MATH 5453

Foundation of Fluid Dynamics

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W1 : Tuesday 12noon
Thursday 12 noon

W2 : Tuesday 12 noon
Wednesday 2pm
Wednesday 3pm

W4 : Tuesday 12 noon
Wednesday 2 pm
Wednesday 3 pm

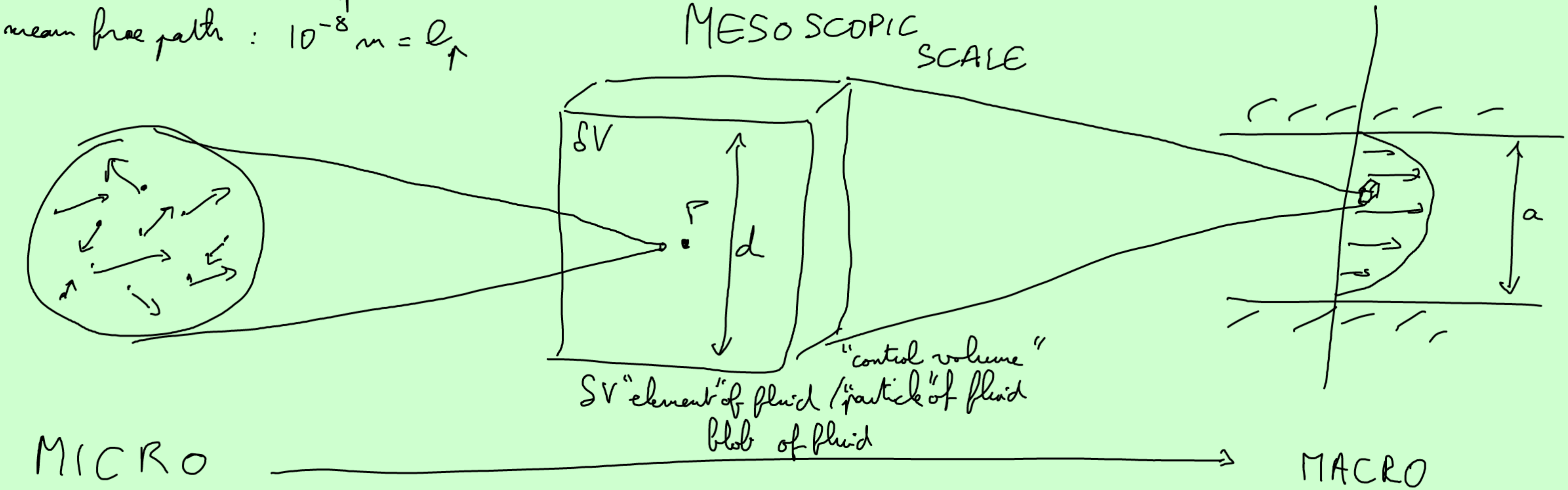
W5 onward : Wednesday 2pm
Thursday 12 noon

20% Homework T
20% Homework N
20% Homework L
40% Exam

Lecture 1

Fluid : substance that flows
gas, liquid, plasma
substance that takes the shape of its container

a fluid molecule $\ell_m = 10^{-10} m = 1 \text{Å}$
1 cm^3 of water : 10^{23} particles
mean free path : $10^{-8} m = \ell_p$



(flow down a pipe)

$d \gg \ell_p$ or $d \gg \ell_m$ & $d \ll a$

$\boxed{\text{Continuum Hypothesis}}$

$$\vec{v}(P,t) = \frac{\sum_{\text{particles in } \delta V} \vec{v}_{\text{particles}}}{\sum_{\text{particles in } \delta V} 1}$$

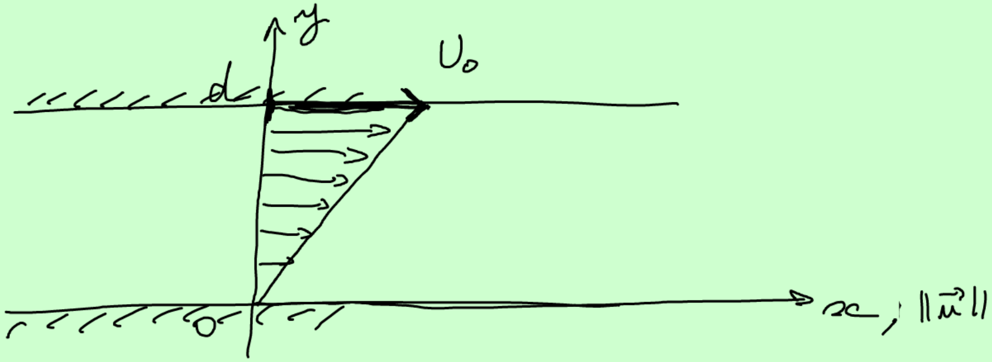
δV is "element" of fluid centered on P

$$\rho = \frac{\sum_{\text{particles in } \delta V} m_{\text{particle}}}{\delta V}$$

② Simple flows

Plane Couette flow :

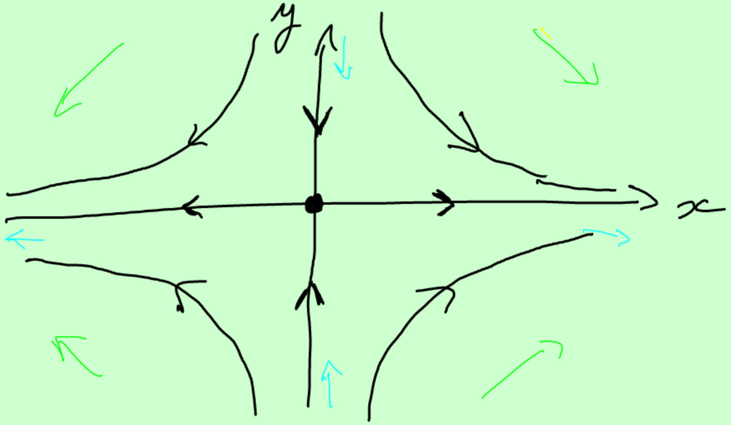
$$\vec{u} = \frac{U_0 y}{d} \vec{e_x}$$



Stagnation point flow :

$$\vec{u} = E x \vec{e_x} - E y \vec{e_y}$$

$$E > 0$$



④ Time-derivative

$\frac{\partial f}{\partial t}$: rate of change with time of f at $\vec{x}$ Euler

$$\frac{Df}{Dt} = \frac{df}{dt}(\vec{x}(t), t)$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \underbrace{\frac{\partial x}{\partial t}}_{u} + \frac{\partial f}{\partial y} \underbrace{\frac{\partial y}{\partial t}}_{v} + \frac{\partial f}{\partial z} \underbrace{\frac{\partial z}{\partial t}}_{w}$$

$$= \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z}$$

$$= \frac{\partial f}{\partial t} + \underline{(\vec{u} \cdot \vec{\nabla}) f}$$

: rate of change with time of f as we follow a particle as it passes through $\vec{x}$

Lagrange

③ Particle paths

visualization using a single particle of a passive tracer.

Releasing a single particle at $t = 0$ and $\vec{x} = \vec{x}_0$

$$\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t)$$

Example: $\vec{u} = Ex\,\vec{e_x} - Ey\,\vec{e_y}$

$$\Rightarrow \begin{cases} \frac{dx}{dt} = Ex \\ \frac{dy}{dt} = -Ey \end{cases}$$

$$\Rightarrow \begin{cases} x = x_0 e^{Et} \\ y = y_0 e^{-Et} \end{cases}$$

$$\Rightarrow \quad xy = x_0 y_0$$

⑤ **Streamline** : line everywhere tangent to the velocity vector.

$$d\vec{x} = \vec{u}\, \underline{ds} \qquad s: \text{ arclength along the streamline}$$

$$\Rightarrow \frac{d\vec{x}}{ds} = \vec{u}$$

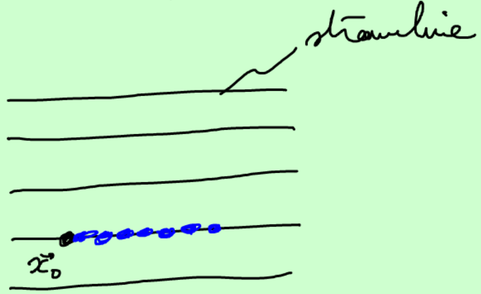
$$\Rightarrow \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = ds$$

If the flow is steady, particle paths and streamlines coincide.

⑥ **Streakline** : Continuous release of dye from $\vec{x} = \vec{x}_0$ from $t = t_0$ to $t = \bar{t}$

$t < \bar{t}$

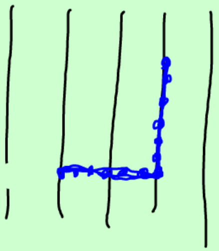
$\vec{u} = u\vec{e}_x \qquad u > 0$



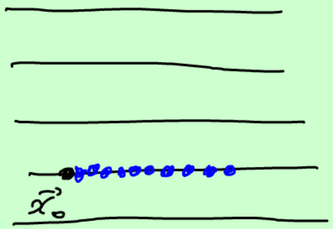
particle path

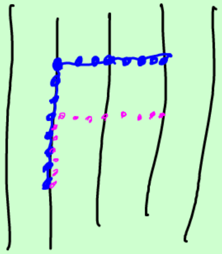
$t \geq \bar{t}$

$\vec{u} = v\vec{e}_y \qquad v > 0$



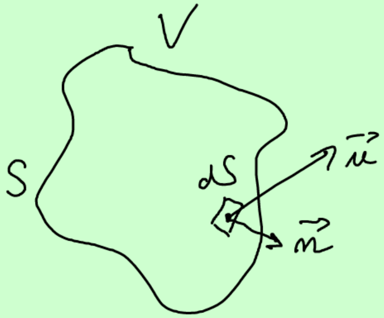
streakline





Lecture 2

① Mass conservation



V is a fixed volume (shape remains the same)

$\vec{n}$: unit outward pointing normal vector

fluid density

total mass in V: $M_V = \int_V \rho \, dV$

rate of change of the total mass in V with time: $\frac{dM_V}{dt} = \frac{d}{dt}\left(\int_V \rho \, dV\right)$

$\Rightarrow \frac{dM_V}{dt} = \int_V \frac{\partial \rho}{\partial t} \, dV$ because V is fixed

The total mass in V may change with time due to mass fluxes:

$$\underline{\frac{dM_V}{dt}} = -\int_S \rho \vec{u} \cdot \vec{n} \, dS$$

$$\Rightarrow \int_{\underline{V}} \frac{\partial \rho}{\partial t} \, dV = -\int_{\underline{S}} \underline{\rho \vec{u}} \, \underline{\cdot \vec{n} \, dS}$$

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t} \, dV = -\int_V \vec{\nabla} \cdot (\rho \vec{u}) \, dV$$ divergence theorem

$$\Rightarrow \int_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) \right] dV = 0$$

Because we did not say anything about the size of V, we can take V to be a blob of fluid.

$$\Rightarrow \left[\frac{\partial \rho}{\partial t} + \underline{\vec{\nabla} \cdot (\rho \vec{u})} = 0 \right.$$ pointwise (everywhere in the macroscopic system).

