MATH5453M Foundations of Fluid Dynamics

Lecture 22: Computational models for turbulent flows

P. A. Davidson: Turbulence, an introduction for scientists and engineers, OUP 2004.
S. B. Pope: Turbulent flows, CUP 2000.

There exists a multitude of computational models for turbulent flows. Most belong to three families: the Reynolds-averaged Navier–Stokes (RANS) methods solve for the large scale mean flow using an ad hoc form for the Reynolds stress; large eddy simulations (LES), as the name suggests, simulate both the mean flow and the larger turbulent eddies, but do not resolve down to the Kolmogorov length-scale but rely on a model for the small scales; and direct numerical simulation (DNS) calculates a single realisation of the full velocity field from the unaveraged Navier–Stokes equations, resolving them until the finest scale. The latter is by far the most computationally expensive method, requiring long runs on supercomputers and being therefore limited to moderate Reynolds numbers. Schemes of the RANS family are comparatively inexpensive to run but their accuracy depends critically on the performance of the Reynolds stress model. Large eddy simulations represent a compromise between the two other familites that seeks an accuracy improvement compared to RANS models but at a fraction of the cost of full DNS.

22.1 RANS models

As derived previously, the RANS equation reads:

$$\rho\left(\frac{\partial\bar{\mathbf{u}}}{\partial t}+(\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}}\right)=-\nabla\bar{p}+\frac{\partial}{\partial x_j}(\bar{\tau}_{ij}+\tau^R_{ij}),$$

where $\bar{\tau}_{ij}=2\rho\nu\bar{E}_{ij}$, and $\bar{E}_{ij}=1/2\,(\partial\bar{u}_i/\partial x_j+\partial\bar{u}_j/\partial x_i)$. This equation requires a model for the Reynolds stress τ^R_{ij} derived from the form of the averaged velocity field $\bar{\mathbf{u}}$. The simplest (and most popular) model introduces an eddy viscosity, ν_T, and the Boussinesq equation:

$$\tau^R_{ij}=2\rho\nu_T\bar{E}_{ij}-\frac{\rho}{3}\langle u'_k u'_k\rangle\delta_{ij}.$$

22.2 Mixing length models

The earliest and simplest model for the eddy viscosity, ν_T, was originally suggested by Prandtl from an analogy with molecular viscosity. The molecular viscosity, ν, is proportional to the product of the molecular speed and mean-free path, thus one might expect that:

$$\nu_t\sim|\mathbf{u}'|l_m, \tag{1}$$

where $|\mathbf{u}'|$ is the speed of the smallest eddies and l_m is the *mixing length*, i.e., the effective distance over which the eddies move. He further reasoned (through a rather questionable argument) that in a shear flow

$$|\mathbf{u}'|\sim l_m\dot{\gamma},$$

where $\dot{\gamma}$ is the shear-rate of the mean-flow, so that

$$\nu_T=l_m^2\dot{\gamma}.$$

This can be generalised to other three-dimensional flows using the Smagorinsky law:

$$\dot{\gamma}=\left(2\bar{E}_{ij}\bar{E}_{ij}\right)^{1/2},$$

or the Baldwin and Lomax law:

$$\dot{\gamma}=\left(2\bar{\Omega}_{ij}\bar{\Omega}_{ij}\right)^{1/2}.$$

This model is not complete as the mixing length, l_m, has not been specified. However, for certain geometries, such as turbulent boundary layers in aeronautics, there are well-established specifications for l_m in terms of distance from walls, pressure gradients, etc. Another issue with this model is that Reynolds stress is determined only from the local instantaneous velocity gradient and takes no account of history effects due to the transport of turbulent energy. As a consequence this model is of limited use for general flows.

22.3 Kinetic energy models

One weakness of the mixing length model lies in the approximation made for $|\mathbf{u}'|$. Both Kolmogorov and Prandtl independently suggested that a better way to estimate this quantity is from the turbulent kinetic energy density per unit mass:

$$k(\mathbf{x},t) = \frac{1}{2}\langle u_i' u_i' \rangle.$$

Proceeding as we did for the Reynolds stress, we can find an evolution equation for k from the fluctuating part of the Navier–Stokes equation:

$$\frac{\bar{D}}{Dt}\left[\frac{1}{2}\langle u_i' u_i' \rangle\right] = \frac{1}{\rho}\tau_{ij}^R \bar{E}_{ij} - 2\nu\langle E_{ij}' E_{ij}' \rangle - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\langle u_i' u_j' u_j' \rangle + \frac{1}{\rho}\langle p' u_i' \rangle - 2\nu\langle u_j' E_{ij}' \rangle\right).$$

