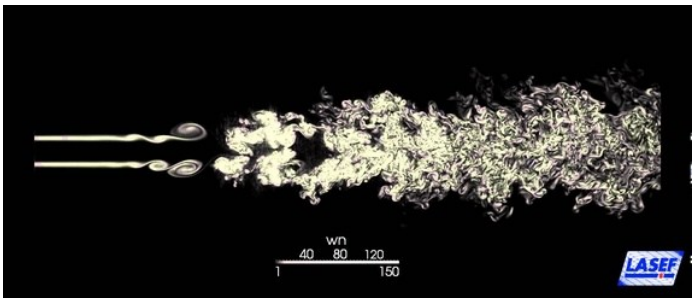
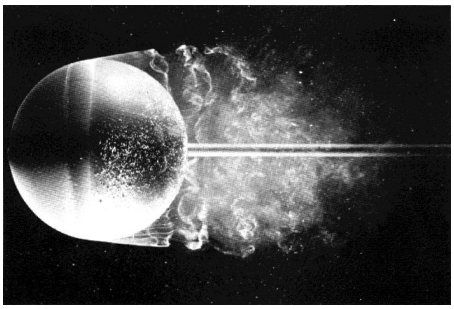
MATH5453M Foundations of Fluid Dynamics

Lecture 21. Introduction to turbulence modelling

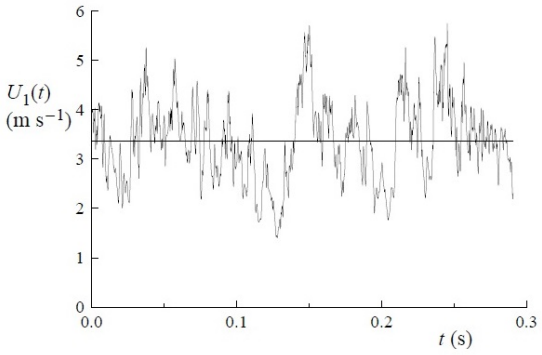
P. A. Davidson:- Turbulence, an introduction for scientists and engineers, OUP 2004. S. B. Pope:- Turbulent flows, CUP 2000.

Turbulence is ubiquitous in high Reynolds number flows. It is characterised by unsteady, seemingly unpredictable motion and a wide continuous range of lengthscales. Here, we will introduce basic ideas used to model turbulent flows in applications.





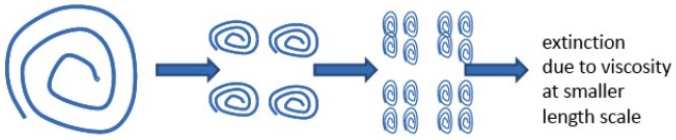
In a turbulent flow, the fluid velocity $\mathbf{u}(\mathbf{x},t)$ fluctuates rapidly in both time and space. Moreover, the system is chaotic and two seemingly identical experiments will not produce the same flow field. Consequently, it might be more useful to view $\mathbf{u}(\mathbf{x},t)$ as a random variable, rather than a deterministic quantity, and to discuss its averaged properties (in a similar way that $\mathbf{u}(\mathbf{x},t)$ is itself an average over the random motion of individual fluid molecules). In particular, we are mainly interested in the large scale fluid motion occuring on the lengthscale of the flow geometry. The small scales are only interesting in the way they affect the large scale motion.



20.1 Scales of turbulence

Due to nonlinearity, energy cascades down from large scale flows to small eddies. As it cascades down, the rate of energy dissipation due to viscous effects increases until no energy is left at a cutoff wavelength. L.F. Richardson summed the process up (borrowing from Swift's original poem):

'Large whirls have smaller whirls which feed on their vorticity, small whirls have lesser whirls and so on until viscosity'



In a turbulent flow, the rate of viscous dissipation does not depend on the value of the viscosity. If the viscosity is lowered, the turbulent cascade goes on to smaller lengthscales, and all the energy gets dissipated at these very small scales. If you look at a photograph of a bonfire, and one of a large forest fire, the flames do not look different once the scale is put aside. Looking more closely, you can see that the forest fire has a much greater range of scales than the bonfire.

