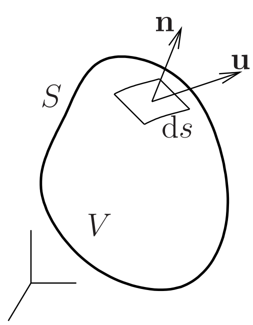
MATH5453M Foundations of Fluid Dynamics

Lecture 3: The Navier–Stokes equation

3.1 The equation of motion: Kundu, p.111–115; Paterson, chapter 8; Acheson, p.207–214

The equation of motion of a fluid is derived in a similar way to the continuity equation but, instead of considering the density of fluid, we look at its momentum.

Consider a fixed volume V with surface S and outward-pointing normal $\mathbf{n}$.



The total momentum in V is

$$\mathbf{q} = \int_V \rho \mathbf{u} \, \mathrm{d}V,$$

where ρ is the density and $\mathbf{u}$ is the velocity vector. The rate of change of momentum in the volume is:

$$\frac{\partial}{\partial t} \int_V \rho \mathbf{u} \, \mathrm{d}V = \int_V \frac{\partial}{\partial t} (\rho \mathbf{u}) \, \mathrm{d}V,$$

since the volume V is fixed in space.

Momentum can change is a variety of ways: (a) it can flow in or out of V, (b) surface forces acting on V, such as the ones generated by pressure and friction and (c) body forces acting on the fluid inside the volume, such as weight, can modify the momentum.

(a) Net momentum flux into the box:

$$-\int_S \rho \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{n}}) \, dS.$$

(b) Surface forces are given by the stress-tensor τ_{ij}. In the absence of viscosity, only normal stress is present in the form of a pressure. In the presence of viscosity, the total surface force is given by

$$\mathbf{f} = \int_S \tau_{ij} \hat{n}_j \, dS.$$

We need here a physical law to relate the stress tensor τ_{ij} to the velocity.

(c) In the absence of other external forces than the weight of fluid, the total body force acting on the fluid element is:

$$\int_V \mathbf{F} \, dv = \int_V \rho \mathbf{g} \, dv,$$

where $\mathbf{F}$ is the force per unit volume.

We use the divergence theorem to turn the surface integrals into volume integrals:

$$-\int_S \rho \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{n}}) \, dS = \int_V -\tfrac{\partial}{\partial x_j} (\rho u_i u_j) \, dv,$$
$$\int_S \tau_{ij} \hat{n}_j \, dS = \int_V \tfrac{\partial}{\partial x_j} \tau_{ij} \, dv.$$

By putting together all the contributions to the momentum budget and by dropping the volume integrals (since the resulting equation has to be valid for any volume), we obtain:

$$\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial}{\partial x_j} \tau_{ij} + F_i.$$

We can simplify this using the mass conservation equation:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial}{\partial x_j} \tau_{ij} + F_i \tag{1}$$

$$\Rightarrow \qquad \rho \frac{\partial u_i}{\partial t} + u_i \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) \right) + \rho u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j} \tau_{ij} + F_i \tag{2}$$

$$\Rightarrow \qquad \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j} \tau_{ij} + F_i, \tag{3}$$

which is the equation of motion in a form valid for both compressible and incompressible flows.

3.2 The constitutive equation

Equation (3) applies to all continuous materials, but it is insufficient to predict motion, since we need an additional equation linking the stress tensor to the deformation of the fluid. This relation is called the constitutive equation.

3.2.1 Ideal Fluid

The simplest constitutive equation for a fluid is that of an ideal or inviscid fluid, for which the only surface force is an isotopic pressure arising from the random thermal motion of the fluid molecules. This force acts along the direction of the normal $\mathbf{n}$ so that

$$\mathbf{f} = -P\hat{\mathbf{n}}.$$

Hence for an ideal fluid $n_j\tau_{ij} = -Pn_i$, so that $\tau_{ij} = -P\delta_{ij}$. Substituting this into the momentum transport equation, we obtain the Euler equation:

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \rho\mathbf{g} - \nabla P. \tag{4}$$

In applications where the effects of viscosity are small, it may be reasonable to neglect viscosity and treat the fluid as an ideal fluid. Whether this approximation is a good one is a tricky issue but it has proven to be reliable in some geophysical and astrophysical settings.

3.2.2 Newtonian Fluid: Kundu, p.111–114; Batchelor, p.141–147

Newton suggested the stress might be linearly related to the rate of shearing of a fluid. However, it was until over century later that Stokes gave a precise definition for a general three-dimensional flow. Since the gradient of velocity describes both the strain rate and the spin of the fluid, Stokes introduced the rate of strain tensor:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

which is the symmetric part of the velocity gradient, or shear tensor: $\partial u_i/\partial x_j$. Stokes proposed that

$$\tau_{ij} = -P\delta_{ij} + 2\mu\left(E_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right), \tag{5}$$

where μ is the dynamic viscosity and has SI units kg·m^{-1}·s^{-1} or Pa·s. Sometimes the dynamic viscosity is expressed in poise (0.1 Pa·s). For example, water has viscosity 0.001 Pa·s at room temperature or 1 centipoise. Any fluid that satisfies equation (5) is called a Newtonian fluid. Fluids composed of small molecules including air and water usually satisfy (5).