This, of course, has the same closure problem as the Reynolds stress. However, we can identify the second term on the right-hand-side, $\epsilon(\mathbf{x},t) = 2\nu\langle E_{ij}' E_{ij}' \rangle$ as the rate of dissipation by the turbulent eddies (per unit mass) and the final term as a flux of kinetic energy:

$$T_i = \frac{1}{2}\langle u_i' u_j' u_j' \rangle + \frac{1}{\rho}\langle p' u_i' \rangle - 2\nu\langle u_j' E_{ij}' \rangle.$$

By analogy with the Fourier law for the thermal energy, $\mathbf{T}$ can be approximated as a diffusive flux:

$$\mathbf{T} = -(\nu_T + \nu)\nabla k.$$

The dissipation ϵ is estimated from the energy generation rate U^3/L as

$$\epsilon = C_D \frac{k^{3/2}}{l_m}, \tag{2}$$

so that

$$\frac{\bar{D}k}{\bar{D}t} = \nabla\cdot(\nu + \nu_T)\nabla k + G - \epsilon, \tag{3}$$

where

$$G = \frac{\tau_{ij}^R \bar{E}_{ij}}{\rho}, \tag{4}$$

$$\tau_{ij}^R = 2\rho\nu_T \bar{E}_{ij} - \frac{2\rho}{3}k\delta_{ij}. \tag{5}$$

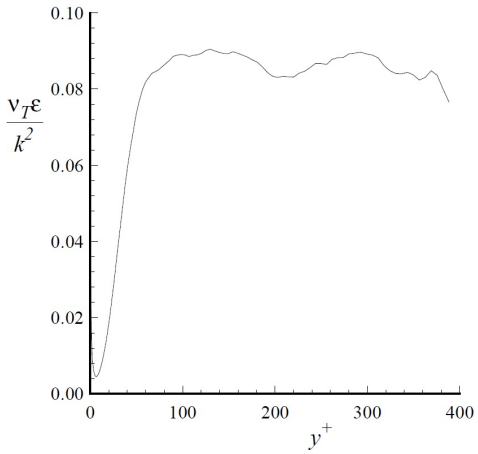
Whilst this model does now include the effects of flow history it is still incomplete.

22.4 The $k-\epsilon$ model

The models introduced so far are incomplete in the sense that we need to define a mixing length l_m in an ad hoc way. If, instead of regarding equation (2) as a definition for ϵ, we viewed it as a definition of the mixing length in terms of ϵ, we can define the eddy viscosity as:

$$\nu_T = C_\mu \frac{k^2}{\epsilon}, \tag{6}$$

for some constant C_μ. In fact, direct numerical simulations show that $\nu_T\epsilon/k^2$ is approximately constant, except in the region close to the wall.



We require a second transport equation for ϵ. Whilst the k transport equation (3) can be justified as a closure approximation to the exact equation, the equation for ϵ is entirely empirical and usually taken to be:

$$\frac{\bar{D}\epsilon}{\bar{D}t} = \nabla\cdot(\nu + \frac{\nu_T}{\sigma_\epsilon})\nabla\epsilon + C_1\frac{G\epsilon}{k} - C_2\frac{\epsilon^2}{k}. \tag{7}$$

The constants σ_ϵ, C_1 and C_2 are determined by comparing with experiments and DNS.

The $k-\epsilon$ model is arguably the simplest complete turbulence model and is widely used in engineering applications. The full problem reads:

$$\nabla \cdot \bar{\mathbf{u}} = 0, \tag{8}$$

$$\rho\left(\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}}\right) = -\nabla \bar{p} + \frac{\partial}{\partial x_j}(\bar{\tau}_{ij} + \tau^R_{ij}), \tag{9}$$

where:

$$\bar{\tau}_{ij} = 2\rho\nu\bar{E}_{ij}, \tag{10}$$

$$\bar{E}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right), \tag{11}$$

$$\tau^R_{ij} = 2\rho\nu_T\bar{S}_{ij} - \frac{2\rho}{3}k\delta_{ij}, \tag{12}$$

$$\nu_T = C_\mu\frac{k^2}{\epsilon}, \tag{13}$$

together with:

$$\frac{\partial k}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)k = \nabla \cdot (\nu + \nu_T)\nabla k + G - \epsilon, \tag{14}$$

where $G = \tau^R_{ij}\bar{E}_{ij}/\rho$, and:

$$\frac{\partial \epsilon}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)\epsilon = \nabla \cdot \left[(\nu + \frac{\nu_T}{\sigma_\epsilon})\right]\nabla\epsilon + C_1\frac{G\epsilon}{k} - C_2\frac{\epsilon^2}{k}. \tag{15}$$

The model constants are usually

$$C_\mu = 0.09 \quad \sigma_\epsilon = 1.3, \quad C_1 = 1.44, \quad C_2 = 1.92.$$

These values are determined so that the $k-\epsilon$ model gives reasonably accurate answers in for a range of flow geometries. However, it is can be wildly inaccurate in some cases such as in regions close to solid boundaries.

