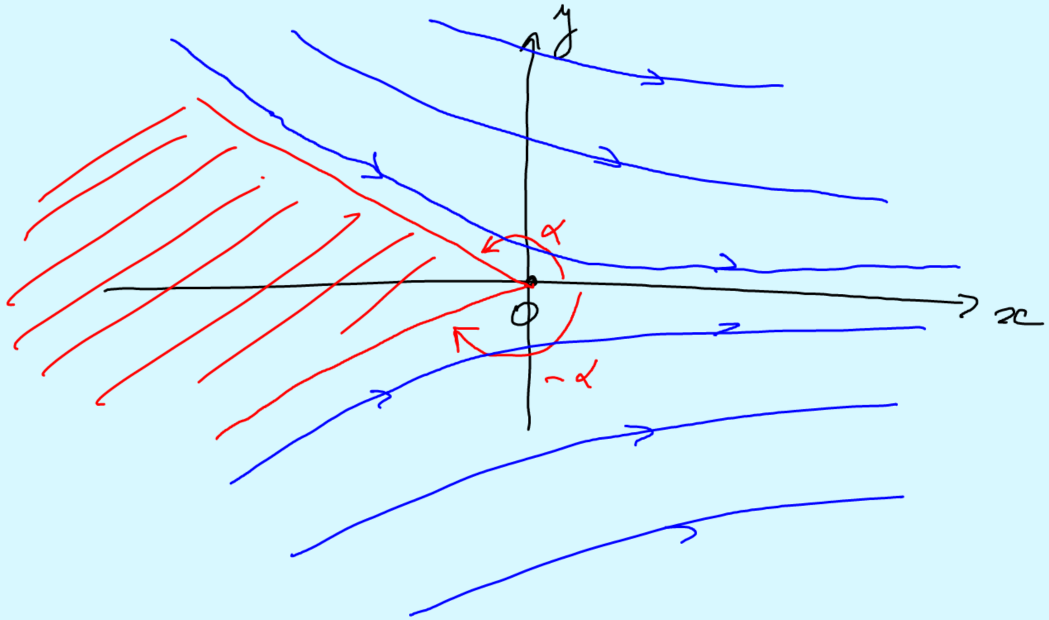
(2) Apply : $Z = z^{\alpha/\pi}$

$$\Rightarrow W(Z) = U_0 Z^{\pi/\alpha}$$

In the z coordinate system, $z = x$ is a streamline $\left(w(x) = U_0 x \in \mathbb{R} \right)$.
What does this streamline become in the Z coordinate system ?

The $y=0$ axis can be parameterized in polar coordinates : $z = r e^{i\theta}$ $\rightarrow$ $\theta = 0$, $\theta = +\pi$, $\theta = -\pi$

* $\theta = 0$: $z = r$ and $Z = r^{\alpha/\pi}$ $\longrightarrow$ remains the right side of the x-axis (with a stretch)
  right side of the x-axis
* $\theta = \pi$: $z = r e^{i\pi}$ and $Z = r^{\alpha/\pi} e^{i\alpha}$ $\rightarrow$ is now at an angle α with $\vec{e_x}$
  left side of the x-axis
* $\theta = -\pi$: $z = r e^{-i\pi}$ and $Z = r^{\alpha/\pi} e^{-i\alpha}$ $\rightarrow$ is now at an angle $-\alpha$ with $\vec{e_x}$
  left side of the x-axis



Example Sheet 5

Problem 3

(b)

$$w = \frac{U}{a-b}\left[a z - b\left(z^2 - a^2 + b^2\right)^{1/2} \right]$$

$$\lim_{|z|\to\infty} w(z) = \lim_{|z|\to\infty} \frac{U}{a-b}\left[a z - b\left(z^2 - a^2 + b^2\right)^{1/2} \right]$$

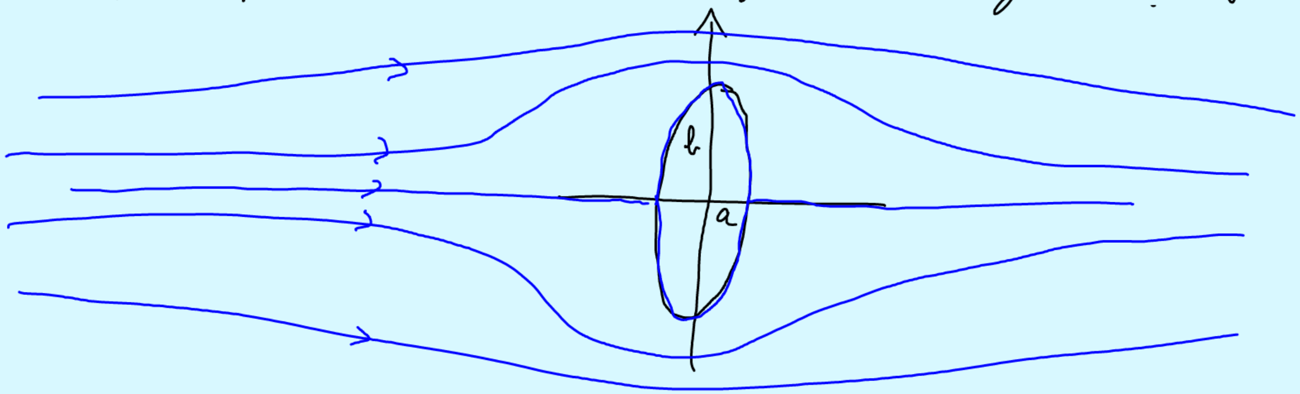
$$= \lim_{|z|\to\infty} \frac{U}{a-b}\left[a z - b z \right]$$