$$\Rightarrow \underline{\frac{\partial \rho}{\partial t}} + \rho \vec{\nabla} \cdot \vec{u} + \underline{\vec{u} \cdot \vec{\nabla}} \rho = 0$$

$$\Rightarrow \boxed{\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{u} = 0}$$ continuity equation

Lecture 3

② Incompressibility condition

Constant density $\Rightarrow$ $\rho\,\vec{\nabla}\cdot\vec{u}=0$

$\Rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{u}=0}$ incompressibility condition/constraint

Remember: $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\psi})=0$ always

$\Rightarrow$ $\vec{u}=\vec{\nabla}\times\vec{\psi}$ ← streamfunction

③ Two-dimensional flows

$$\vec{u}=u(x,y)\,\vec{e}_x+v(x,y)\,\vec{e}_y \longrightarrow \vec{\psi}=\psi(x,y)\,\vec{e}_z$$

$$u=\frac{\partial\psi}{\partial y} \qquad v=-\frac{\partial\psi}{\partial x}$$

3.1 Streamlines

lines where $\psi=\text{cst.}$

- $$\vec{u}\cdot\vec{\nabla}\psi = u\frac{\partial\psi}{\partial x}+v\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} = 0$$

$$\Longrightarrow \vec{u}\perp\vec{\nabla}\psi$$

- $d\psi=0 \Rightarrow \dfrac{\partial\psi}{\partial x}dx+\dfrac{\partial\psi}{\partial y}dy=0$

$(dx,dy)=d\vec{l}$ element of displacement along a streamline

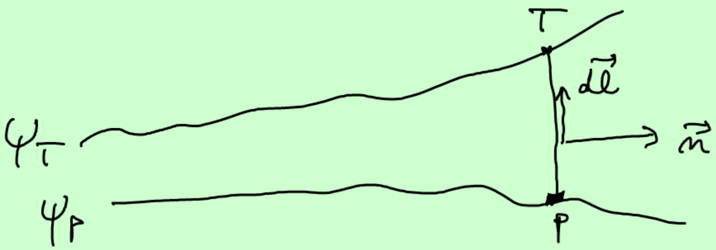
$$\Rightarrow -v\,dx + u\,dy = 0$$

$$\Rightarrow \vec{u} \times \vec{dl} = 0$$

$$\Rightarrow \vec{u} \parallel \vec{dl}$$

$\Rightarrow$ streamlines are parallel to the velocity

3.2 Flux between streamlines



$$Q = \int_P^T \vec{u} \cdot \vec{n}\, ds$$

$$\vec{dl} = dx\, \vec{e_x} + dy\, \vec{e_y} \quad \swarrow$$

$$\vec{n}\, \underline{ds} = dy\, \vec{e_x} - dx\, \vec{e_y} \quad \swarrow$$

$$= ds \left(\frac{\partial y}{\partial s} \vec{e_x} - \frac{\partial x}{\partial s} \vec{e_y} \right) \quad \swarrow$$

$$\Rightarrow \quad Q = \int_P^T \left(u \frac{\partial y}{\partial s} - v \frac{\partial x}{\partial s} \right) ds$$

$$= \int_P^T \left(\frac{\partial \psi}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial \psi}{\partial x} \frac{\partial x}{\partial s} \right) ds$$

$$= \int_P^T \frac{d\psi}{ds}\, ds$$

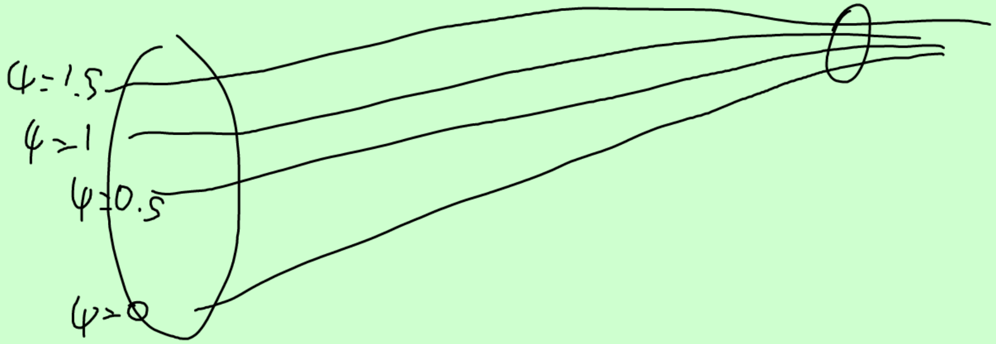
$$= \int_P^T d\psi$$

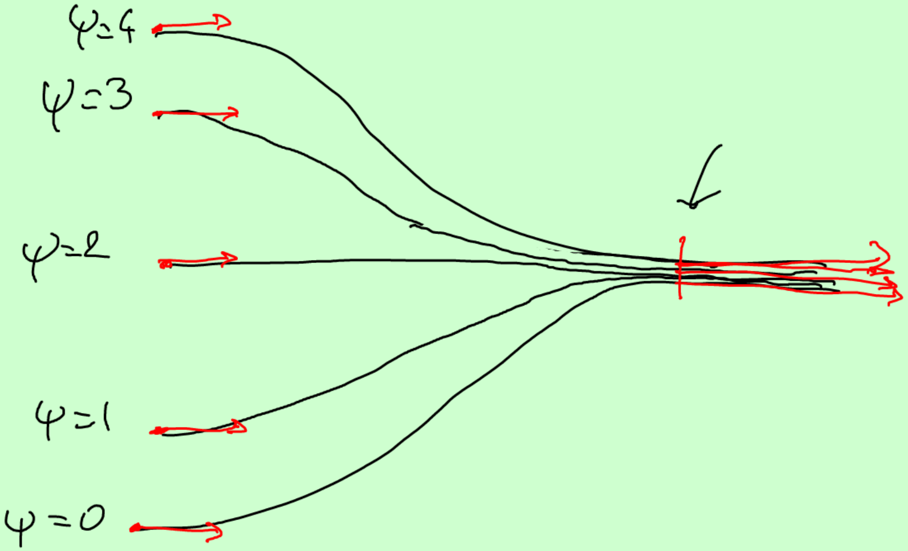
$$= \psi_T - \psi_P$$

$$\|\vec{u}\| = \left\| \frac{\partial \psi}{\partial y} \vec{e_x} - \frac{\partial \psi}{\partial x} \vec{e_y} \right\|$$

$$= \| \vec{\nabla}\psi \|$$

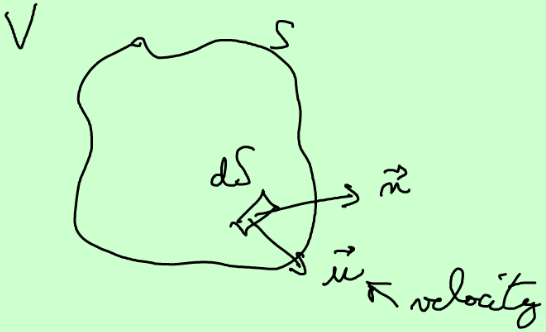
$$\underline{u} = \frac{\partial \psi}{\partial y}$$





Chapter 3

① Equation of motion



V is fixed

$\vec{n}$: unit outward pointing normal vector total momentum in V: $\int_V \vec{q}\, dV = \int_V \rho \vec{u}\, dV$

$\uparrow$ momentum density

rate of change of the total momentum in V with respect to time:

$$\frac{d}{dt}\int_V \rho \vec{u}\, dV \Rightarrow \int_V \frac{\partial}{\partial t}(\rho \vec{u})\, dV$$

Newton's second law:

$$\underbrace{\begin{pmatrix}\text{rate of change of}\\ \text{momentum}\\ \text{in V}\end{pmatrix}}_{\vec{I}} = \underbrace{\begin{pmatrix}\text{flux of momentum}\\ \text{through S}\end{pmatrix}}_{\vec{II}} + \underbrace{\begin{pmatrix}\text{net force}\\ \text{acting on V}\end{pmatrix}}_{\vec{III}}$$

$$I_i = \int_V \frac{\partial}{\partial t}(\rho u_i)\, dV$$

$$\underline{II_i = -\int_S q_i \overbrace{u_j n_j}\, dS} \qquad -\int_S \vec{q}\,\vec{u}\cdot\vec{n}\, dS$$

$$\rho\,\vec{u}\,\vec{u}\cdot\vec{n}\, dS$$

$$= -\int_V \frac{\partial}{\partial x_j}(q_i u_j)\, dV$$

$$= -\int_V \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) dV$$

$$\vec{III} = \int_V \vec{F}_{body}\, dV + \int_S \vec{F}_{local}\, dS$$

force densities

total stress tensor

$$\vec{F}_{body} = \rho \vec{g} \quad (+\ \vec{other})$$

$$F_{body_i} = \rho g_i$$

$$F_{local_i} = n_j \tau_{ji}$$

$$\Rightarrow \int_S F_{local_i}\, dS = \int_S \tau_{ji} \underline{n_j\, dS}$$

$$= \int_V \frac{\partial}{\partial x_j}\left(\tau_{ji}\right) dV$$

$$\int_S \vec{F}_{local}\, dS = \int_S \vec{n}\cdot\bar{\bar{\tau}}\, dS$$

$$= \int_V \vec{\nabla}\cdot\bar{\bar{\tau}}\, dV$$

$$\Rightarrow \int_V \frac{\partial}{\partial t}\left(\rho u_i\right) dV = -\int_V \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) dV + \int_V \rho g_i\, dV + \int_V \frac{\partial}{\partial x_j}\left(\tau_{ji}\right) dV$$

V can be as small as a blob of fluid:

$$\underline{\frac{\partial}{\partial t}}\left(\rho u_i\right) + \underline{\frac{\partial}{\partial x_j}\left(\rho u_i u_j\right)} = \rho g_i + \frac{\partial}{\partial x_j}\left(\tau_{ji}\right) \qquad (1)$$

Mass conservation : $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$

$[\tau]_{ji}$

$$(1) \Rightarrow \rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} + u_i \frac{\partial}{\partial x_j}(\rho u_j) + \rho u_j \frac{\partial}{\partial x_j} u_i = \rho g_i + \frac{\partial}{\partial x_j}(\tau_{ji})$$

$$\Rightarrow \boxed{\begin{aligned} &\rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} u_i\right] = \rho g_i + \frac{\partial}{\partial x_j}\tau_{ji} \\ &\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = \rho \vec{g} + \vec{\nabla}\cdot\bar{\bar{\tau}} \end{aligned}}$$

momentum equation , where $\bar{\bar{\tau}}$ is the second rank tensor called

total stress tensor
to be specified by a "constitutive equation"

② Constitutive equations

2.1 Ideal fluid

$\vec{F}_{local} = -P\vec{n}$ $\Rightarrow$ $n_j \tau_{ji} = -P n_i$

total pressure

$\Rightarrow$ $\tau_{ij} = -P\delta_{ij}$

$(\tau_{ij} = -P$ if $j=i$, 0 otherwise$)$

$$\bar{\bar{\tau}} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

$$\Rightarrow \quad \rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} u_i\right] = \rho g_i + \frac{\partial}{\partial x_j}\left(-P\delta_{ji}\right)$$

$$\boxed{\begin{aligned} &\Rightarrow \rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} u_i\right] = \rho g_i - \frac{\partial P}{\partial x_i} \\ &\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = \rho \vec{g} - \vec{\nabla} P \end{aligned}}$$