Since a complete turbulence model only requires two quantities of different dimensions, other two-variable eddy-viscosity models have been proposed. For example, the $k-\omega$ model replaces the equation for ϵ with one for ω, the dissipation rate. The resulting model is, in particular, better at handling flows at lower Reynolds numbers and near solid boundaries.

There are also other, more complex, RANS models that model the Reynolds stress anisotropically and which require more fitting parameters. A recent development in RANS models is the use of machine learning from DNS to develop Reynolds stress models.

22.5 Large eddy simulation models

Large eddy simulations can be regarded as a middle way between RANS and full DNS. The idea is to simulate the larger scales, which contain most of the energy, but not resolve the finer scale structures responsible for viscous dissipation. As with the RANS model, we separate the velocity field: $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$, but, here, we use a spatial filter to average over the small scales:

$$\bar{\mathbf{u}}(\mathbf{x}, t) = \int G(\mathbf{x}, \mathbf{x}')\mathbf{u}(\mathbf{x}', t)\, d^3\mathbf{x}', \tag{16}$$

where $G(\mathbf{x}, \mathbf{x}')$ is a filter function with the property that

$$\int G(\mathbf{x}, \mathbf{x}')\, d^3\mathbf{x}' = 1.$$

Popular choices include the Gaussian filter:

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{\pi^{3/2}L^3}\exp\left[-\frac{(\mathbf{x} - \mathbf{x}')^2}{L^2}\right],$$

which smooths out eddies with length-scales less than L, and the box filter:

$$G(\mathbf{x}, \mathbf{x}') = 1/L^3,$$

for points $\mathbf{x}'$ within a cube of volume L^3 centred around $\mathbf{x}$. The filtered variables contain all the eddies with length scales bigger than the cut-off scale L. Therefore, the resolution of the numerical scheme must be chosen so that we are

able to resolve scales down to L. LES is computationally intensive, as we want to resolve down to the smallest eddy size we can manage.

The decomposition into filtered $\bar{\mathbf{u}}$ and residual fields $\mathbf{u}'$ appears to be similar to the averaged and perturbed quantities in the RANS equations, however, unlike for the RANS decomposition, $\overline{\mathbf{u}'} \neq 0$ here. Nevertheless, filtering the Navier–Stokes equation results in an equation with the same terms as its Reynolds-averaged counterpart:

$$\rho\left(\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}}\right) = -\nabla \bar{p} + \rho\nu\nabla^2\bar{\mathbf{u}} + \nabla \cdot \tau^R, \tag{17}$$

except that the definition of τ_{ij}^R, the residual stress, is slightly different:

$$\tau_{ij}^R = \rho\left(\bar{u}_i\bar{u}_j - \overline{u_i u_j}\right). \tag{18}$$

In practice, though, this makes no difference since we usually model this term with an eddy viscosity:

$$\tau_{ij}^R = 2\rho\nu_T\bar{E}_{ij} + \frac{1}{3}\delta_{ij}\tau_{kk}^R. \tag{19}$$

We still need to specify ν_T, but this is less important than in RANS as this concerns only the scales with the least amount of energy. Consequently, it is common to use a simple mixing length model such as the Smagorinsky model with L as the mixing length:

$$\nu_T = C_s^2 L^2 \left(2\bar{E}_{ij}\bar{E}_{ij}\right)^{1/2}. \tag{20}$$

Here, C_s is the Smagorinsky constant and $C_s = 0.1$ is a popular choice. Note that ν_T is small for small L, which is fine since we only need a small eddy viscosity to damp down small eddies. Moreover, the isotropy assumption is not problematic since it tends to hold well on small scales (at least away from boundaries), while the anisotropy of the larger scales is handled by the resolved modes. The downside of LES is its computational cost: even with filtering, the flows are unsteady and three-dimesional. LES is usually performed on high performance parallel computing facilities. The LES model does not work so well near boundaries, so some codes introduce ad hoc modifications to ν_T near boundaries.

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