$$= \lim_{|z|\to\infty} \frac{Uz\,(a-b)}{a-b}$$

$$= Uz$$

↳ far from the origin, the flow resembles a uniform flow of speed U in the $\vec{e_x}$ direction.

Remember (question (a)), the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a streamline. This is an ellipse of axis lengths a and b. The resulting flow is the flow past an ellipse of semi-axis a and b generated by a uniform flow in the $\vec{e_x}$ direction.



Problem 4

$$w(z) = f(z) + \overline{f(-\bar{z})}$$

Imaginary axis : $z = iy \implies w(iy) = f(iy) + \overline{f(-\overline{iy})}$

$$= f(iy) + \overline{f(iy)}$$

$$\in \mathbb{R}$$

$\implies$ the imaginary axis is a streamline

$$f(z) = -\frac{i\Gamma}{2\pi} \ln(z - z_0) \quad , \; z_0 = a + ib$$

$$\hookrightarrow \quad w(z) = -\frac{i\Gamma}{2\pi} \ln(z - z_0) + \overline{-\frac{i\Gamma}{2\pi} \ln(-\bar{z} - z_0)}$$

$$= -\frac{i\Gamma}{2\pi} \ln(z - z_0) + \frac{i\Gamma}{2\pi} \ln\left(-z - \bar{z}_0\right)$$

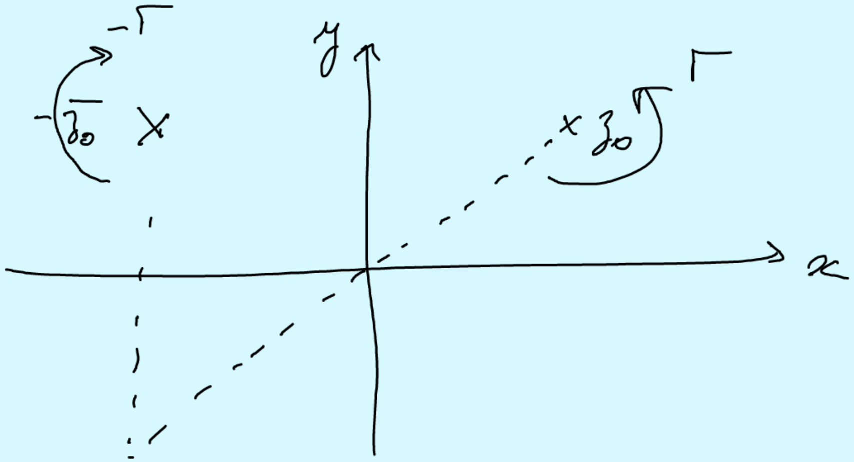
$$= -\frac{i\Gamma}{2\pi} \ln(z - z_0) + \frac{i\Gamma}{2\pi} \ln\left(z + \bar{z}_0\right) + \frac{i\Gamma}{2\pi} \ln(\overset{e^{i\pi}}{-1})$$

$$= -\frac{i\Gamma}{2\pi} \ln(z - z_0) + \frac{i\Gamma}{2\pi} \ln(z + \bar{z}_0) - \frac{\Gamma}{2}$$

↑ does not matter because a potential

is defined up to a constant

$$w' = -\frac{i\Gamma}{2\pi}\ln(z-z_0) + \frac{i\Gamma}{2\pi}\ln(z+\bar{z}_0)$$



$F(z) = w'(z) + \overline{w'(\bar{z})}$ guarantees a wall at $z = x$

Remember $w'(z) = f(z) + \overline{f(-\bar{z})}$

$$\Rightarrow F(z) = \underbrace{f(z) + \overline{f(-\bar{z})}}_{w'(z)} + \underbrace{\overline{f(\bar{z}) + \overline{f(-\bar{\bar{z}})}}}_{\overline{w'(\bar{z})}}$$

$$= f(z) + \overline{f(-\bar{z})} + \overline{f(\bar{z})} + f(-z)$$