Euler equation

(motion of an ideal fluid)

1.2 Newtonian fluid

(• stress is a linear response to strain
• stress is isotropic)

dynamic viscosity

$$\tau_{ji} = -P\delta_{ij} + \underbrace{2\mu\left[E_{ij} - \frac{1}{3}\delta_{ij}\cancel{\frac{\partial u_k}{\partial x_k}}\right] \quad \left(+\mu_b \frac{\partial u_k}{\partial x_k} \ldots\right)}_{\text{viscous stress tensor } \bar{\bar{\sigma}}}$$

$\rightarrow 0$

total stress tensor

incompressible fluid : $\vec{\nabla}\cdot\vec{u} = 0$

$$\frac{\partial u_k}{\partial x_k} = 0$$

$$\tau_{ji} = -P\delta_{ij} + 2\mu E_{ij}$$

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parenthesis

$$K_{ij} = \frac{\partial u_i}{\partial x_j} = \underset{\text{symmetric}}{E_{ij}} + \underset{\text{antisymmetric}}{\Omega_{ij}}$$

velocity gradient tensor

$$E_{ij} = E_{ji} \qquad \Omega_{ij} = -\Omega_{ji}$$

$$E_{ij} = \frac{1}{2}\left(K_{ij} + K_{ji}\right) \qquad \Omega_{ij} = \frac{1}{2}\left(K_{ij} - K_{ji}\right)$$

(A) 2D flow incompressible :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

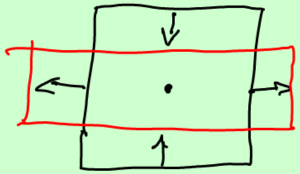
$$\overline{\overline{K}} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \quad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

3 unknowns

$$\overline{\overline{E}} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \qquad \overline{\overline{\Omega}} = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$$

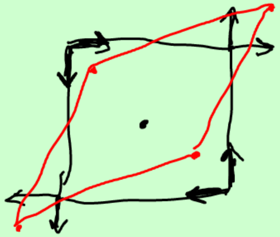
- $$\overline{\overline{E}}_{a=1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = -1$$

$$\overline{\overline{\Omega}}_{a=1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$



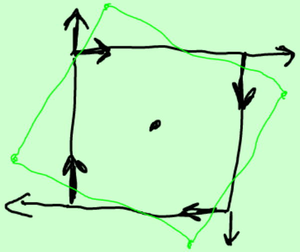
DEFORMATION

- $$\overline{\overline{E}}_{b=1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial x} = 1$$



DEFORMATION

$$\cdot\ \bar{\bar{\Omega}}_{C=1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial x} = -1$$



SOLID BODY ROTATION

$\bar{\bar{E}}$ → deformation : strain rate tensor ← linked to forces

$\bar{\bar{\Omega}}$ → solid body rotations : vorticity tensor ← linked to torques

\ parenthesis

$$\tau_{ji} = -P\delta_{ij} + 2\mu E_{ij}$$

$$= -P\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\Rightarrow \rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} u_i\right] = \rho g_i + \frac{\partial}{\partial x_j}\left[-P\delta_{ij} + \mu\frac{\partial u_i}{\partial x_j} + \mu\frac{\delta u_j}{\partial x_i}\right]$$

$$= \rho g_i - \frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2} + \underbrace{\mu\frac{\partial^2 u_j}{\partial x_i \partial x_j}}$$

$$\mu\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} = 0$$ because of the incompressibility condition

$$\Rightarrow \rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} u_i\right] = \rho g_i - \frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2}$$

$$\Rightarrow \boxed{\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = \rho\vec{g} - \vec{\nabla}P + \mu\vec{\nabla}^2\vec{u}}$$ Navier-Stokes equation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = \vec{g} - \frac{1}{\rho}\vec{\nabla}P + \nu\vec{\nabla}^2\vec{u}$$

$\nu = \frac{\mu}{\rho}$ kinematic viscosity.

Remark: no flow, no forces : $\rho\vec{g} - \vec{\nabla}P_H = 0$ ↖ hydrostatic pressure

$\Rightarrow P_H = \rho\vec{g}\cdot\vec{x} + P_0$ ↖ reference pressure

Let us write $P = P_H + p$ ↖ dynamic pressure

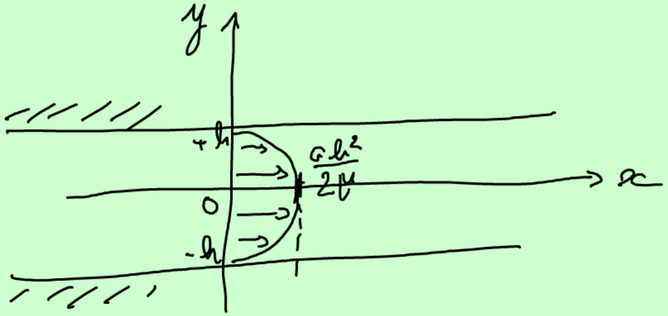
$\uparrow$ total $\quad \uparrow$ hydrostatic

$$\hookrightarrow \quad \rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = -\vec{\nabla}p + \mu\vec{\nabla}^2\vec{u}$$ ←

Chapter 4

① Plane Poiseuille flow

POISEUILLE



$\leftarrow \vec{\nabla}p = -G\vec{e}_x \qquad G > 0$

↖ $\frac{\partial p}{\partial x} = -G$

* Equations : Navier-Stokes in 2D , incompressibility condition

$$\begin{cases} \rho \dfrac{\partial u}{\partial t} + \rho u \dfrac{\partial u}{\partial x} + \rho v \dfrac{\partial u}{\partial y} = - \dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u}{\partial x^2} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ \rho \overset{\swarrow}{\dfrac{\partial v}{\partial t}} + \rho u \overset{\swarrow}{\dfrac{\partial v}{\partial x}} + \rho v \overset{\swarrow}{\dfrac{\partial v}{\partial y}} = - \dfrac{\partial p}{\partial y} + \mu \dfrac{\partial^2 v}{\partial x^2} + \mu \dfrac{\partial^2 v}{\partial y^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

* BC : $y = -h \quad u = 0 \xleftarrow{\text{no-slip}} \quad v = 0 \Leftarrow$ impenetrability

$y = +h \quad u = 0 \quad v = 0$

* Hypotheses : • steady flow $\dfrac{\partial}{\partial t} = 0 \Leftarrow$

• $v \equiv 0 \leftarrow$

↳ Solution :

$$\begin{cases} \rho u \dfrac{\partial u}{\partial x} = - \dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u}{\partial x^2} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ 0 = - \dfrac{\partial p}{\partial y} \\ \dfrac{\partial u}{\partial x} = 0 \end{cases}$$

$y = \pm h \ , \ u = v = 0$

$$\Rightarrow \begin{cases} 0 = - \dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ 0 = \dfrac{\partial p}{\partial y} \\ \dfrac{\partial u}{\partial x} = 0 \end{cases}$$

$$\Rightarrow \underbrace{\mu \frac{\partial^2 u}{\partial y^2}}_{\text{varies with } y \text{ only}} = \underbrace{\frac{\partial p}{\partial x}}_{\text{varies with } x \text{ only}} = k_1$$

$$\Rightarrow \mu \frac{\partial^2 u}{\partial y^2} = - G$$

$$\Rightarrow \quad u = -\frac{G}{2\mu} y^2 + k_2 y + k_3$$

$$\text{BC}: \quad y = -h \,, \quad u = 0 \quad \Rightarrow \quad -\frac{G h^2}{2\mu} - k_2 h + k_3 = 0$$

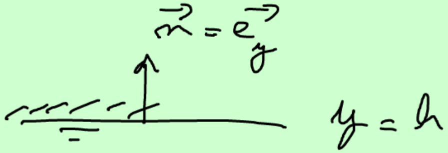
$$y = h \,, \quad u = 0 \quad \Rightarrow \quad -\frac{G h^2}{2\mu} + k_2 h + k_3 = 0$$

$$\Rightarrow \quad k_2 = 0 \,, \quad k_3 = \frac{G h^2}{2\mu}$$

$$\Rightarrow \quad \boxed{u = \frac{G}{2\mu}\left(h^2 - y^2\right)}$$

Force density excerted by the wall onto the fluid at $y = +h$

$$\overrightarrow{F^{bT}_{w\to f}} = \vec{n} \cdot \bar{\bar{\tau}} \Big|_{y=h}$$



$$= \vec{e_y} \cdot \bar{\bar{\tau}} \Big|_{y=h}$$

$$\left(= \sum_{i=x,y} \bar{\bar{\tau}}_{yi} \Big|_{y=h} \vec{e_i} \right)$$

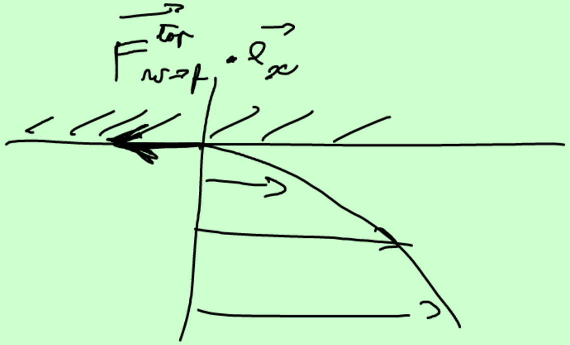
$$= \tau_{yx} \Big|_{y=h} \vec{e_x} + \tau_{yy} \Big|_{y=h} \vec{e_y}$$

$$\bar{\bar{\tau}} = 2\mu \bar{\bar{E}} \left(-p\bar{\bar{I}}\right)$$

Let us look at τ_{yx}: $\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right)$

at $y=h$: $\tau_{yx}(h) = \mu \frac{\partial u}{\partial y} = -Gh$

↳ $\overrightarrow{F^{top}_{w\to f}} = -Gh\,\vec{e_x}\ \left(+\ \vec{e_y}\right)$



⑥ Stokes first problem

* BC: $y=0$: $u=U$, $v=0$

$y\to\infty$: $u\to 0$, $v\to 0$

* Equations: NS + incompressibility

$$\begin{cases} \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} \\ \rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu\frac{\partial^2 v}{\partial x^2} + \mu\frac{\partial^2 v}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{cases}$$

* IC: $t=0$, $u=0$, $v=0$

* Hypotheses:
- $v = 0$
- $\vec{\nabla}_p = \vec{0}$ because the flow is entirely driven by the motion of the wall.

