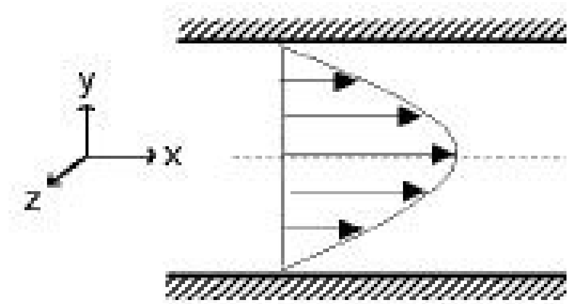
# MATH5453M Foundations of Fluid Dynamics

Lecture 4: Some exact solutions of the Navier–Stokes equation

## 4.1 Plane Poiseuille flow

Let us consider the steady flow along an infinite channel driven by a pressure gradient. We define the following Cartesian coordinates: $x$ is the streamwise direction and $y$ is the wall-normal direction bounded by $y = \pm h$, and $z$ is the spanwise direction.




The fluid velocity, $\mathbf{u}$ satisfies the no-slip boundary condition at the walls, so that $\mathbf{u} = \mathbf{0}$ at $y = \pm h$. We further assume that the flow is one-dimensional and only varies in the wall-normal direction:

$$\mathbf{u} = (u(y), 0, 0).$$

Note that this automatically satisfies $\nabla \cdot \mathbf{u} = 0$. Furthermore, $(\mathbf{u} \cdot \nabla)\mathbf{u} = 0$ and, since the flow is steady, $\partial \mathbf{u}/\partial t = 0$. Thus, the Navier–Stokes equation reduces to:

$$\begin{cases} 0 = -\dfrac{\partial p}{\partial x} + \mu \dfrac{d^2 u}{dy^2} \\ 0 = -\dfrac{\partial p}{\partial y} \\ 0 = -\dfrac{\partial p}{\partial z} \end{cases}$$

The dynamic pressure, $p$, is therefore a function of $x$ alone, so the $x$-component of the Navier–Stokes equation implies that

$$\begin{aligned} \mu \frac{d^2 u}{dy^2} &= \frac{\partial p}{\partial x} \\ &= -G \end{aligned}$$

for some constant $G$, which is the pressure gradient that drives the flow. Integrating the equation for $u$ and applying the boundary conditions at $y = \pm h$, we obtain the solution

$$u(y) = \frac{G}{2\mu}\left(h^2 - y^2\right),$$

where the pressure in the channel is given by $p(x) = p_0 - Gx$.

The fluid exerts a viscous force on the boundaries at $y = \pm h$. On the top boundary, the unit normal is $\hat{\mathbf{y}}$ and the viscous force is in the $\hat{\mathbf{x}}$ direction. The viscous force acting on the fluid is thus $F_f = \tau_{xy} = 2\mu E_{xy} = \mu \partial u/\partial y = -Gh$. As a result, the force exerted by the fluid onto the top wall is $F_w = Gh$. Due to the opposite direction of the outward-pointing normal at the bottom wall, the viscous force on that wall is equal to that acting on the top wall.

## 4.2 Hagen–Poiseuille flow

We consider pipe flow which we describe using the cylindrical polar coordinates $(R, \phi, z)$, with $z$ in the streamwise direction and $R = a$ the pipe wall. We assume that the flow takes the simple form $(0, 0, w(R))$, where $w$ is a function of $R$ to be determined. The flow is steady, and we can easily verify that $(\mathbf{u} \cdot \nabla)\mathbf{u} = 0$, so there is no acceleration.