We can estimate the smallest scales in a turbulent flow from a scale analysis. Let U and L represent the typical scales for the velocity and length in the large scale flow. The Reynolds number is $Re = UL/\nu$, where ν is the kinematic viscosity of the fluid. The kinetic energy per unit mass associated with this flow is of order U^2 and the large scale eddy turnover time is of order L/U. This provides a power input into the turbulence of order $U^2/(L/U) = U^3/L$ per unit mass. This must be balanced by the rate at which energy is dissipated by viscosity at the smallest scale, where the effective value of the Reynolds number is unity and where the velocity and length scales are denoted by u and l respectively. The dissipation rate per unit mass is:

$$\epsilon \sim \nu \left(\frac{u}{l}\right)^2 \sim \frac{U^3}{L}.$$

Hence combing with

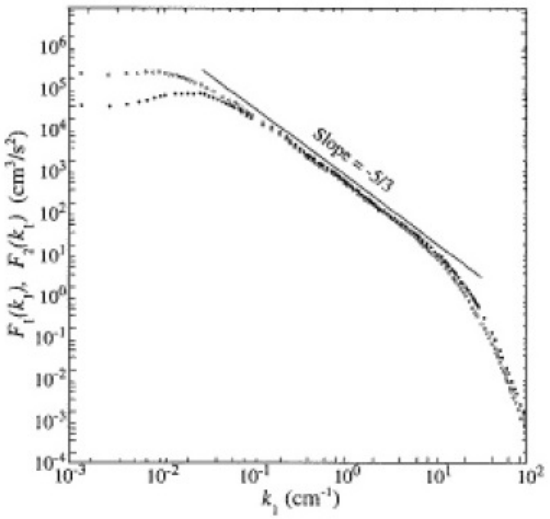
$$Re = \frac{UL}{\nu}, \qquad \text{and} \qquad \frac{ul}{\nu} = 1,$$

we obtain estimates for the smallest length and velocity scales as

$$l = LRe^{-3/4}, \qquad \text{and} \qquad u = URe^{-1/4}.$$

The scales l and u are called the *Kolmogorov scales* of turbulence. If we have a flow of water with a length scale of 1 metre at 10 metres per second ($Re = 10^4$), we would need to resolve down to scales of 1 millimetre to capture the full turbulent motion. Whilst this might just about to be possible with modern computers, flows with larger Reynolds numbers are clearly impractical.

At sufficiently high Reynolds numbers, Kolmogorov hypothesised that for intermediate lengthscales between L and l, turbulence is self-similar. The kinetic energy density associated with the turbulent motion depends only on ϵ and the intermediate lengthscale r. The square of the velocity fluctuations at these (and smaller) scales, v^2, must then be proportional to $\epsilon^{2/3}r^{2/3}$. This result is usually expressed via the wavenumber, $k \sim 1/r$ and in terms of the energy density $E(k) = \frac{d}{dk}(v^2)$. It follows that $E(k) = \alpha\epsilon^{2/3}k^{-5/3}$, a law known as Kolmogorov's five-thirds law. This scaling law can be found in well-developed turbulence at high Reynolds.



20.2 Averaged equations

All turbulence theories start by defining an averaging operator, $\langle\cdot\rangle$, to obtain equations for the large scale dynamics. We can write $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$ where $\bar{\mathbf{u}} = \langle\mathbf{u}\rangle$ is the average velocity and $\mathbf{u}' = \mathbf{u} - \langle\mathbf{u}\rangle$ is its fluctuation. The averaging process is linear, so for any quantities a and b: $\langle a + b\rangle = \langle a\rangle + \langle b\rangle$. Also, we assume the averaging process commutes with differentiating both in time and space, so that:

$$\langle\frac{\partial\mathbf{u}}{\partial t}\rangle = \frac{\partial}{\partial t}\langle\mathbf{u}\rangle, \qquad \langle\nabla\mathbf{u}\rangle = \nabla\langle\mathbf{u}\rangle.$$

Lastly, $\langle\langle a\rangle b\rangle = \langle a\rangle\langle b\rangle$.

