MATH5453M Foundations of Fluid Dynamics

Lecture 6: Dynamical similarity and the Reynolds number

6.1 Dynamical Similarity: Tritton, p.89–96; Kundu, p.143–151

Suppose we are interested in the flow past an obstacle such as a cylinder. We might then solve the Navier-Stokes equations

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} = -\nabla p + \mu\nabla^2\mathbf{u},$$
$$\nabla\cdot\mathbf{u} = 0,$$

usually numerically. The input parameters are the flow speed far from the cylinder, the radius of the cylinder, the density and the viscosity of the fluid.

Having solved the problem for one set of parameter values, we might be interested in its solution for other parameter values. The logical way to do so is to rerun the code using the new setup. The aim of dynamical similarity is to reduce the number of system parameters so that each numerical run provides as much information as possible.

6.2 Nondimensionalization

To make the problem dimensionless, we need to choose suitable units for each problem variable. Let us consider the flow past a cylinder. The velocity far away from the cylinder is uniform, $U_0\hat{\mathbf{x}}$, and the axis of the cylinder of radius a is in the z-direction. A sensible unit for the velocity is, therefore, U_0: $\mathbf{u} = U_0\mathbf{u}^*$, where the asterisk denotes the dimensionless variable. The dimensionless velocity far away from the cylinder is: $\mathbf{u}^* = \hat{\mathbf{x}}$. The radius of the cylinder, a, is a natural quantity to use for nondimensionalizing distances. From these two units, we can define a unit of time: a/U_0: $t = at^*/U_0$. Since the coordinates x, y and z have units of length, we write $x = ax^*$, $y = ay^*$, $z = az^*$. The nabla operator then becomes:

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right) = \frac{1}{a}\left(\frac{\partial}{\partial x^*},\frac{\partial}{\partial y^*},\frac{\partial}{\partial z^*}\right) \qquad \text{or} \qquad \nabla = \frac{1}{a}\nabla^*.$$

Upon replacing the dimensional variables in the Navier–Stokes equation and dividing by ρ, we get

$$\frac{U_0^2}{a}\frac{\partial\mathbf{u}^*}{\partial t^*} + \frac{U_0^2}{a}(\mathbf{u}^*\cdot\nabla^*)\mathbf{u}^* = -\frac{1}{a\rho}\nabla^* p + \frac{\mu U_0}{a^2\rho}\nabla^{*2}\mathbf{u}^*.$$

We simplify this equation by multiplying by $a^2\rho/\mu U_0$:

$$\frac{U_0 a\rho}{\mu}\left(\frac{\partial\mathbf{u}^*}{\partial t^*} + (\mathbf{u}^*\cdot\nabla^*)\mathbf{u}^*\right) = -\frac{a}{\mu U_0}\nabla^* p + \nabla^{*2}\mathbf{u}^*.$$

At this stage, the pressure p is still dimensional. We can nondimensionalize by anything that has the right dimensions, but setting $p = \mu U_0 p^*/a$, i.e. the unit of pressure is $\mu U_0/a$ (dimensions $\mathrm{ML}^{-1}\mathrm{T}^{-2}$, i.e, it is a force per unit area) would make the resulting equation friendlier:

$$\frac{U_0 a\rho}{\mu}\left(\frac{\partial\mathbf{u}^*}{\partial t^*} + (\mathbf{u}^*\cdot\nabla^*)\mathbf{u}^*\right) = -\nabla^* p^* + \nabla^{*2}\mathbf{u}^*,$$

which only shows one dimensionless group, the Reynolds number: $Re = U_0 a\rho/\mu$. Dropping the asterisks, we obtain a nondimensinoal version of the Navier–Stokes equation:

$$Re\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \nabla^2\mathbf{u}.$$

This equation highlights the fact that, fo rthe flow pas a cylinder, there is only one important parameter: the Reynolds number.

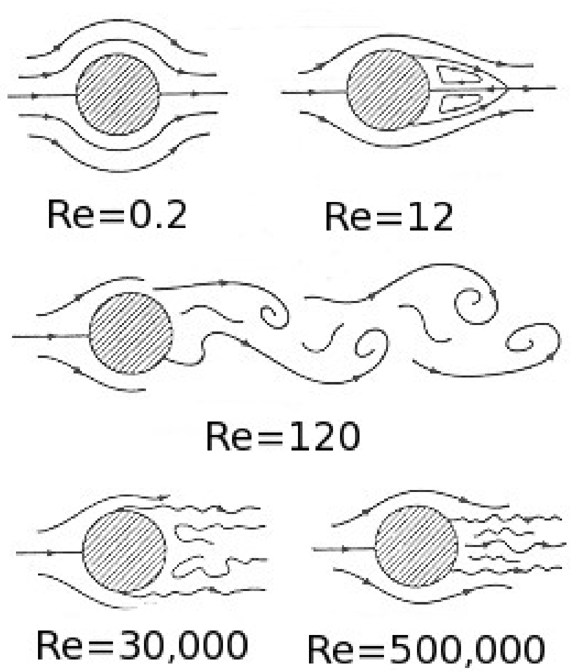
6.3 Reynolds number and dynamical similarity

Given the nondimensionalization performed in the previous Section, we now know that there is only one parameter in the problem of the flow past a cylinder. This has important repercussions. If we know the solution for a given set of parameter values, e.g. $a = 0.1$m, $U_0 = 1$m·s^{-1}, $\mu = 0.001$ Pa·s and density $\rho = 10^3$ kg·m^{-3}, it is pointless to solve the problem again for $a = 0.1$m, $U_0 = 2$m·s$^-1$, $\mu = 0.002$ Pa·s and $\rho = 10^3$ kg·m^{-3} because the Reynolds number in both cases is the same. The nondimensional result in both cases will be the same and it is the conversion from nondimensional to dimensional that will produce different dimensional results. Using dynamical similarity can save a lot of unnecessary work.

Using dynamical similarity is not restricted to the flow past a cylinder (or to fluid dynamics!). In general, physical problems are more complicated and involve more than one dimensionless parameter.

6.4 Flow past a cylinder at varying Reynolds number: Tritton, p.23, p.75–80

We consider the flow pas a cylinder. Below are sketch of the flow at various values of $Re = \rho U d/\mu$, where d is the cylinder diameter.



- $Re = 0.2$: viscous effects dominate over inertia. This is called Stokes flow or creeping flow. The flow is steady and presents fore/aft and top/bottom symmetries.
- $Re = 12$: the flow is no longer dominated by viscous effects and there is a balance between them and inertia. The flow is still top/bottom symmetric but has lost its fore/aft symmetry. Two recirculation rolls have formed at the back of the cylinder that are consequences of the detachment of the boundary layer on the sides of the cylinder due to inertia.
- $Re = 120$: the flow is time-periodic and has lost all its symmetries. The recirculation rolls are advected so strongly that they alternately detach and travel down the wake. This is known as a von Kàrmàn street or von Kàrmàn vortices.
- $Re = 30,000$: the flow is turbulent. The boundary layers are laminar and detach early from the cylinder to yield a large turbulent wake.
- $Re = 500,000$: the flow is turbulent. the boundary layers are turbulent and detach from the cylinder at a point further downstream. The turbulent wake is therefore thinner.

You will find some interesting results from numerical simulation of the flow past a cylinder at: imechanica.org/files/Sato_MDAC_final_2232012.pdf.

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