The $R$ and $\phi$ components of the Navier–Stokes equation imply that $p$ is a function of $z$ only and:

$$\frac{\partial p}{\partial z} = \mu \frac{1}{R}\frac{d}{dR}\left(R\frac{dw}{dR}\right) = -G, \quad \text{with } G = \text{ cst.}$$

It follows that the general solution is

$$w = -\frac{G}{4\mu}R^2 + B\ln R + C.$$

To keep the solution physical at $R = 0$, we impose $B = 0$. Furthermore, $C = Ga^2/4\mu$ to satisfy the no-slip boundary condition $w = 0$ at $R = a$. The solution is

$$w = \tfrac{G}{4\mu}(a^2 - R^2),$$
$$p = p_0 - Gz.$$

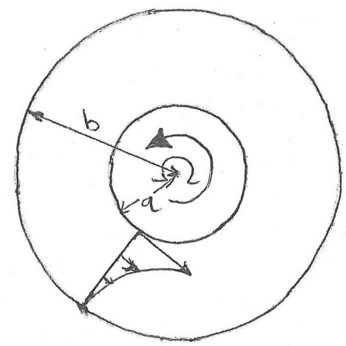
The flow rate in thus:

$$\int_0^a 2\pi R w(R)\, dR = \frac{\pi G a^4}{8\mu}.$$

This formula works well for slow/viscous flows, but if the flow regime changes to become turbulent, for example, then this result overestimates the actual flow rate.

## 4.3 Taylor–Couette flow

Taylor–Couette flow is the flow between two concentric rotating cylinders of outer radius $R = b$ and inner cylinder radius $R = a$. In general, both cylinders can rotate but we look here at the simplest configuration, where the outer cylinder is at rest and the inner cylinder rotates at angular velocity $\Omega$. We use cylindrical polar coordinates, with $z$ being the streamwise direction and assume that $\mathbf{u} = U(R)\hat{\boldsymbol{\phi}}$, so that there is no streamwise flow.




The Navier–Stokes equation has axisymmetric solutions if $p$ is independent of $\phi$ and $z$. The projections of the Navier–Stokes equation onto the $R$ and $\phi$ directions lead to:

$$-\rho\frac{U^2}{R} = -\frac{\partial p}{\partial R}, \tag{1}$$
$$\nabla^2 U - \frac{U}{R^2} = 0. \tag{2}$$

Expressing the Laplacian in cylindrical coordinates:

$$\nabla^2 U - \frac{U}{R^2} = 0$$
$$\Rightarrow \qquad \frac{1}{R}\frac{d}{dR}\left(R\frac{dU}{dR}\right) - \frac{U}{R^2} = 0.$$

This equation is a Cauchy equation, so we look for a solution in the form $U = R^n$. The general solution is

$$U = AR + \frac{B}{R}.$$

Applying the boundary conditions ($U = 0$ at $R = b$ and $U = \Omega a$ at $R = a$), we finally obtain:

$$U = \frac{\Omega a^2 b^2}{R(b^2 - a^2)} - \frac{\Omega a^2 R}{b^2 - a^2}.$$

We can obtain the pressure from equation (1). The surface force exerted by the outer cylinder onto the fluid is:

$$\begin{aligned}\tau_{R\phi} &= \mu\left(\frac{dU}{dR} - \frac{U}{R}\right)\\ &= -\frac{2\mu B}{R^2}\\ &= -\frac{2\mu\Omega a^2 b^2}{R^2(b^2 - a^2)}.\end{aligned}$$

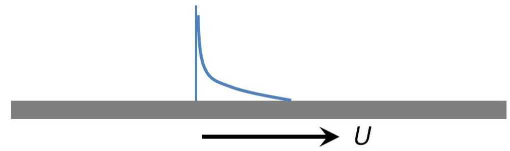
Hence, the torque per unit length in the $z$ direction is

$$T = \frac{4\pi\mu\Omega a^2 b^2}{(b^2 - a^2)}.$$

This gives a convenient method to measure viscosity (the Couette viscometer).

## 4.4 Stokes first problem: Acheson, p.35–38

Consider the flow above a solid wall at $y = 0$. Initially, the fluid is at rest but, at time $t = 0$, the wall starts moving at velocity $U$ in the $x$ direction.
We assume that flow is one-dimensional and independent of the streamwise and spanwise directions so that $\mathbf{u} = (u(y,t), 0, 0)$. Since this flow is exclusively driven by the motion of the boundary, we can assume that $\partial p/\partial x = 0$.




