MATH5453M Foundations of Fluid Dynamics

Lecture 5: Vorticity

5.1 Vorticity: Kundu, p.171–191; Paterson, p.71–83; Tritton, p.81–88

Vorticity is defined as $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. It is a very useful quantity that measures the amount of particular spin or local rotation in the fluid.

Imagine a bucket of uniformly rotating fluid with angular velocity Ω. The velocity writes $\mathbf{u} = \Omega R\hat{\boldsymbol{\phi}}$ in polar coordinates or $\mathbf{u} = (-\Omega y, \Omega x, 0)$ in Cartesian coordinates. The vorticity, $\nabla \times \mathbf{u} = (0, 0, 2\Omega)$, is parallel to the rotation axis and twice its value. So, a uniformly rotating fluid has uniform vorticity equal to twice its angular velocity.

5.2 The Rankine vortex: Paterson, p.80

The Rankine vortex is defined as

$$\begin{cases} \mathbf{u} = \dfrac{\omega_0 R}{2}\hat{\boldsymbol{\phi}}, & \text{for } R < a \\ \mathbf{u} = \dfrac{\omega_0 a^2}{2R}\hat{\boldsymbol{\phi}}, & \text{for } R > a \end{cases}.$$

Since the vorticity for this flow is $(1/R)(\partial/\partial R)(RU_\phi)$, it is equal to ω_0 for $R \leq a$ and vanishes for $R > a$. In other words, this is the flow generated by a patch of vorticity contained within a circle of radius a. If you put a rubber duck in such a flow, it goes round but keeps pointing in the same direction when it is outside the vorticity patch, but starts rotating around itself when it reaches the vorticity region.

To find out what the pressure is, we need to look at the radial component of the Navier–Stokes equation:

$$-\rho\frac{u_\phi^2}{R} = -\frac{\partial p}{\partial R},$$

which gives:

$$\begin{cases} p = p_0 - \dfrac{\rho\omega_0^2 a^4}{8R^2} & \text{for } R > a \\ p = p_0 - \dfrac{\rho\omega_0^2 a^2}{4} + \dfrac{\rho\omega_0^2 R^2}{8} & \text{for } R < a \end{cases}.$$

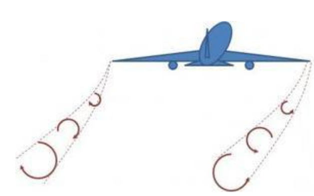
Here, p_0 is the pressure at infinity, i.e., away from the vortex. Note that the pressure inside the vortex is lower than the ambient pressure p_0.

5.3 The line vortex

Since the vorticity inside the Rankine vortex is ω_0 and since it vanishes outside, $\int_A \boldsymbol{\omega}\cdot\hat{\mathbf{n}}\,dS = \pi a^2\omega_0$ for any area A enclosing $R < a$. By using Stokes theorem, we find that the circulation is also $\Gamma = \int_C \mathbf{u}\cdot d\mathbf{l} = \pi a^2\omega_0$. If we let a shrink to zero while keeping the Γ finite, we get a line vortex. Line vortices are easier to handle mathematically, but they are an idealization: any real vortex has a finite core. If we let $\Gamma = \pi a^2\omega_0$, the line vortex has $\mathbf{u} = \Gamma/(2\pi R)\hat{\boldsymbol{\phi}}$ for $R > 0$. The streamfunction for a line vortex is $\psi = -(\Gamma/2\pi)\log R$, where R is the distance from the vortex.

5.4 Aircraft wing vortices

The lift on an aircraft wing is associated with a net circulation around the wing, i.e., with vorticity production parallel to the wing. This vorticity comes off the end of each wing to produce two trailing vortices, which rotate in opposite directions and whose axis trails along a roughly parallel line to the direction of flight. The low pressure in the vortices is associated with low temperature (this is explained later in the thermodynamics section), which brings the air below its dew-point and creates a vapour trail.