↳ Solutions:
$$\begin{cases} \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} = 0 \end{cases}$$

$$\Rightarrow \rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}}$$ heat equation (diffusion equation)

2 methods:
- separation of variables (cF MATH 3620, plane Couette flow startup)
- similarity solution

Similarity solution: $\underset{\sim}{\eta} = y\, t^{a}$ ← ETA (similarity variable) $u(y,t) = f(\eta)$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial t} \Rightarrow \frac{\partial}{\partial t} = a t^{a-1} y \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial t} \Rightarrow \frac{\partial}{\partial y} = t^{a} \frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(t^{a} \frac{\partial}{\partial \eta} \right) \Rightarrow \frac{\partial^2}{\partial y^2} = t^{2a} \frac{\partial^2}{\partial \eta^2}$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad \Rightarrow \quad a\, t^{a-1}\, y \frac{\partial f}{\partial \eta} = \nu\, t^{2a} \frac{\partial^2 f}{\partial \eta^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial \eta^2} = \frac{a\, t^{-a-1}\, y}{\nu} \frac{\partial f}{\partial \eta} \quad \leftarrow$$

We could turn this equation into an equation on one variable (η), if:

$$\boxed{t^{-a-1}\, y = \eta} = y\, t^{a}$$

$$\Rightarrow t^{-a-1} = t^{a}$$

$$\Rightarrow t^{-2a-1} = 1$$

$$\Rightarrow -2a-1 = 0$$

$$\Rightarrow a = -\frac{1}{2}$$

$$\hookrightarrow \boxed{\frac{d^2 f}{d\eta^2} = -\frac{1}{2} \frac{\eta}{\nu} \frac{df}{d\eta}}$$

ODE of one variable

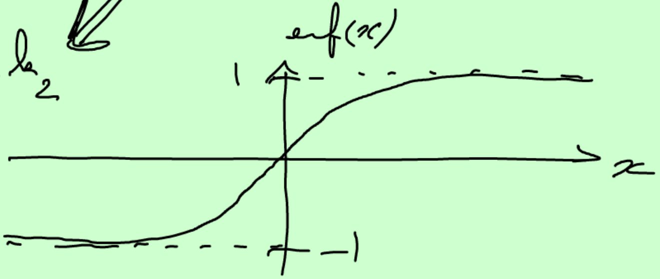
$$\nearrow \eta = y\, t^{-1/2}$$

$$= \frac{y}{\sqrt{t}}$$

ν: kinematic viscosity

$$f'' = -\frac{1}{2}\frac{\eta}{\nu} f' \Rightarrow f' = k_1 \exp\left(-\frac{\eta^2}{4\nu}\right)$$

$$\Rightarrow f = k_1 \underbrace{\int_0^\eta \exp\left(-\frac{\xi^2}{4\nu}\right) d\xi}_{} + k_2$$



$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-y^2)\, dy$$

↑ error function

change of variables: $y = \frac{\xi}{2\sqrt{\nu}} \Rightarrow \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{2\sqrt{\nu}x} \exp\left(-\frac{\xi^2}{4\nu}\right) \frac{d\xi}{2\sqrt{\nu}}$

$$= \frac{1}{\sqrt{\pi\nu}} \int_0^{2\sqrt{\nu}x} \exp\left(-\frac{\xi^2}{4\nu}\right) d\xi$$

$$\eta = 2\sqrt{\nu}\, x \Rightarrow \mathrm{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) = \frac{1}{\sqrt{\pi\nu}} \underbrace{\int_0^\eta \exp\left(-\frac{\xi^2}{4\nu}\right) d\xi}_{}$$

$$\Rightarrow f = \underbrace{k_1 \sqrt{\pi\nu}}_{} \,\mathrm{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) + k_2$$

$$\Rightarrow f = k_3\, \mathrm{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) + k_2 \rightarrow u = k_3\, \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) + k_2$$

BC: $y = 0 : u = U$

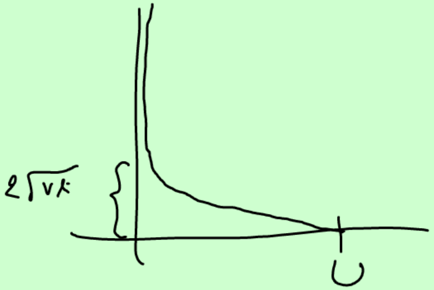
$y \to \infty : u \to 0$

$y = 0 \longrightarrow k_2 = U$

$y \to \infty : \longrightarrow k_3 + k_2 = 0 \Rightarrow k_3 = -U$

$$\Rightarrow \boxed{u = U\left[1 - \mathrm{erf}\left(\frac{y}{\underline{2\sqrt{\nu t}}}\right)\right]}$$

Remember : IC : $t = 0$, $u \equiv 0$ → automatically satisfied which validates the similarity variable approach



$$u \ll U \Rightarrow \frac{y}{2\sqrt{\nu t}} \gg 1$$

$$\Rightarrow y \gg \underline{2\sqrt{\nu t}}$$

$$\mathrm{erf}(2) \simeq 0.99 \Rightarrow \underline{u \lesssim 0.01\, U} \text{ for } y \gg 4\sqrt{\nu t}$$

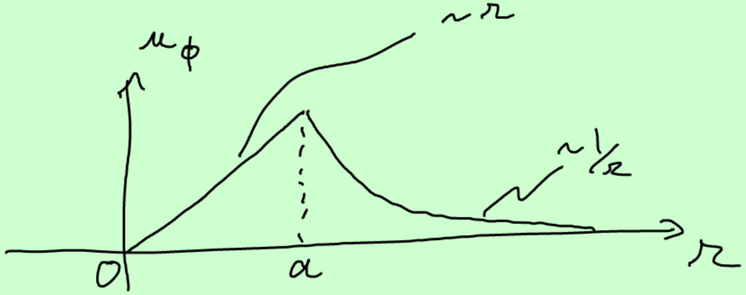
We can define the thickness of the viscous boundary layer by using the 99% criterion:

$$\underline{\delta_{99} = 4\sqrt{\nu t}}$$

Chapter 5

② The Rankine vortex

RANKINE



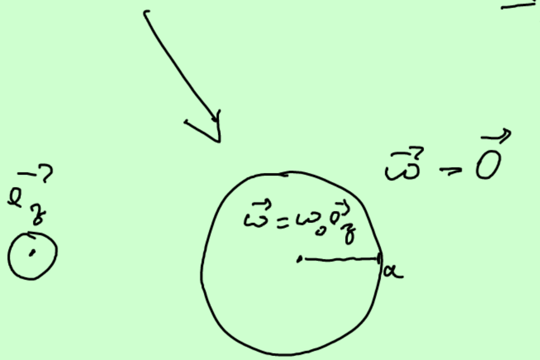
$$\vec{u} = k r \vec{e}_\phi \qquad r \leq a$$

$$\vec{u} = \frac{k a^2}{r} \vec{e}_\phi \qquad r > a$$

vorticity : $\vec{\omega} = \vec{\nabla} \times \vec{u}$

2D, velocity in $\vec{e_\phi}$ only : $\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r}\,\partial_r (r\, u_\phi) \end{pmatrix}$

$$\Rightarrow \vec{\omega} = \begin{cases} 2k\, \vec{e_z} & \text{for } r \leq a \\ \vec{0} & \text{for } r > a \end{cases}$$



$$\Rightarrow \omega_0 = 2k$$

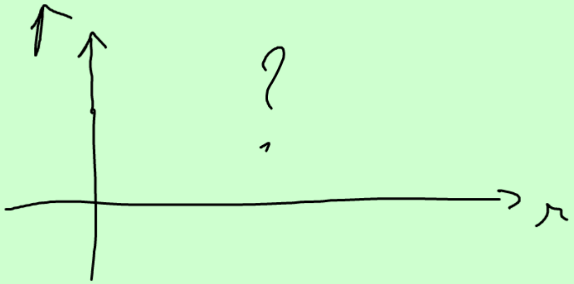
We can define the Rankine vortex by :

$$\vec{u} = \begin{cases} \frac{\omega_0 r}{2}\, \vec{e_\phi} & r \leq a \\ \frac{\omega_0 a^2}{2r}\, \vec{e_\phi} & r > a \end{cases}$$

An interesting calculation involves the pressure :

$\vec{NS} \cdot \vec{e_r}$: $\rho \frac{u_\phi^2}{r} = \frac{\partial p}{\partial r}$

find the pressure

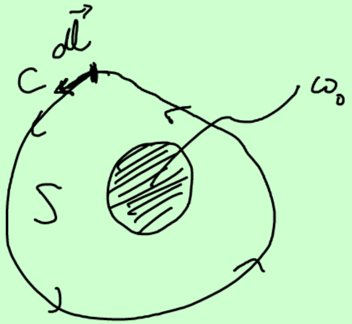


3 Line / point vortex

Around the Rankine vortex: $\int_S \vec{\omega} \cdot \vec{n}\, dS = \pi a^2 \omega_0$

disk sufficiently large compared to the size of the vortex

By Stokes theorem : $\int_S \vec{\omega} \cdot \vec{n}\, dS = \underbrace{\oint_C \vec{u} \cdot \vec{dl}}_{=\Gamma \text{ circulation}} = \pi a^2 \omega_0$



$$\Gamma = \pi a^2 \omega_0$$

Taking a to 0 but keeping Γ finite leads to a point vortex (line vortex in 3D)

↳ potential flow theory

5 Vorticity equation

↙

$$\rho\left[\frac{\partial \vec{u}}{\partial t} + \underline{\underline{(\vec{u}\cdot\vec{\nabla})\vec{u}}}\right] = -\vec{\nabla}p + \mu\vec{\nabla}^2\vec{u} + \vec{F}$$

- $\vec{u}\times(\vec{\nabla}\times\vec{u})\Big|_i = \varepsilon_{ijk}\, u_j\, (\vec{\nabla}\times\vec{u})_k$

$$* \quad = \varepsilon_{ijk}\, u_j\, \varepsilon_{klm}\frac{\partial}{\partial x_l} u_m$$

$$= \varepsilon_{kij}\,\varepsilon_{klm}\, u_j \frac{\partial}{\partial x_l} u_m$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) u_j \frac{\partial}{\partial x_l} u_m$$

$$= u_j \frac{\partial}{\partial x_i} u_j - u_j \frac{\partial}{\partial x_j} u_i$$

$$= \frac{1}{2}\frac{\partial}{\partial x_i}\left(u_j^2\right) - u_j \frac{\partial}{\partial x_j} u_i$$

$$= \frac{1}{2}\vec{\nabla}\left(\vec{u}^2\right)\Big|_i - \underline{\underline{(\vec{u}\cdot\vec{\nabla})\vec{u}}}\Big|_i$$

In NS: $$\rho\left[\frac{\partial\vec{u}}{\partial t} + \frac{1}{2}\vec{\nabla}(\vec{u}^2) - \vec{u}\times\vec{\omega}\right] = -\vec{\nabla}p + \mu\vec{\nabla}^2\vec{u} + \vec{F}$$

$$\Rightarrow \rho\left[\frac{\partial\vec{u}}{\partial t} - \vec{u}\times\vec{\omega}\right] = -\vec{\nabla}\left(p + \frac{1}{2}\rho\vec{u}^2\right) + \mu\vec{\nabla}^2\vec{u} + \vec{F}$$

Taking the curl: $$\rho\left[\frac{\partial\vec{\omega}}{\partial t} - \vec{\nabla}\times\vec{u}\times\vec{\omega}\right] = \vec{0} + \mu\vec{\nabla}^2\vec{\omega} + \vec{0}$$

