MATH5453M Foundations of Fluid Dynamics

Lecture 7: Potential flows and the Bernoulli equation

For high enough Reynolds numbers, vorticity is often confined to boundary layers near solid surfaces so that, away from these areas, the flows can be considered to be vorticity-free.

7.1 Potential flow: Paterson, p.205–240

Suppose we have vorticity-free flow. We can then define a potential φ, such that $\mathbf{u} = \nabla\varphi$ and

$$\varphi = \int_{\mathbf{x_0}}^{\mathbf{x}} \mathbf{u} \cdot d\mathbf{l}.$$

We choose the potential to be zero at $\mathbf{x_0}$, which defines a unique potential.

Proof Normally, the value of φ would depend on the path chosen to get from $\mathbf{x_0}$ to $\mathbf{x}$, but if we define two different paths, C_1 and C_2, then the Stokes theorem stipulates:

$$\int_{C_1} \mathbf{u} \cdot d\mathbf{l} + \int_{-C_2} \mathbf{u} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{u} \cdot \hat{\mathbf{n}}\, ds,$$

where S is the surface enclosed by C_1 and C_2. For irrotational flows, the right-hand-side of this equation vanishes and:

$$\int_{C_1} \mathbf{u} \cdot d\mathbf{l} = \int_{C_2} \mathbf{u} \cdot d\mathbf{l},$$

regardless of the paths C_1 and C_2. So the velocity potential is uniquely defined.

For example consider the stagnation point flow, $\mathbf{u} = (x, -y, 0)$. Note that this flow is irrotational and incompressible. The velocity potential can be solved for by integrating $\partial\varphi/\partial x = x$: $\varphi = x^2/2 + f(y)$, then substituting into $\partial\varphi/\partial y = -y$. We get $\varphi = x^2/2 - y^2/2 + C$, where the velocity potential is defined up to a constant, C, which can be chosen to be zero without impacting the velocity. The velocity potential then writes: $\varphi = x^2/2 - y^2/2$.

If a velocity potential exists and the fluid is incompressible, $\nabla \cdot \mathbf{u} = 0$ implies $\nabla^2\varphi = 0$. This is the Laplace equation, a very well known and studied equation in applied mathematics. It is also possible to transform solutions of the Laplace equation into other solutions. For example, there is a simple solution to potential flow past a cylinder of radius a in polar coordinates (R, ϕ):

$$\varphi = U_0 R \cos\phi + \frac{U_0 a^2 \cos\phi}{R}.$$

By making a suitable transformation of coordinates, known as the Joukowski transformation (see: Paterson, p.462; Acheson, p.134), we can turn this into potential flow past a shape like an aerofoil.

7.1.1 Complex variable methods: Paterson, p.396–432; Acheson, p.124

For two-dimensional steady flows, we can also define the streamfunction ψ and, since $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, it follows from the irrotationality condition that $\nabla^2\psi = 0$. So φ and ψ both satisfy the Laplace equation for steady 2D irrotational flows. We can also relate the velocity potential and the streamfunction via:

$$u = \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \tag{1}$$

$$v = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{2}$$

The right-hand pair of equalities in the above equations are known as the *Cauchy–Riemann* equations and have a connection with complex variable theory.

Let f be a complex function of the complex variable z, where $z = x + iy$. The derivative of f with respect to z is defined in the usual way:

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}.$$

This definition requires that the limit be the same for all infinitesimal increments δz whichever their direction in the complex plane. As a consequence, differentiability of a complex function imposes restrictions on its real and imaginary parts. If $f(z) = g(x, y) + ih(x, y)$ is complex differentiable, where g and h are both real functions of real variables, then taking $\delta z = \delta x$ gives:

$$\frac{df}{dz} = \frac{\partial g}{\partial x} + i\frac{\partial h}{\partial x}, \tag{3}$$

while taking $\delta z = i\delta y$ gives:

$$\begin{aligned} \frac{df}{dz} &= \frac{1}{i}\frac{\partial g}{\partial y} + \frac{\partial h}{\partial y} && (4) \\ &= \frac{\partial h}{\partial y} - i\frac{\partial g}{\partial y}. && (5) \end{aligned}$$