5.5 The vorticity equation

Given the identity:

$$\mathbf{u}\times\boldsymbol{\omega} = \nabla\left(\frac{1}{2}\mathbf{u}^2\right) - (\mathbf{u}\cdot\nabla)\mathbf{u},$$

the Navier–Stokes equation can be written

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} - \mathbf{u}\times\boldsymbol{\omega}\right) = -\nabla(p+\frac{1}{2}\rho\mathbf{u}^2) + \mathbf{F} + \mu\nabla^2\mathbf{u}. \tag{1}$$

Assuming ρ is constant, the curl of equation (1) gives:

$$\rho\left(\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla\times(\mathbf{u}\times\boldsymbol{\omega})\right) = \nabla\times\mathbf{F} + \mu\nabla^2\boldsymbol{\omega}.$$

We can also use the identity (provided the flow incompressibility: $\nabla\cdot\mathbf{u}=0$ and $\boldsymbol{\nabla}\cdot\omega=0$):

$$\nabla\times(\mathbf{u}\times\boldsymbol{\omega}) = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\boldsymbol{\omega},$$

to obtain:

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}, \tag{2}$$

where we have assumed that $\nabla\times\mathbf{F}=\mathbf{0}$, as it is the case for the weight. This is the standard form of the vorticity equation for an incompressible fluid. It does not include any vorticity generation mechanism: vorticity is created by external forces or by the boundaries.

5.6 Vortex stretching

Equation (2) is forced by $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$ but what does this term do? Suppose $\boldsymbol{\omega}$ is in the z direction and consider the z component of the vorticity equation, ignoring diffusion temporarily,

$$\frac{D\omega_z}{Dt} = \omega_z\frac{\partial u_z}{\partial z}.$$

If $\partial u_z/\partial z > 0$, then the fluid is stretching in the z-direction, and this stretching acts as a positive growth rate for ω_z. Vorticity is amplified by stretching.

The bath-plug vortex is a classic example of this phenomenon: u_z increases down the plug-hole due to gravity. So $\partial u_z/\partial z > 0$ and any small seed of vorticity in the flow gets amplified.

Here are the vorticity sources in common phenomena

- Bath-plug vortex: any residual vorticity in the fluid that has not dissipated. This could be from the pouring of the fluid into the bathtub, from the removal of an object (or you!) from the bath, etc.
- Smoke ring: vorticity is created as the air flows past your lips. Suppose a stream of air moving at speed U_0 in the x-direction past a flat plate starting at $x=0$, with its normal in the y direction. At the plate surface, the no-slip boundary condition means $u=0$ at $y=0$. As the fluid velocity must reach the free stream velocity U_0 as y increases, there is a strong shear near $x=0$ and the vorticity in the z direction, $-du/dy$ is non-zero. As the fluid moves along the plate, vorticity diffuses and contaminates the rest of the flow. When the fluid gets to the end of the plate (or of your lips), the vorticity that has been created is carried by the fluid (advection) into the room and forms a vortex ring.
- Aircraft wing: the aircraft's surface moving against still air creates the vorticity in a similar manner to the way it is produced in the case of the smoke ring.
- Hurricane: there are various sources of vorticity, in particular the shearing layer that is created when atmospheric current meet. The ambient vorticity is then amplified by vortex stretching and planetary rotation.
- Atmospheric convection: most atmospheric flow (e.g. cloud production) are driven by convection and produce ascending and descending currents. This convection creates rolls in which the flow rotation around a horizontal axis and, thus, vorticity. Vorticity is created at the boundary with the ground but also in a more subtle way: the term $\nabla\times\rho\mathbf{g}$ only vanishes if the density is constant. However, the atmosphere is stratified both horizontally and vertically (hotter fluids are typically less dense). As a result, this term acts as non-zero forcing into the vorticity equation.
- Turbulence: turbulence is full of vorticity. Once vorticity has been created, the $(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$ term can move it anywhere, and can change its direction. This makes three-dimensional turbulence a very difficult problem!

5.7 The Kelvin circulation theorem: Paterson, p.85–88 and p.177–180; Batchelor, p.266–277

We consider an inviscid, incompressible flow for which the body forces derive from a potential V. For C, a closed curve made up of material fluid particles, the Kelvin circulation theorem stupilates that

$$\frac{D}{Dt}\int_C \mathbf{u}\cdot d\mathbf{l} = 0.$$

Proof As the curve C is made of moving fluid particles, $d\mathbf{l}$ changes with time:

$$\frac{D}{Dt}\int_C \mathbf{u}\cdot d\mathbf{l} = \int_C \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{l} + \int_C \mathbf{u}\cdot\frac{D}{Dt}d\mathbf{l}.$$