↖ we take these forces to derive from a potential ↙

$$\vec{F}=\vec{\nabla}V$$
$$\vec{\nabla}\times\vec{F}=\vec{\nabla}\times\vec{\nabla}V$$
$$=\vec{0}$$

$$\vec{\nabla}\times\vec{u}\times\vec{\omega}\Big|_i=\varepsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{u}\times\vec{\omega})\Big|_k$$

$$=\varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\varepsilon_{klm}u_l\omega_m\right)$$

$$=\varepsilon_{kij}\varepsilon_{klm}\frac{\partial}{\partial x_j}(u_l\omega_m)$$

$$=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\frac{\partial}{\partial x_j}(u_l\omega_m)$$

$$=\frac{\partial}{\partial x_j}(u_i\omega_j)-\frac{\partial}{\partial x_j}(u_j\omega_i)$$

$$=u_i\cancel{\frac{\partial\omega_j}{\partial x_j}}+\omega_j\frac{\partial u_i}{\partial x_j}-u_j\frac{\partial\omega_i}{\partial x_j}-\omega_i\cancel{\frac{\partial u_j}{\partial x_j}}$$

→ 0 incompressible $\vec{\nabla}\cdot\vec{u}=0$

$=0$ incompressible

$$=(\vec{\omega}\cdot\vec{\nabla})\vec{u}\Big|_i-(\vec{u}\cdot\vec{\nabla})\vec{\omega}\Big|_i$$

$$\hookrightarrow\boxed{\rho\left[\frac{\partial\vec{\omega}}{\partial t}+(\vec{u}\cdot\vec{\nabla})\vec{\omega}\right]=\rho(\vec{\omega}\cdot\vec{\nabla})\vec{u}+\mu\vec{\nabla}^2\vec{\omega}}$$

VORTICITY EQUATION

rate of change with time of vorticity

advection of vorticity

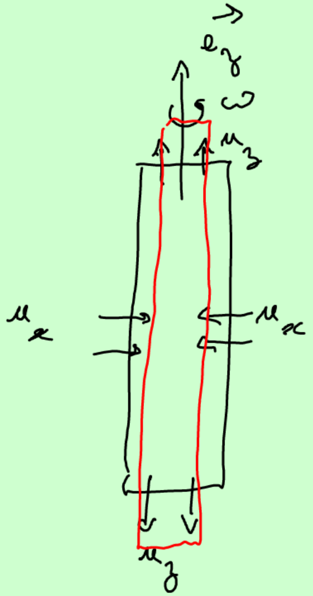
vortex stretching

dissipation of vorticity due to viscous effects

To understand $\rho(\vec{\omega}\cdot\vec{\nabla})\vec{u}$, let us consider $\vec{\omega} = \omega\,\vec{e_z}$

$\overrightarrow{\text{vorticity eq}}\cdot\vec{e_z}$ $$\rho\left[\frac{\partial\omega}{\partial t} + (\vec{u}\cdot\vec{\nabla})\omega\right] = \rho\,\omega\frac{\partial u_z}{\partial z} + \mu\,\vec{\nabla}^2\omega$$

Let us focus on $\rho(\vec{\omega}\cdot\vec{\nabla})\vec{u}$: $\frac{\partial\omega}{\partial t} = \omega\frac{\partial u_z}{\partial z} \Rightarrow \frac{1}{\omega}\frac{\partial\omega}{\partial t} = \frac{\partial u_z}{\partial z}$



$$\frac{\partial u_z}{\partial z} > 0 \Rightarrow \frac{1}{\omega}\frac{\partial\omega}{\partial t} > 0 \Rightarrow |\omega| > 0$$

7 Kelvin circulation theorem

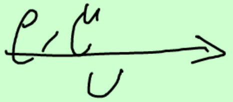
$\Gamma = \oint_C \vec{u}\cdot d\vec{l}$ circulation

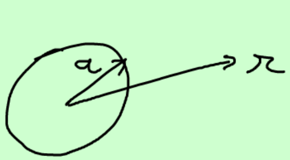
inviscid flow ($\nu = 0$), incompressible flow, forces derive from a potential

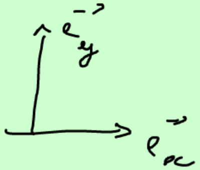
$$\boxed{\frac{d\Gamma}{dt} = 0}$$

Lecture 8

Chapter 6 : Dynamic similarity







NS: $\boxed{\rho}\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = -\vec{\nabla}p + \boxed{\mu}\,\vec{\nabla}^2\vec{u}$

incompressibility: $\vec{\nabla}\cdot\vec{u} = 0$

BC: $r \to \infty$ $\quad \vec{u} = \boxed{U}\,\vec{e}_x$

$r = \boxed{a}$ $\quad \vec{u} = \vec{0}$

Non-dimensionalization: $\vec{x} = a\,\vec{x}^*$

$[x] = L$

$[a] = L$

$[x^*] = 1$ ← nondimensional

$\vec{u} = U\,\vec{u}^*$

$[u] = L.T^{-1}$

$[U] = L.T^{-1}$

$[u^*] = 1$

$t = \frac{a}{U}\,t^* \longrightarrow \left[\frac{a}{U}\right] = T$

$p = P\,p^*$

P ← NOT the total pressure, but a pressure scale.

$\nabla \to \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}$

$\frac{\partial}{\partial(a x^*)}$

$\frac{\partial}{a\,\partial x^*}$

$\frac{1}{a}\frac{\partial}{\partial x^*}$

$\to \frac{1}{a}\vec{\nabla}^*$

$$\vec{\nabla}\cdot\vec{u} = 0 \Rightarrow \frac{1}{a}\vec{\nabla}^*\cdot(U\vec{u}^*) = 0$$

$$\Rightarrow \frac{U}{a}\vec{\nabla}^*\cdot\vec{u}^* = 0$$

$$\Rightarrow \vec{\nabla}^*\cdot\vec{u}^* = 0$$

$$\rho\left[\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = -\vec{\nabla}p + \mu\vec{\nabla}^2\vec{u} \Rightarrow \rho\left[\frac{U^2}{a}\frac{\partial\vec{u}^*}{\partial t^*} + \frac{U^2}{a}(\vec{u}^*\cdot\vec{\nabla}^*)\vec{u}^*\right] = -\frac{P}{a}\vec{\nabla}^*p^* + \mu\frac{U}{a^2}\vec{\nabla}^{*2}\vec{u}^*$$

$$\Rightarrow \rho\frac{U^2}{a}\left[\frac{\partial\vec{u}^*}{\partial t^*} + (\vec{u}^*\cdot\vec{\nabla}^*)\vec{u}^*\right] = -\frac{P}{a}\vec{\nabla}^*p^* + \mu\frac{U}{a^2}\vec{\nabla}^{*2}\vec{u}^*$$

$$\Rightarrow \frac{\partial\vec{u}^*}{\partial t^*} + (\vec{u}^*\cdot\vec{\nabla}^*)\vec{u}^* = -\frac{P}{\rho U^2}\vec{\nabla}^*p^* + \underbrace{\frac{\mu}{\rho U a}}\vec{\nabla}^{*2}\vec{u}^*$$

$\left[\frac{\mu}{\rho U a}\right] = 1$, therefore it's called a dimensionless group or number.

The Reynolds number: $Re = \frac{\rho U a}{\mu}$

$$Re \simeq \frac{\text{inertia}}{\text{viscous effects}}$$

$$\frac{\partial\rho}{\partial t}$$

$$\swarrow$$

$$\underbrace{\rho = \boxed{\rho_0} + \alpha T} + \ldots$$

$$\Rightarrow \quad \frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^* \cdot \vec{\nabla}^*)\vec{u}^* = -\frac{P}{\rho U^2}\vec{\nabla}^* p^* + \frac{1}{Re}\vec{\nabla}^{*2}\vec{u}^*$$

case 1: $P = \rho U^2$

case 2: $P = \frac{\mu U}{a}$

$$\frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^* \cdot \vec{\nabla}^*)\vec{u}^* = -\vec{\nabla}^* p^* + \frac{1}{Re}\vec{\nabla}^{*2}\vec{u}^*$$

$$Re\left[\frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^* \cdot \vec{\nabla}^*)\vec{u}^*\right] = -\vec{\nabla}^* p^* + \vec{\nabla}^{*2}\vec{u}^*$$

BC: $r^* \to \infty \quad \vec{u}^* = \vec{e}_x$

$r^* = 1 \quad \vec{u}^* = \vec{0}$

Chapter 7: Potential flows

Lecture 9

2D incompressible and irrotational flows:

$\vec{\nabla}\cdot\vec{u}=0$ $\quad\quad$ $\vec{\nabla}\times\vec{u}=0$

* incompressible flow: $\vec{\nabla}\cdot\vec{u}=0$

streamfunction ψ : $u=\frac{\partial\psi}{\partial y}$, $v=-\frac{\partial\psi}{\partial x}$

automatically satisfies the incompressibility condition

$\vec{u}=\vec{\nabla}\psi$

$$\vec{\nabla}\times\vec{u}=0 \Longrightarrow \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$$

irrotationality

$$\Rightarrow -\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}=0$$

$$\Rightarrow \boxed{\vec{\nabla}^2\psi=0}$$

* irrotationality condition: $\vec{\nabla}\times\vec{u}=0$

$\vec{\nabla}\times(\vec{\nabla}V)=\vec{0}$

potential

velocity potential: φ : $u=\frac{\partial\varphi}{\partial x}$, $v=\frac{\partial\varphi}{\partial y}$

$\vec{u}=\vec{\nabla}\varphi$

incompressibility condition: $\vec{\nabla}\cdot\vec{u}=0 \Rightarrow \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

$$\Rightarrow \underbrace{\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}}=0$$

$$\Rightarrow \boxed{\vec{\nabla}^2\varphi=0}$$

$$\boxed{\begin{aligned} u &= \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \varphi}{\partial y} = - \frac{\partial \psi}{\partial x} \end{aligned}}$$

Cauchy - Riemann equations

Complex analysis: $\dfrac{df}{dx} = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x}$

$$\frac{df}{dz} = \lim_{|\delta z| \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \qquad z = x + iy$$

↖ has to take the same value no matter the direction of δz.

$$f : \mathbb{C} \longmapsto \mathbb{C} \qquad f(z) = g(x,y) + i\, h(x,y)$$

$$g : \mathbb{R}^2 \longmapsto \mathbb{R}$$

$$h : \mathbb{R}^2 \longmapsto \mathbb{R}$$

$$\frac{df}{dz} = \frac{\partial g}{\partial x} + i \frac{\partial h}{\partial x}$$ in the δx direction

$$\frac{df}{dz} = \frac{\partial g}{i\, \partial y} + i \frac{\partial h}{i\, \partial y}$$ in the $i\delta y$ direction

$$= \frac{\partial h}{\partial y} - i \frac{\partial g}{\partial y}$$

$\Rightarrow$ for f to be differentiable, we need

$$\frac{\partial g}{\partial x} + i \frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} - i \frac{\partial g}{\partial y}$$

$$\Rightarrow \begin{cases} \dfrac{\partial g}{\partial x} = \dfrac{\partial h}{\partial y} \\[2ex] \dfrac{\partial h}{\partial x} = - \dfrac{\partial g}{\partial y} \end{cases}$$

Based on this little complex analysis detour, we can introduce the complex potential:

$$w = \varphi + i\psi$$

complex potential ↗ w

velocity potential ↑ φ

↖ streamfunction ψ

w is complex differentiable ⟹ the flow represented by φ and ψ is incompressible and irrotational

$$\frac{dw}{dz} = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$= u - i v$$

(what about the $i\delta y$ direction?!)