We will assume that the flow is incompressible, so that ρ is constant and $\nabla\cdot(\bar{\mathbf{u}} + \mathbf{u}') = 0$. It follows that $\nabla\cdot\bar{\mathbf{u}} = 0$ and $\nabla\cdot\mathbf{u}' = 0$, so both the mean flow and the fluctuations are individually incompressible.

The Navier–Stokes equation becomes

$$\rho\left(\frac{\partial}{\partial t}(\bar{\mathbf{u}} + \mathbf{u}') + (\bar{\mathbf{u}} + \mathbf{u}')\cdot\nabla(\bar{\mathbf{u}} + \mathbf{u}')\right) = -\nabla(\bar{p} + p') + \rho\nu\nabla^2(\bar{\mathbf{u}} + \mathbf{u}').$$

Upon averaging, we obtain:

$$\rho\left(\frac{\partial\bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}} + \langle(\mathbf{u}'\cdot\nabla)\mathbf{u}'\rangle\right) = -\nabla\bar{p} + \rho\nu\nabla^2\bar{\mathbf{u}}. \tag{1}$$

This is referred to as the *Reynolds-averaged Navier–Stokes* or RANS equations. The structural difference between this and the usual Navier–Stokes equation is the additional term $\rho\langle(\mathbf{u}'\cdot\nabla)\mathbf{u}'\rangle$. Writing this term in its conservative form and using the incompressibility condition, we get:

$$\rho\langle(\mathbf{u}'\cdot\nabla)\mathbf{u}'\rangle = \rho\frac{\partial}{\partial x_j}\langle u_i'u_j'\rangle,$$

so this additional term can be thought of as the divergence of a stress, referred to as the Reynolds stress tensor:

$$\tau_{ij}^R = -\rho\langle u_i'u_j'\rangle. \tag{2}$$

The Reynolds averaged Navier–Stokes equation can then be written as:

$$\rho\left(\frac{\partial\bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}}\right) = -\nabla\bar{p} + \frac{\partial}{\partial x_j}(\bar{\tau}_{ij} + \tau_{ij}^R), \tag{3}$$

where $\bar{\tau}_{ij} = 2\rho\nu\bar{E}_{ij}$, and $\bar{E}_{ij} = 1/2\,(\partial\bar{u}_i/\partial x_j + \partial\bar{u}_j/\partial x_i)$.

The problem of predicting the large scale turbulent flow boils down to predicting this term accurately. In essence, we require a constitutive equation that determines the Reynolds stress from the variables describing the large scale flow. This has so far proved to be a difficult problem. Physically, the Reynolds stress represents the forcing on the mean flow from the small-scale turbulent motion.

20.3 The moment closure problem

A natural idea, which unfortunately does not work, is to consider the fluctuations by subtracting off the averaged Navier–Stokes equation from the full Navier–Stokes equation:

$$\rho\left(\frac{\partial\mathbf{u}'}{\partial t} + (\bar{\mathbf{u}}\cdot\nabla)\mathbf{u}' + (\mathbf{u}'\cdot\nabla)\bar{\mathbf{u}} + (\mathbf{u}'\cdot\nabla)\mathbf{u}' - \langle(\mathbf{u}'\cdot\nabla)\mathbf{u}'\rangle\right) = -\nabla p' + \rho\nu\nabla^2\mathbf{u}'.$$

We can write this as

$$\rho\left(\frac{\partial u_i'}{\partial t} + (\bar{\mathbf{u}}\cdot\nabla)u_i' + (\mathbf{u}'\cdot\nabla)\bar{u}_i + (\mathbf{u}'\cdot\nabla)u_i' - \langle(\mathbf{u}'\cdot\nabla)u_i'\rangle\right) = -\nabla_i p' + \rho\nu\nabla^2 u_i'$$

and a similar equation for u_j'. Multiplying the u_i' equation by u_j', the u_j' equation by u_i', adding them together and averaging, we obtain an equation for the second moment $\langle u_i'u_j'\rangle$ and, hence, the Reynolds stress, τ_{ij}^R. We get a complicated equation (Davidson equation (4.8)) that starts by:

$$\frac{\bar{D}}{Dt}[\rho\langle u_i'u_j'\rangle] = -\frac{\bar{D}}{Dt}\tau_{ij}^R = \tau_{ik}^R\frac{\partial\bar{u}_j}{\partial x_k} + \tau_{jk}^R\frac{\partial\bar{u}_i}{\partial x_k} + \frac{\partial}{\partial x_k}\left(-\rho\langle u_i'u_j'u_k'\rangle\right) + \cdots.$$