We also have $Dd\mathbf{l}/Dt = d\mathbf{l}\cdot\nabla\mathbf{u}$, which is not obvious. Suppose that at $t=0$, the vector $d\mathbf{l}$ goes from $\mathbf{x_1}$ to $\mathbf{x_2}$. Then at $t=\delta t$, $\mathbf{x_1}$ has moved to $\mathbf{x_1}+\mathbf{u}(\mathbf{x_1})\delta t$ and $\mathbf{x_2}$ has moved to $\mathbf{x_2}+\mathbf{u}(\mathbf{x_2})\delta t$. The new $d\mathbf{l}$ is the difference between these displacement vectors: $(\mathbf{u}(\mathbf{x_2})-\mathbf{u}(\mathbf{x_1}))\delta t$. By the Taylor theorem, this is $d\mathbf{l}\cdot\nabla\mathbf{u}\delta t$, so that the rate of change of $d\mathbf{l}$ is indeed $d\mathbf{l}\cdot\nabla\mathbf{u}$. We thus have:

$$\begin{aligned}\frac{D}{Dt}\int_C \mathbf{u}\cdot d\mathbf{l} &= \int_C \frac{Du_i}{Dt}dl_i + \int_C u_i dl_j\frac{\partial u_i}{\partial x_j}\\ &= \int_C \frac{Du_i}{Dt}dl_i + \int_C \frac{1}{2}dl_j\frac{\partial u_i^2}{\partial x_j}.\end{aligned}$$

We use the Euler equation for Du_i/Dt:

$$\frac{D}{Dt}\int_C \mathbf{u}\cdot d\mathbf{l} = \int_C -\nabla\frac{p}{\rho}\cdot d\mathbf{l} + \int_C \nabla V\cdot d\mathbf{l} + \int_C \nabla\frac{1}{2}\mathbf{u}^2\cdot d\mathbf{l}.$$

Note that all the gradients integrated around the closed curve C do not contribute, because

$$\int_A^B \nabla V\cdot d\mathbf{l} = V(B)-V(A).$$

The result from removing these terms is the Kelvin circulation theorem.

Using the Stokes therorem, the Kelvin theorem becomes:

$$\frac{D}{Dt}\int_S \boldsymbol{\omega}\cdot\hat{\mathbf{n}}dS = 0,$$

where S is the surface enclosed by C. The amount of vorticity within any material surface remains constant and, if the surface shrinks, the vorticity must increase in proportion. This is a different view point on vortex stretching.

What happens if there is viscosity? The vorticity can diffuse into weaker vorticity regions much like temperature would diffuse along a cold metallic bar.

5.8 Interacting line vortices

By the Kelvin theorem, line vortices move with the fluid. This makes it relatively simple to see how line vortices interact. With finite core vortices, the effect of one vortex on another is to change their shape into something more complicated. Finite core vortex interactions is an active research area.

5.8.1 The vortex pair: Paterson, p.90–91

Consider vortex lines of opposite vorticity in the z direction and located at $x=0$ and $y=\pm a$. These vortices will move in the x direction, under each other's influence so it is easier change referential: we keep the vortices fixed and let the fluid to move sideways at the corresponding speed instead. The resulting streamfunction is:

$$\psi = Uy + \frac{\kappa}{2\pi}\ln\left[x^2+(y-a)^2\right]^{1/2} - \frac{\kappa}{2\pi}\ln\left[x^2+(y+a)^2\right]^{1/2},$$

which is found by adding the streamfunctions due to the vortex pair to the streamfunction for the uniform flow. Since $u=\partial\psi/\partial y$,

$$u = U + \frac{\kappa(y-a)}{2\pi(x^2+(y-a)^2)} - \frac{\kappa(y+a)}{2\pi(x^2+(y+a)^2)}.$$

The movement of the upper vortex at $(0, a)$ is due to the motion induced by the lower vortex and the uniform stream, $u = U - \kappa/4\pi a$, so to keep it at rest, we must have $U = \kappa/4\pi a$. This value for U keeps the upper vortex still as well. So now moving to the frame in which the fluid is at rest and the vortices move, the pair moves with speed $\kappa/4\pi a$. Notice that the further apart the vortices, the slower they move. It is easy to show that the y velocity of both vortices is zero, so they stay at the same distance away from each other.

5.9 The vortex ring: Lamb, p.236–243

It is also possible to study the axisymmetric vortex ring, where the vorticity is concentrated into a thin circular structure. Small vortex rings move faster than large ones, so that it is possible to make a small vortex ring pass through a larger one. As this happens, their interaction makes the small vortex ring grow and slow down while the large one contracts and speeds up. In a well-controlled setting, it is possible to see vortex rings pass through each other several times, as shown in `www.youtube.co.uk/watch?v=mHyTOcfF99o`. Vortex rings are a phenomenon which dolphins are known to play with.

Dr. Cédric Beaume – c.m.l.beaume@leeds.ac.uk