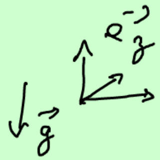
$$\overline{\frac{dw}{dz}} = u + i v$$

↖ ¯ : complex conjugate

The Bernoulli equations

Inviscid flow → Euler equation

$$\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = -\vec{\nabla}P - \rho g\,\vec{e}_z$$



$$\Rightarrow \rho\left[\frac{\partial \vec{u}}{\partial t} - \vec{u}\times\vec{\omega}\right] = -\vec{\nabla}\left(P + \frac{1}{2}\rho\vec{u}^2\right) - \rho g\,\vec{e}_z$$

$$\Rightarrow \boxed{\rho\left[\frac{\partial \vec{u}}{\partial t} - \vec{u}\times\vec{\omega}\right] = -\vec{\nabla}\left(P + \frac{1}{2}\rho\vec{u}^2 + \rho g z\right)}$$

case 1 : steady , irrotational

$$\Rightarrow \vec{\nabla}\left(P+\frac{1}{2}\rho\vec{u}^2+\rho g z\right)=\vec{0}$$

$$\Rightarrow P+\frac{1}{2}\rho\vec{u}^2+\rho g z = cst$$

case 2 : steady , (along a streamline) ↖ hypothesis to come later in calculation ←

$$\rho\,\vec{u}\times\vec{\omega} = \vec{\nabla}\left(P+\frac{1}{2}\rho\vec{u}^2+\rho g z\right)$$

$$\Rightarrow \rho\,\vec{u}\cdot(\vec{u}\times\vec{\omega}) = \vec{u}\cdot\vec{\nabla}\left(P+\frac{1}{2}\rho\vec{u}^2+\rho g z\right)$$

$$\Rightarrow 0 = \vec{u}\cdot\vec{\nabla}\left(P+\frac{1}{2}\rho\vec{u}^2+\rho g z\right)$$

$$\Rightarrow P+\frac{1}{2}\rho\vec{u}^2+\rho g z = cst(\psi)$$ in the direction of $\vec{u}$

$=$ along a streamline

case 3 : unsteady , irrotational

$$\rho\frac{\partial\vec{u}}{\partial t} = -\vec{\nabla}\left(P+\frac{1}{2}\rho\vec{u}^2+\rho g z\right)$$

irrotational $\Rightarrow \vec{u} = \vec{\nabla}\varphi$

$$\Rightarrow \rho\frac{\partial\vec{\nabla}\varphi}{\partial t} = \vec{\nabla}\left(\rho\frac{\partial\varphi}{\partial t}\right) = -\vec{\nabla}\left(P+\frac{1}{2}\rho\vec{u}^2+\rho g z\right)$$

$$\Rightarrow \vec{\nabla}\left(\rho\frac{\partial\varphi}{\partial t}+P+\frac{1}{2}\rho\vec{u}^2+\rho g z\right)=0$$

$$\Rightarrow \boxed{\rho \frac{\partial \varphi}{\partial t} + P + \frac{1}{2} \rho \vec{u}^2 + \rho g z = cst(t)}$$

Lecture 19: High Re flows

$$\frac{\partial \vec{u}^*}{\partial t^*} + \left(\vec{u}^* \cdot \vec{\nabla}^*\right)\vec{u}^* = -\vec{\nabla}^* p^* + \frac{1}{Re}\,\underline{\underline{\vec{\nabla}^{*2}\vec{u}^*}}$$

Even if Re >> 1, we should not neglect this term

(singular perturbations)

Boundary layers will develop where $\vec{\nabla}^{*2}\vec{u}^* \sim Re$ in such a way that $\frac{1}{Re}\vec{\nabla}^{*2}\vec{u}^*$ enters the dominant balance (and both boundary conditions can be applied).

In the boundary layer: $\rho\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} \sim \mu\vec{\nabla}^2\vec{u}$

$$\Rightarrow \rho\frac{U^2}{L} \sim \frac{\mu U}{\delta^2}$$

← thickness of the boundary layer.

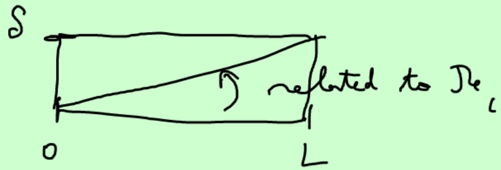
advective length

$$\Rightarrow \delta^2 \sim \frac{\mu L}{\rho U}$$

$$\Rightarrow \delta \sim L\sqrt{\frac{\mu}{\rho U L}}$$

$$\Rightarrow \delta \sim L\, Re_L^{-1/2}$$

$$\frac{\delta}{L} \sim Re_L^{-1/2}$$

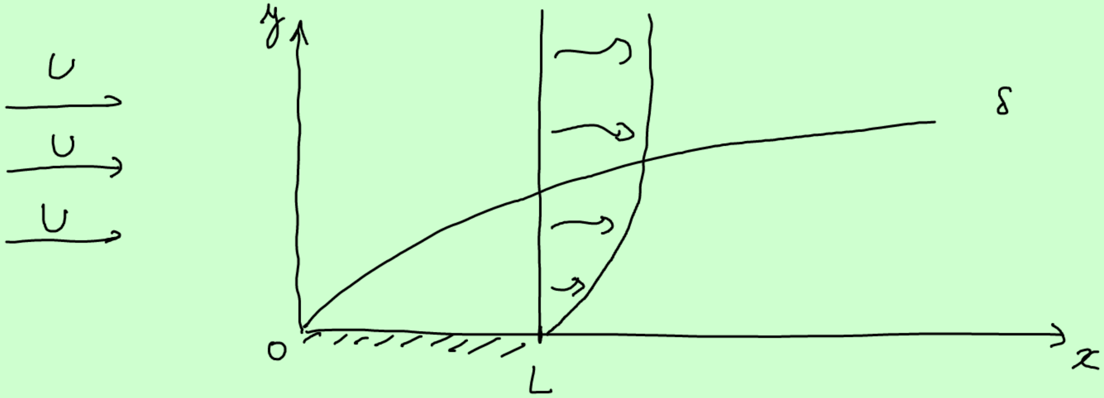


Blasius boundary layer :

(Prandtl & Blasius 1903)

Hypothesis : • 2D
• steady
• High Re

$\varepsilon = \frac{\delta}{L} \ll 1$



Equations :

$$\begin{cases} \rho u \partial_x u + \rho v \partial_y u = -\partial_x p + \mu \partial_{xx} u + \mu \partial_{yy} u \\ \rho u \partial_x v + \rho v \partial_y v = -\partial_y p + \mu \partial_{xx} v + \mu \partial_{yy} v \\ \partial_x u + \partial_y v = 0 \end{cases}$$

3 equations, 3 unknowns (u, v, p)

2D (x, y)

BC :

$$\begin{cases} y = 0 & u = v = 0 \\ y \to \infty & u \to U \quad v \to 0 \end{cases}$$

Nondimensionalization :

$$x = L\, x^*$$
$$y = \delta\, y^* = \varepsilon L\, y^*$$
$$u = U u^*$$
$$v = \varepsilon U v^* \quad \text{(incompressibility)}$$
$$p = \rho U^2 p^*$$

$$\begin{cases} \frac{\rho U^2}{L}\left[u^* \partial_{x^*} u^* + v^* \partial_{y^*} u^*\right] = -\frac{\rho U^2}{L}\partial_{x^*} p^* + \frac{\mu U}{L^2}\partial_{x^*x^*} u^* + \frac{\mu U}{\varepsilon^2 L^2}\partial_{y^*y^*} u^* \\ \frac{\rho \varepsilon U^2}{L}\left[u^*\partial_{x^*} v^* + v^*\partial_{y^*} v^*\right] = -\frac{\rho U^2}{\varepsilon L}\partial_{y^*}p^* + \frac{\mu\varepsilon U}{L^2}\partial_{x^*x^*}v^* + \frac{\mu U}{\varepsilon L^2}\partial_{y^*y^*}v^* \end{cases}$$

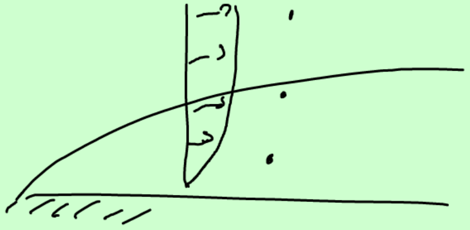
$$\Rightarrow \begin{cases} u^* \partial_{x^*} u^* + v^* \partial_{y^*} u^* = -\partial_{x^*} p^* + \cancel{\frac{1}{Re_L}\partial_{x^*x^*} u^*} + \frac{1}{\varepsilon^2 Re_L}\partial_{y^*y^*} u^* \\ \cancel{u^*\partial_{x^*} v^*} + \cancel{v^*\partial_{y^*} v^*} = -\frac{1}{\varepsilon^2}\partial_{y^*}p^* + \cancel{\frac{1}{Re_L}\partial_{x^*x^*}v^*} + \cancel{\frac{1}{\varepsilon^2 Re_L}\partial_{y^*y^*}v^*} \end{cases}$$

To keep advection <u>and</u> diffusion in the dominant balance, we impose $Re_L = \varepsilon^{-2}$

$$\Rightarrow \frac{\delta^2}{L^2} = Re_L^{-1}$$

$$\Rightarrow \delta = L\, Re_L^{-1/2}\ . \quad \textcircled{!}$$

$$\Rightarrow \begin{cases} \rho u \partial_x u + \rho v \partial_y u = -\partial_x p + \mu \partial_{yy} u \\ 0 = -\partial_y p \\ \partial_x u + \partial_y v = 0 \end{cases}$$



If we look infinitely far away from the boundary, the flow is steady and irrotational, so we can apply the Bernoulli law: $p + \frac{1}{2}\rho \vec{u}^2 = cst$ $\Rightarrow p + \frac{1}{2}\rho U^2 = cst$
(1)

$$\Rightarrow p = cst$$

$$\Rightarrow \partial_x p = 0$$

$$\Rightarrow \begin{cases} \rho\, u\, \partial_x u + \rho\, v\, \partial_y u = \mu\, \partial_{yy} u \\ \partial_x u + \partial_y v = 0 \end{cases}$$