The Newtonian constitutive law is based on the assumptions that the stress is linearly dependent on the shear and has no preferred direction, i.e., it is isotropic. The Stokes formula (5) is not quite the most general possible law satisfying these constraints, and it was later shown that, for compressible fluids, it is possible to add in another term called the bulk viscosity μ_v and form the more general law:

$$\tau_{ij} = -P\delta_{ij} + 2\mu(\sigma_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}) + \mu_v\delta_{ij}\frac{\partial u_k}{\partial x_k}.$$

The additional bulk viscosity term is usually small and equation (5) remains widely used, even for compressible fluids.

In the case of an incompressible fluid of constant viscosity,

$$\frac{\partial}{\partial x_j}\tau_{ij} = -\frac{\partial P}{\partial x_i} + \mu\nabla^2 u_i,$$

so the Navier–Stokes equation for an incompressible flow reads:

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + F_i + \mu\nabla^2 u_i. \tag{6}$$

This equation is more commonly encountered in its vectorial form:

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla P + \mathbf{F} + \mu\nabla^2\mathbf{u},$$

and is perhaps the most fundamental equation in fluid dynamics and the starting point of most fluids research.

3.2.3 Non-Newtonian fluids

Fluids that do not satisfy equation (5) are described as being non-Newtonian. A great range of fluids fall into this category, including fluids containing polymers, multiphase fluids, such as colloids and foams, and many biological fluids. Non-Newtonian fluid dynamics is an active research area, but out of the scope of this lecture.

3.3 Hydrostatic and Dynamic Pressure

In the absence of flow and where the only external force acting on the fluid is gravity, the Navier–Stokes equation reduces to a balance between the gravitational force and the pressure:

$$-\nabla P + \rho \mathbf{g} = 0.$$

The resulting pressure solution,

$$P_H = P_0 + \rho \mathbf{g} \cdot \mathbf{x},$$

is referred to as hydrostatic pressure. In flows with free-surfaces, such as rivers or ocean waves, weight is often one of the driving forces. However in flows without free-surfaces the weight is simply balanced by the hydrostatic pressure. As a consequence it is often useful to subtract off the hydrostatic pressure by writing the pressure in the form,

$$P = P_H + p \tag{7}$$

where p is referred to as the *dynamic pressure*. This reduces the Navier–Stokes equation to

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \mathbf{u}. \tag{8}$$

3.4 Boundary conditions

To solve the equations for a fluid flow, we need to know what boundary conditions to apply on the bounding surfaces. In general, both the velocity and forces must be continuous at a fluid boundary, however, how these are applied depends upon the nature of the boundary.

3.4.1 Solid boundaries

Where a viscous fluid is in contact with a solid surface moving at velocity $\mathbf{U}$, friction between the solid surface and the fluid prevent the latter to slip along the former. As a result, the velocity of the fluid, $\mathbf{u}$, must be equal to the velocity of the solid surface on the boundary:

$$\mathbf{u} = \mathbf{U}.$$

As well as matching the velocity, we also have a boundary condition for the stress. The surface force density $\mathbf{f}$ applied by the boundary on the fluid is equal to

$$\mathbf{f} = \hat{\mathbf{n}} \cdot \tau \tag{9}$$

where $\hat{\mathbf{n}}$ is the unit outward-pointing normal vector, directed out of the fluid. By Newton's third law, the fluid imposes an equal and opposite force density on the solid boundary, so that additional boundary condition is automatically satisfied.

3.4.2 Free surfaces

Where a fluid is in contact with air (or a fluid of much lower viscosity), the only force exerted by the air on the fluid is due to the pressure P_{atm}. In the absence of surface forces, such as surface tension, the force applied by the air onto the fluid is $-P_{\mathrm{atm}}\hat{\mathbf{n}}$. The force balance implies that:

$$\hat{\mathbf{n}} \cdot \tau = -P_{\mathrm{atm}}\hat{\mathbf{n}}. \tag{10}$$

Consequently, $\mathbf{n} \cdot \sigma$ must be parallel to $\mathbf{n}$ and there is no force parallel to the surface. Free surfaces (unlike a solid surface) cannot support shear.

We still require one additional boundary condition. If the position of the surface is given by $\eta(\mathbf{x}, t) = 0$, then, since all points on the surface must remain on the surface, we can impose:

$$\frac{D\eta}{Dt} = 0.$$

This is often called a kinematic constraint. In particular, if the surface remains fixed in time, then:

$$\mathbf{u} \cdot \nabla \eta = 0,$$

where $\nabla \eta = \mathbf{n}$ is the normal to the surface, implying that the normal component of the velocity must vanish:

$$\mathbf{u} \cdot \hat{\mathbf{n}} = 0.$$

3.4.3 Boundary between immiscible fluids

At a boundary between two fluids of different viscosities, both the velocity $\mathbf{u}$ and force density $\hat{\mathbf{n}} \cdot \tau$ must be continuous. Furthermore, the surface between the fluids must satisfy the kinematic boundary condition.

3.4.4 Boundary conditions for the Euler equation

For an ideal fluid, the absence of viscosity implies that the fluid cannot exert any tangential stress. This results in the impossibility to apply some of the boundary conditions expressed above. This is a consequence of the fact that the Navier–Stokes equation is of higher differential order than the Euler equation and, therefore, requires more boundary conditions.

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