This is an evolution equation for the quantity we want, however, the right-hand-side also contains derivatives of the triple correlation $\langle u_i'u_j'u_k'\rangle$ which is a 3^{rd} order tensor we do not yet know anything about. Trying to get an equation for this tensor in a similar way would require knowledge of a fourth order tensor, and so on. This recursive problem is called a closure problem and shows that it is impossible to develop a predictive, statistical model of turbulence by simply manipulating the equation of motion via averaging techniques. Since we cannot get an exact form for the Reynolds stress, we have to model it somehow.

20.4 Eddy viscosity models

A simple approach to close the equations, dating back to Boussinesq and Prandtl, is to assume the Reynolds stress term looks like the ordinary viscous term, but with a larger coefficient of viscosity, the *eddy viscosity*. This has a number of key advantages. Since the extra term is of the similar form to the ordinary viscous term, the structure of the Navier–Stokes equation remains the same. This term dissipates energy and Boussinesq suggested that the Reynolds stress could be approximated by:

$$\begin{aligned}
\tau_{ij}^R &= -\rho\langle u_i'u_j'\rangle &(4)\\
&= \rho\nu_T\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right) - \frac{\rho}{3}\langle u_k'u_k'\rangle\delta_{ij} &(5)\\
&= 2\rho\nu_T\bar{E}_{ij} - \frac{\rho}{3}\langle u_k'u_k'\rangle\delta_{ij}. &(6)
\end{aligned}$$

The last term on the right is needed since $\nabla\cdot\bar{\mathbf{u}} = 0$ but, since it is isotropic, it can be incorporated into the pressure. This formula is the basis of many turbulence models, but it leaves unspecified how to choose the numerical value of the eddy viscosity. The Reynolds averaged Navier–Stokes equations become

$$\rho\left(\frac{\partial\bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}}\right) = -\nabla p^* + (\nu+\nu_T)\nabla^2\bar{\mathbf{u}}, \tag{7}$$

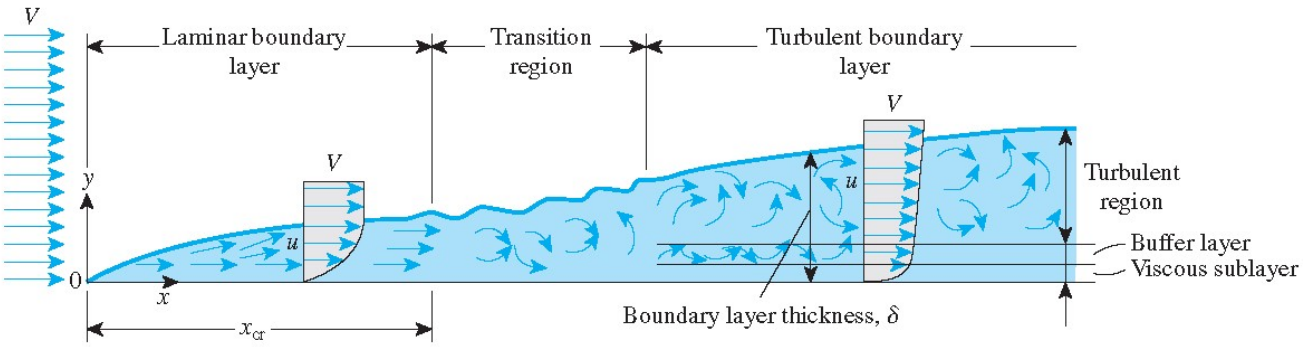
where $p^* = \bar{p} + \rho\langle u_k'u_k'\rangle/3$ is the modified pressure. Most turbulence models, including the $k-\epsilon$ model and "Large Eddy Simulations" use this form of the averaged Navier–Stokes equation.