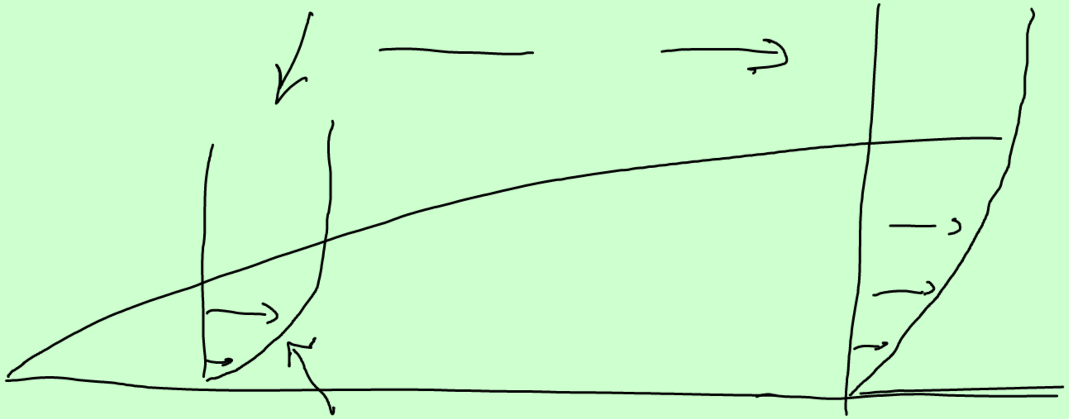
Streamfunction : $u = \frac{\partial \psi}{\partial y}$ $\qquad v = -\frac{\partial \psi}{\partial x}$

$$\boxed{\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}}$$

$\nu = \frac{\mu}{\rho}$ kinematic viscosity

1 equation, 1 unknown (ψ)

2D (x, y)



Stokes 1st problem

$\eta = y\, t^{-1/2}$

$u(x,y) = f(\eta)$

Similarity variable: $\underline{\eta = y\left(\frac{U}{\nu x}\right)^{1/2}} = \frac{y}{\delta}$ $\qquad \delta(x) = \left(\frac{x\nu}{U}\right)^{1/2}$

$$u = U f'(\eta) \Rightarrow U f' = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} \leftarrow \frac{1}{\delta}$$

$$\Rightarrow \frac{\partial \psi}{\partial \eta} = U \delta f'$$

$$\Rightarrow \boxed{\psi = U\delta f} \swarrow$$

$$\psi = U\delta(x) f(\eta)$$

$$\eta = \frac{y}{\delta}$$

$$\frac{\partial\eta}{\partial y} = \frac{1}{\delta} \swarrow$$

$$\frac{\partial\eta}{\partial x} = -\frac{y\delta'}{\delta^2}$$

$$\partial_x \psi = U\delta' f + U\delta \frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial x}$$

$-v$

$$= U\delta' f - \frac{U y \delta'}{\delta} f'$$

$$\partial_y \psi = U\delta \frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial y}$$

u

$$= U f'$$

$$\partial_{yy}\psi = \partial_y (U f') = \frac{U f''}{\delta}$$

$$\partial_{yyy}\psi = \frac{U f'''}{\delta^2}$$

$$\partial_{xy}\psi = \partial_x (U f') = U \frac{\partial f'}{\partial\eta}\frac{\partial\eta}{\partial x} = -\frac{U y \delta'}{\delta^2} f''$$

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \nu \frac{\partial^3\psi}{\partial y^3}$$

$$\Rightarrow U f'\left[-\frac{U y \delta'}{\delta^2} f''\right] - \left[U\delta' f - \frac{U y \delta'}{\delta} f'\right]\frac{U f''}{\delta} = \frac{\nu}{\delta^2} U f'''$$

$$\Rightarrow \; -\cancel{\frac{U^2 y g'}{g^2}} f f'' - \frac{U^2 g'}{g} f f'' + \cancel{\frac{U^2 y g'}{g^2}} f f'' = \frac{\nu}{g^2} U f'''$$

$$\Rightarrow \; f''' + \frac{U g' g}{\nu} f f'' = 0$$

$$g = \left(\frac{\nu x}{U}\right)^{1/2} \quad , \quad g' = \frac{1}{2}\left(\frac{\nu}{U x}\right)^{1/2}$$

$$\Rightarrow \; g g' = \frac{1}{2} \, \frac{\nu}{U}$$

$\hookrightarrow$ $\boxed{f''' + \frac{1}{2} f f'' = 0}$ Blasius boundary layer equation

1 equation, 1 unknown (f)
1D (η)

$$\text{BC:} \quad \begin{array}{lll|ll} y = 0 & u = 0 & & \eta = 0 & f = 0 \\ & v = 0 & & & f' = 0 \\ y \to \infty & u \to U & & \eta \to \infty & f' \to 1 \\ & v \to 0 & & & \end{array}$$

no need for extra BC, problem is 3rd order.

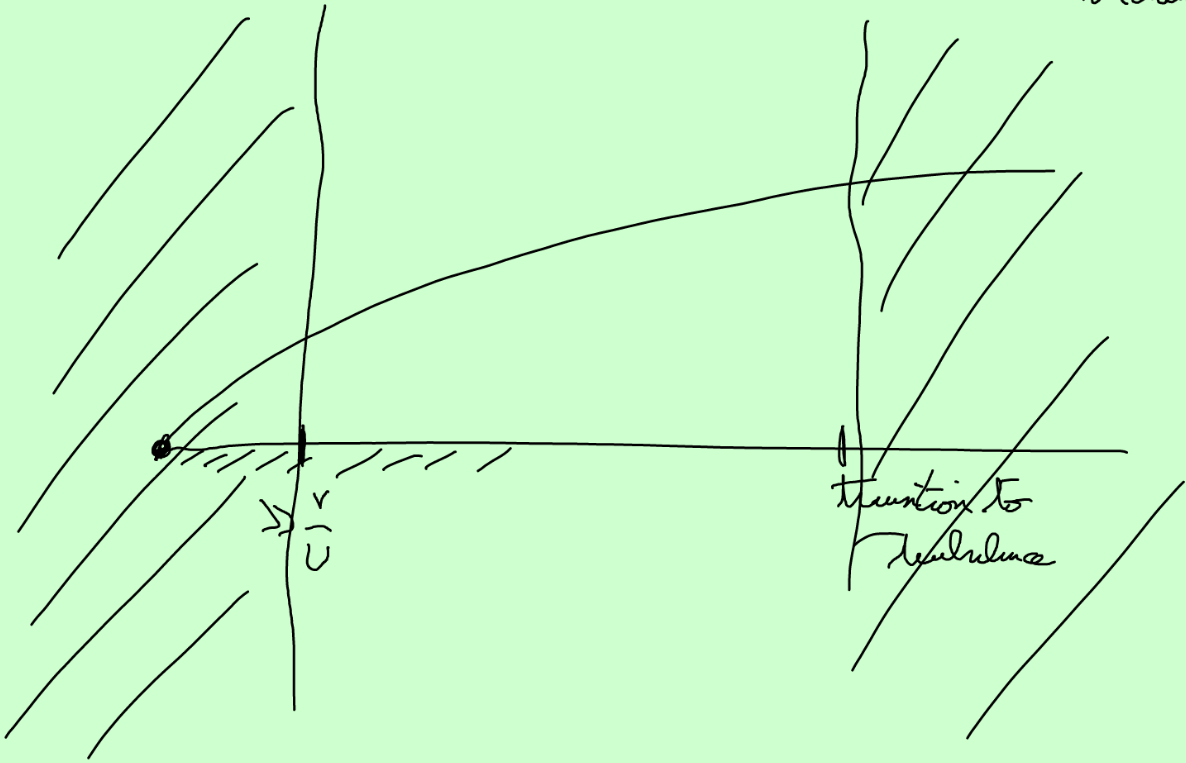
Boundary layer thickness: $\frac{u}{U} = 0.99 \rightarrow f' = 0.99 \Rightarrow \eta_{99} \sim 5$ (numerically)

$$\Rightarrow \boxed{\delta_{99}\left(\frac{U}{\nu x}\right)^{1/2} \sim 5} \leftarrow$$

$$\Rightarrow \frac{\delta_{99}}{x} \sim 5 \left(\frac{U x}{\nu}\right)^{-1/2}$$

BL thickness $\Rightarrow \frac{\delta_{99}}{x} \sim 5\, Re_x^{-1/2}$

within 10% of numerical results for the laminar BL



$Re_L \gg 1$

$\Rightarrow \frac{UL}{\nu} \gg 1$

$\Rightarrow L \gg \frac{\nu}{U}$

Lecture 20: From laminar to turbulent flows

Toy problem for turbulence:

$$\left(\frac{\partial u}{\partial t}=\right)\dot{u} = u\left[1-\frac{1}{r}-(u-1)^2\right] \quad , \; r \geq 0$$

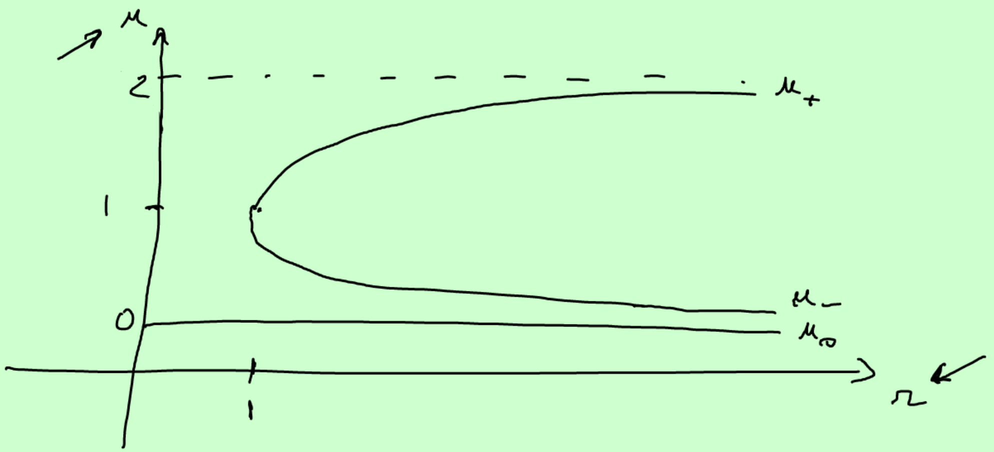
parameter

state variable
scalar

* steady states:

- $u_0 = 0$
- $u_+ = 1+\sqrt{1-\frac{1}{r}}$
- $u_- = 1-\sqrt{1-\frac{1}{r}}$

$\left.\right\} r \geq 1$



bifurcation diagram

* stability: $u = U + \tilde{u}$

u_0, u_+ or u_-
base state

perturbation

$$\dot{\tilde{u}} = (U+\tilde{u})\left[1-\frac{1}{r}-(U+\tilde{u}-1)^2\right]$$

$$= \underbrace{U\Big[1-\frac{1}{R}-(U-1)^2\Big]}_{=0} + U\Big[-\tilde{u}^2 - 2\tilde{u}(U-1)\Big]$$

$$+ \tilde{u}\Big[1-\frac{1}{R}-(U+\tilde{u}-1)^2\Big]$$

$$= -U\tilde{u}^2 - 2\tilde{u}U^2 + 2\tilde{u}U + \tilde{u} - \frac{\tilde{u}}{R} - U^2\tilde{u} - \tilde{u}^3 - \tilde{u}$$

$$- 2U\tilde{u}^2 + 2U\tilde{u} + 2\tilde{u}^2$$

$$= \tilde{u}\Big[-2U^2 + 2U + \underbrace{1 - \frac{1}{R} - U^2 -}_{} 1 + 2U\Big] + \tilde{u}^2\Big[-U - 2U + 2\Big] - \tilde{u}^3$$

$$\dot{\tilde{u}} = \tilde{u}\Big[-\frac{1}{R} + 4U - 3U^2\Big] + \tilde{u}^2\Big[2 - 3U\Big] - \tilde{u}^3$$