Comparing the real and imaginary terms in equations (3) and (5), we obtain the Cauchy–Riemann equations:

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{\partial h}{\partial y}, && (6) \\ \frac{\partial g}{\partial y} &= -\frac{\partial h}{\partial x}. && (7) \end{aligned}$$

Hence, if we represent points in the $x - y$ plane as complex numbers, $z = x + iy$, and introduce the complex potential $w(z) = \varphi(x, y) + i\psi(x, y)$, then, provided $w(z)$ is a complex diffentiable, φ and ψ will satisfy the Cauchy–Riemann equations and, hence, be related by equations (1) and (2). The fluid velocity is given by

$$\frac{dw}{dz} = u - iv.$$

The limitation of this beautiful theory is that it is only useful for two-dimensional irrotational inviscid flows.

7.2 Bernoulli equation: Paterson, p.180–196; Kundu, p.128–134

We start here from the hypothesis of an inviscid flow. The Euler equation reads:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega}\right) = -\nabla(P + \frac{1}{2}\rho\mathbf{u}^2) - \rho g\hat{\mathbf{z}},$$

where the only body force considered is gravity. There are several versions of the Bernoulli equation depending on the hypotheses made.

7.2.1 Bernoulli v.1 – Steady irrotational flow

For a steady and irrotational flow, all the terms on the left hand side vanish and, since $\rho g\hat{\mathbf{z}} = \nabla(\rho g z)$, we have:

$$\nabla\left(P + \frac{1}{2}\rho\mathbf{u}^2 + \rho g z\right) = 0,$$

so

$$P + \frac{1}{2}\rho\mathbf{u}^2 + \rho g z = cst$$

, which is the simplest form of the Bernoulli equation.

Flow out of a barrel: Paterson, p.181

Suppose water flows out of a barrel through a tap. There is no obvious vorticity source, so we can assume that the flow is irrotational as a first approximation. Assume the top of the water is a height h above the tap and that the water level drops very slowly. At the top surface, the pressure is that of the atmosphere, P_{atm}, and the flow speed is very small. The Bernoulli constant can be obtained at the surface: $c = P_{atm} + \rho g h$. At the tap ($z = 0$), the pressure is also that of the atmosphere because the water coming out is in contact with it, and the velocity is u. Note that the velocity of the water coming out of the tap may not be exactly constant, as water near the tap edges may be moving more slowly. However, applying the Bernoulli equation gives a simple approximate answer, that can sometimes be accurate. So:

$$P_{atm} + \rho g h = P_{atm} + \frac{1}{2}\rho u^2,$$

or $u = \sqrt{2gh}$. This result is known as the Torricelli law.

7.2.2 Bernoulli v.2 – Steady flow along a streamline: Batchelor, p.386–396

By writing the right-hand-side of the Euler equation in its potential form, we have:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} - \mathbf{u}\times\boldsymbol{\omega}\right) = -\nabla\left(P + \frac{1}{2}\rho\mathbf{u}^2 + \rho g z\right).$$

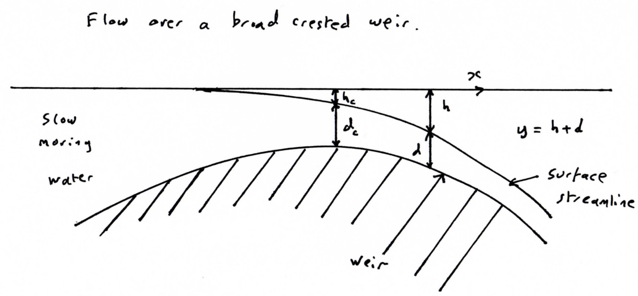
Even if the flow is not irrotational, $\mathbf{u}\cdot(\mathbf{u}\times\boldsymbol{\omega}) = 0$, so, along a streamline:

$$P + \frac{1}{2}\rho\mathbf{u}^2 + \rho g z = cst.$$

Broad crested weir: Batchelor, p.391–392
We can apply the Bernoulli law to the surface streamline of the flow going over a weir. Assuming the velocity is a function of x only and that the pressure on the surface is that of the atmosphere, the Bernoulli equation gives $u^2/2 - gh = cst$. Behind the weir, the velocity is slow, so $u^2 = 2gh$. Also, the flux of water per unit length across the weir, $Q = ud$, must be a constant independent of x. So, $y = h + d = Q/u + u^2/2g$. At the top of the weir, $x = x_c$, $dy/dx = 0$:

$$-\frac{Q}{u_c^2} + u_c/g = 0 \quad\Rightarrow\quad u_c = (Qg)^{1/3}$$
$$\Rightarrow\quad d_c = Q^{2/3}g^{-1/3}$$
$$\Rightarrow\quad h_c = \frac{Q^{2/3}}{2g^{1/3}}.$$

The easiest height to measure is $y_c = h_c + d_c$, the height of the pool behind the weir above the top of the weir. Then the flux of water over the weir is $(2y_c/3)^{3/2}g^{1/2}$.



7.2.3 Bernoulli v.3 – Unsteady irrotational flows: Paterson, p.228–231

Starting again from the Euler equation:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} - \mathbf{u}\times\boldsymbol{\omega}\right) = -\nabla\left(P + \frac{1}{2}\rho\mathbf{u}^2 + \rho g z\right),$$

we impose the irrotationality condition, $\boldsymbol{\omega} = 0$, to get:

$$\nabla\left(\rho\frac{\partial\varphi}{\partial t} + P + \frac{1}{2}\rho\mathbf{u}^2 + \rho g z\right) = 0,$$

which yields:

$$\frac{\partial\varphi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\mathbf{u}^2 + gz = C(t),$$

where $C(t)$ is a constant that depends on time.

Collapsing bubble
We consider a bubble of fluid of radius $R(t)$ in a liquid with pressure p_0 far from the bubble. We neglect gravity and the gas pressure inside the bubble. We also assume the flow is purely radial, $u_r(r,t)$, and, consequently, irrotational. We then need to solve $\nabla^2\varphi = 0$ and the only spherically symmetric solution of the Laplace equation that vanishes at infinity is:

$$\varphi = \frac{A(t)}{r},$$

giving

$$u_r = -\frac{A(t)}{r^2}.$$

At the surface of the bubble, $u_r = \dot{R}$, so

$$\varphi = -R^2\dot{R}/r.$$

The Bernoulli equation for an unsteady flow reads:

$$\frac{\partial \varphi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2 = k_1,$$

where p is the dynamic pressure and k_1 a constant to be determined.

For $r \to \infty$, we get $k_1 = p_0/\rho$. Expressing the Bernoulli equation at $r = R$, where $p = 0$, yields:

$$-R\ddot{R} - 2\dot{R}^2 + \tfrac{1}{2}\dot{R}^2 = \frac{p_0}{\rho}$$

$$\Rightarrow \qquad -R\ddot{R} - \tfrac{3}{2}\dot{R}^2 = \frac{p_0}{\rho},$$

which is an ODE for R. Multiplying by $2R^2\dot{R}$ we can integrate once to get:

$$\dot{R}^2 = \frac{2p_0}{3\rho} + \frac{k_2}{R^3},$$

where k_2 is a constant of integration. Since $\dot{R} = 0$ at $R = R_0$, we have:

$$\dot{R}^2 = \frac{2p_0}{3\rho}\left(\frac{R_0^3}{R^3} - 1\right).$$

There is no analytic solution of this equation but, as R gets smaller, $\dot{R}$ increases, so the bubble collapse accelerates. If a small amount of gas is trapped inside the bubble, it heats up because due to the compression. A striking illustration of this is *sonoluminesence* (https://www.youtube.com/watch?v=HE0k7Z2xoiw), where ultrasound causes a bubble to initial expand and then collapse. The collapse is sufficiently sudden that the heated gas inside the bubble gives off a flash of light. The process then repeats, so that the bubble gives the appearance of steady source of light, akin to "a star in a jar".

Dr. Cédric Beaume – c.m.l.beaume@leeds.ac.uk