Whilst the idea of an eddy viscosity is somewhat empirical, Prandtl provided a physical justification based on an analogy with the kinetic theory of gases. In a gas, molecules have a characteristic speed, v (which is of the order of the sound speed), and travel an average a distance λ, the mean free path, before colliding and transferring their momentum to another molecule. This leads to a kinematic viscosity: $\nu \sim v\lambda$. Prandtl's idea was to think of the small-scale eddies in a turbulent flow as small parcels of fluid with typical velocity $|u'|$ travelling a distance l before interacting with other eddies, and then transferring their momentum, hence giving $|u'|l$ as the eddy viscosity. Although the eddies travel more

slowly than individual fluid molecules, their interaction distance is much longer than the mean free path and the eddy viscosity is typically larger than the molecular viscosity.

The eddy viscosity model does have limited physical justification. The derivation of viscosity in kinetic theory relies on the assumptions that the mean free-path is small compared to the flow lengthscales and that the velocities have a distribution close to thermodynamic equilibrium. In particular, the latter assumption requires the velocities to be isotropically distributed. This is not always true of turbulent flows, as there often is a preferred direction induced by gravity, rotation or boundaries.

20.5 Turbulent flow near a wall



Just as with laminar flows, there is a boundary layer structure for turbulent flow near a rigid boundary. It is more complex and is made up of multiple layers.

For a steady large scale flow, we can derive boundary layer equations equivalent to the laminar boundary layer equations for the flow above a rigid boundary at $y = 0$. Within the boundary layer, we assume that averaged quantities vary in the x direction over a lengthscale L that is much larger than the thickness of the boundary layer. The dominant terms in the streamwise projection of the Reynolds averaged Navier–Stokes equation are:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial}{\partial y}\langle u'v'\rangle. \tag{8}$$

In this equation, the advection terms and the pressure gradient are of size U^2/L and, as with the case of the laminar boundary layer, the averaged pressure $\bar{p}$ does not vary significantly over the thickness of the boundary. We can then write:

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} = \bar{U}_0\frac{d\bar{U}_0}{dx}$$

where $\bar{U}_0$ is the averaged velocity in the x direction above the boundary layer.

The no-slip boundary condition implies that, at the wall, both the averaged and the fluctuating velocity must vanish. Hence, close to the wall, we can neglect the advection (and pressure terms) and equation (8) reduces to

$$\frac{\partial}{\partial y}\left(\nu\frac{\partial \bar{u}}{\partial y} - \langle u'v'\rangle\right) = 0. \tag{9}$$

Within this region, we have:

$$\nu\frac{\partial \bar{u}}{\partial y} - \langle u'v'\rangle = \nu\left.\frac{\partial \bar{u}}{\partial y}\right|_{y=0} \tag{10}$$

$$= \frac{\tau_w}{\rho}, \tag{11}$$

where τ_w is the wall shear stress. As a result, the sum of the tangential components of the viscous and Reynolds stresses remains constant in this region. This result also provides a scale for the velocity perturbations, $u_* = \sqrt{(\tau_w/\rho)}$, called the friction velocity, and an associated lengthscale, ν/u_*, which represents the height above the wall where viscosity is important. At sufficiently high Reynolds numbers, this lengthscale is much smaller than the total thickness of the boundary layer, δ.

We can divide the boundary layer into a number of different sublayers:

- **Viscous sublayer:** $0 < y \leq \nu/u_*$. This is the region closest to the wall where the fluid viscosity, ν, is important. There, advection can be neglected. The mean flow, $\bar{u}$, only depends on the distance from the wall:

$$\bar{u} = u_* f\left(\frac{yu_*}{\nu}\right), \tag{12}$$

for some dimensionless function, f, such that $f = 0$ and $f' = 1$ at $y = 0$. This is sometimes known as the law of the wall.