Let us assume that $\tilde{u}$ is small : $\tilde{u} = \varepsilon\hat{u}$, $\varepsilon \ll 1$, $\hat{u} = O(1)$

$$\hookrightarrow \quad \varepsilon\dot{\hat{u}} = \varepsilon\hat{u}\Big[-\frac{1}{R} + 4U - 3U^2\Big] + O(\varepsilon^2)$$

$$\Rightarrow \quad \dot{\hat{u}} \simeq \hat{u}\Big[-\frac{1}{R} + 4U - 3U^2\Big] \longleftarrow$$

The solution is $\underline{\hat{u} = k e^{\lambda t}}$

growth rate : $\lambda < 0$ → perturbation decays → U is stable

$\lambda > 0$ → perturbation grows → U is unstable

$$\hookrightarrow \quad k\lambda e^{\lambda t} = k e^{\lambda t}\Big[-\frac{1}{R} + 4U - 3U^2\Big]$$

$$\Rightarrow \quad \lambda = -\frac{1}{R} + 4\underline{U} - 3\underline{U}^2$$ (dispersion relation)

* for u_0: $\lambda = -\frac{1}{r} + 4\times 0 - 3\times 0^2$

$= -\frac{1}{r}$

Since $r > 0$, $\lambda < 0 \Longrightarrow u_0$ is stable

* for u_+: $\lambda = -\frac{1}{r} + 4\left(1+\sqrt{1-1/r}\right) - 3\left(1+\sqrt{1-1/r}\right)^2$

$= -\frac{1}{r} + 4 + 4\sqrt{1-1/r} - 3 - 3 + \frac{3}{r} - 6\sqrt{1-1/r}$

$= \frac{2}{r} - 2 - 2\sqrt{1-1/r}$

$= -2\left[1 - \frac{1}{r} + \sqrt{1-1/r}\right]$

$< 0 \quad (r \geq 1)$

$\Rightarrow u_+$ is stable

* for u_-: $\lambda = -\frac{1}{r} + 4\left(1-\sqrt{1-1/r}\right) - 3\left(1-\sqrt{1-1/r}\right)^2$

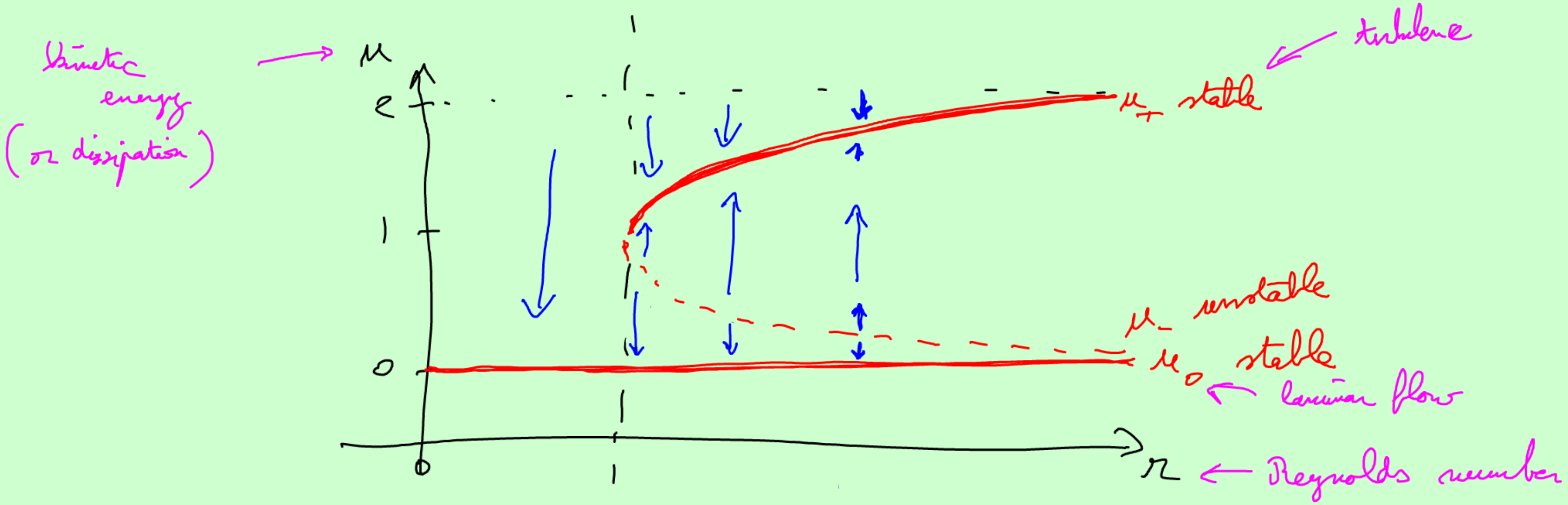
$= -\frac{1}{r} + 4 - 4\sqrt{1-1/r} - 3 - 3 + \frac{3}{r} + 6\sqrt{1-1/r}$

$= \frac{2}{r} - 2 + 2\sqrt{1-1/r}$

$= 2\left[\underline{\sqrt{1-1/r}} - \left(1 \underset{\uparrow}{-} 1/r\right)\right]$

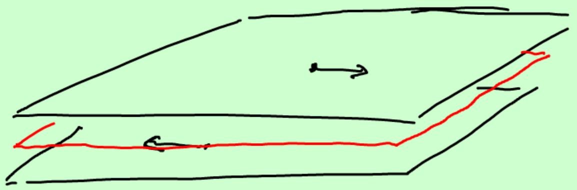
> 0

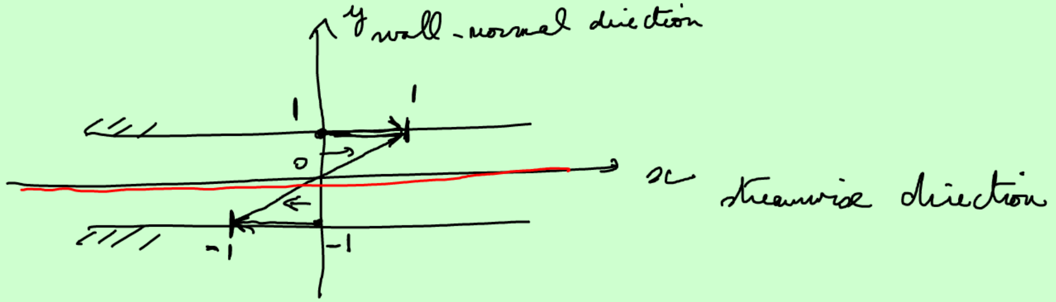
$\Rightarrow u_-$ is unstable

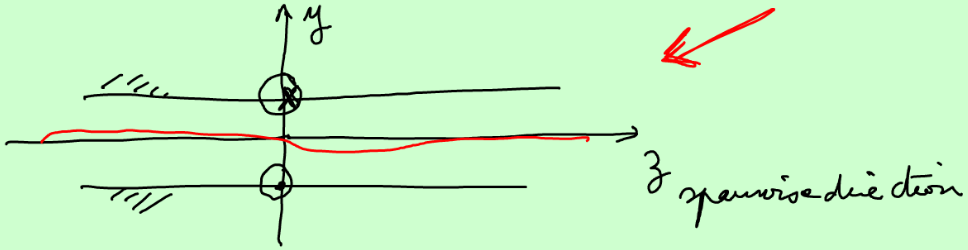


Transition to turbulence in shear flows

Plane Couette flow:







$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}p + \underset{\uparrow}{\dfrac{1}{Re}}\,\vec{\nabla}^2\vec{u} \\ \vec{\nabla}\cdot\vec{u} = 0 \end{cases}$$

$$\left.\begin{array}{l} x\text{ - direction} \\ z\text{ - direction} \end{array}\right\} \text{periodic boundary conditions}$$

$y = 1 \quad \vec{u} = 1\,\vec{e_x}$

$y = -1 \quad \vec{u} = -1\,\vec{e_x}$

Lecture 22

High Reynolds number flows

Kolmogorov (1941)

$$\frac{\partial \vec{u}}{\partial t} = -(\vec{u}\cdot\vec{\nabla})\,\vec{u}\; \overbrace{- \frac{1}{\rho}\vec{\nabla} p + \nu \vec{\nabla}^2 \vec{u}}^{\text{linear}}$$

redistribution of the energy between wavenumbers

ensures incompressibility

dissipates energy

* Large scale flow : $Re = \frac{UL}{\nu} \gg 1$

Kinetic energy : $E_k \sim U^2$

Eddy turnover time : $\tau = \frac{L}{U}$

Power input at large scale : $\mathcal{P} = \frac{E_k}{\tau} = \frac{U^3}{L} \sim L^2 . T^{-3}$

* Smallest scale in the flow should be such that advection and diffusion are in balance :

$$Re_\ell = 1 \Rightarrow \boxed{\frac{u\ell}{\nu} = 1}$$

dissipation at small scales : $\varepsilon = \nu\left(\frac{u}{\ell}\right)^2 \sim L^2 . T^{-3}$

$\nu(\vec{\nabla}\times\vec{u})^2$

$$\varepsilon = \frac{\nu^3}{\ell^4} \quad \text{or} \quad \varepsilon = \frac{u^4}{\nu}$$

* $\mathcal{P} = \varepsilon$ for an established flow

$$\hookrightarrow \quad \frac{U^3}{L} = \frac{\nu^3}{\ell_k^4} \qquad\qquad \Bigg| \qquad \frac{U^3}{L} = \frac{u_k^4}{\nu}$$

$$\Rightarrow \ell = L\left(\frac{\nu^3}{U^3L^3}\right)^{1/4} \qquad\qquad \Rightarrow u = U\left(\frac{\nu}{UL}\right)^{1/4}$$

$$\Rightarrow \boxed{\ell = L\ \mathcal{R}e^{-3/4}} \qquad\qquad \Rightarrow \boxed{u = U\ \mathcal{R}e^{-1/4}}$$

Kolmogorov scale

Kolmogorov laws

* At intermediate scales, turbulence is self-similar

dissipation is proportional to the available energy at all wavenumber.

$$\varepsilon \sim E_k \ ?$$

$$\varepsilon \rightarrow L^2.T^{-3} \qquad E_k \rightarrow L^2.T^{-2}$$

NO

At intermediate scales: $E_k \sim v^2$ — intermediate velocity scale

$$\varepsilon \sim v^2\,\frac{v}{r}$$ — intermediate length scale

$$\Rightarrow \quad \varepsilon = \frac{v^3}{r}$$

$$\Rightarrow \quad v^3 = \varepsilon r$$

$$\Rightarrow \quad v^2 = \varepsilon^{2/3} r^{2/3}$$

We introduce the wavenumber: $k \sim \frac{1}{r}$

$$\Rightarrow \quad v^2 = \varepsilon^{2/3} k^{-2/3}$$

Results are often represented using the density of kinetic energy:

$$\boxed{E = \frac{d(v^2)}{dk} \sim \varepsilon^{2/3} k^{-5/3}}$$ Kolmogorov $\frac{5}{3}$ law.