The Navier–Stokes equation reduces to

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2}, \tag{3}$$

together with: the boundary conditions, $u = U$ on $y = 0$ and $U \to 0$ as $y \to \infty$; and the initial condition, $u = 0$ at $t = 0$.

The streamwise velocity hence satisfies the diffusion equation with diffusivity $\nu = \mu/\rho$, where $\nu$ is called the kinematic viscosity. This problem is equivalent to that of finding the temperature distribution in a semi-infinite bar, when the temperature of one end is suddenly increased to a different constant. There exists a number of different methods for solving diffusion problems but, in this case, we shall find the solution by seeking a *similarity solution*. We observe that the equation and boundary conditions are conserved under the transformation $y \mapsto ay$, $t \mapsto a^2 t$, which suggests that solutions exist of the form

$$u(y,t) = f(\eta), \text{ where } \eta = yt^{-1/2}.$$

Substituting into the equation (3), we obtain

$$\frac{d^2 f}{d\eta^2} + \frac{\eta}{2\nu}\frac{df}{d\eta} = 0.$$

Substituting $v = df/d\eta$, this equation becomes

$$\frac{dv}{d\eta} + \frac{\eta}{2\nu} v = 0$$

$$\Rightarrow \qquad v = A \exp\left(-\frac{\eta^2}{4\nu}\right).$$

Integrating again, we get

$$f = A \int_0^\eta \exp\left(-\frac{\eta^2}{4\nu}\right) d\eta + B.$$

The above integral can be expressed in terms of the error function:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-y^2\right) dy.$$

Substituting $y = \eta/(2\sqrt{\nu})$, we have

$$f = A\sqrt{\nu\pi}\operatorname{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) + B,$$

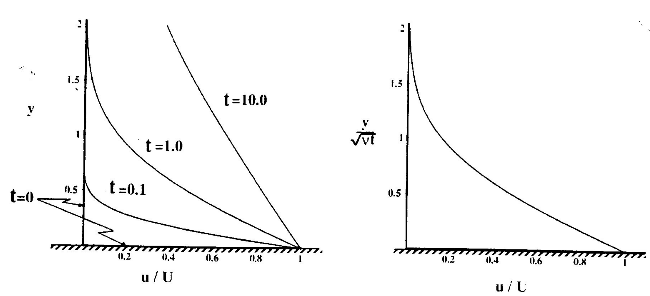
so that:

$$u(y,t) = A\sqrt{\nu\pi}\operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) + B.$$

In order to satisfy the boundary conditions on $y = 0$, $B = U$ and, since $\operatorname{erf}(x) \to 1$ as $x \to \infty$, $A\sqrt{\nu\pi} = -U$:

$$u(y,t) = U\left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right]. \tag{4}$$

Finally, we need to check that $u(y,0) = 0$ for all $y > 0$, so that the initial condition is verified. This condition holds because the error function tends to 1 when its argument tends to 0.




The velocity is approximately zero wherever $y/(2\sqrt{\nu t})$ is large. So, for a fixed value of $y$, the velocity remains lower than $0.01U$ until a time $t$ such that $y \approx 4\sqrt{\nu t}$. Hence, at time $t$, the fluid is only moving within a narrow region of height

$4\sqrt{\nu t}$. This narrow region is called the *viscous boundary layer*. Note that the boundary layer thickness is independent of $U$.

For example, let us consider water ($\nu \approx 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$) in the vicinity of an suddenly moved boundary. After 1s, the boundary layer thickness is around 4mm and, after 100s, it is still only 4cm thick. The effects of the boundary are only felt in this narrow boundary layer.

Dr. Cédric Beaume – c.m.l.beaume@leeds.ac.uk