- **Inertial sublayer:** $\nu/u_* \ll y \ll \delta$. In this region, the Reynolds stress dominates over viscosity, so the latter can be neglected. This region links together the viscous sublayer and the outer boundary layers.
- **Outer or defect layer:** $y \sim \delta$. In this region, the Reynolds stress acts in a similar way to the viscous term in the laminar boundary layer equations. Thus, on dimensional grounds, we expect that the departure of $\bar{u}$ from the outer flow U_0 depends only on the turbulent intensity and the distance to the wall relative to the thickness of the boundary layer, δ, so that:

$$\bar{u} - \bar{U}_0 = u_* F\left(\frac{y}{\delta}\right), \tag{13}$$

for some other function F such that $F \to 0$ as $y/\delta \to \infty$.

Due to the different dominating terms in the boundary layer equations, we can construct solutions in the viscous sublayer and in the outer layer, which should them be matched in the inertial sublayer. Formally, this is a problem of matched asymptotic expansions but von Kàrmàn was able to deduce the form of the solution in 1930, long before the method of matched asymptotics was established.

We can obtain the solution in the inertial sublayer in two different ways. One way is to take the limit in the viscous sublayer as $yu_*/\nu \to \infty$; the other is to take the limit in the outer layer as $y/\delta \to 0$. From equation (13), the velocity gradient reads

$$\frac{d\bar{u}}{dy} = \frac{u_*}{\delta} F'(\eta), \quad \text{where} \quad \eta = \frac{y}{\delta},$$

while, from equation (12), we have,

$$\frac{d\bar{u}}{dy} = \frac{u_*^2}{\nu} f'(\xi), \quad \text{where} \quad \xi = \frac{yu_*}{\nu}.$$

Equating these two expressions and multiplying by y/u_*, we have

$$\eta F'(\eta)|_{\eta \to 0} = \xi f'(\xi)|_{\xi \to \infty}.$$

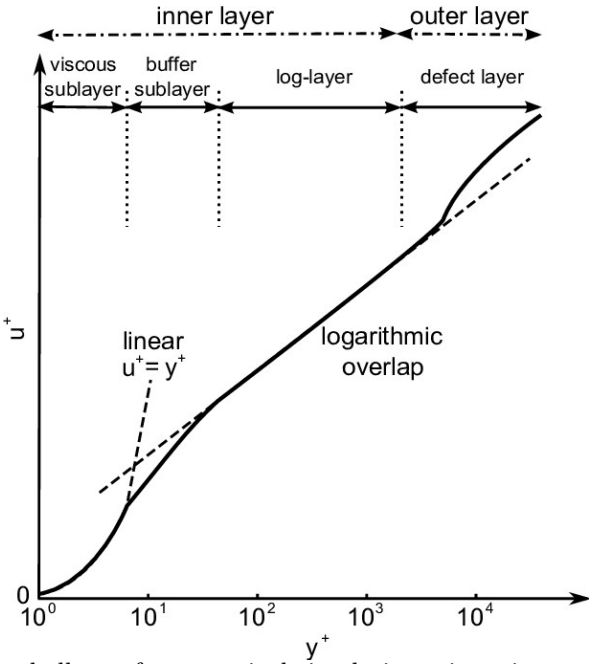
Since η is independent of ν and since ξ is independent of δ, both sides are equal to a constant, usually introduced as $1/\kappa$, where κ is the dimensionless von Kàrmàn constant. In the inertial sublayer:

$$\frac{d\bar{u}}{dy} = \frac{u_*}{\kappa y},$$

so that:

$$\bar{u} = \frac{u_*}{\kappa} \log\left(\frac{yu_*}{\nu}\right) + C.$$

This logarithmic behaviour is seen in experiments of high Reynolds number turbulent flows and the prediction of this logarithmic velocity profile was a major landmark in the development of turbulence theory. Experiments also showed that κ takes values around 0.4.



The complex structure of turbulent boundary layers presents a challenge for numerical simulations since, in many cases, the viscous sublayer, in which the no-slip boundary condition is satisfied, is much smaller than the resolution of the simulation can be. As a result, different boundary conditions, called wall functions, need to be used. Additionally, the roughness of the boundary may exceed the height of the viscous sublayer (for example in atmospheric flows where the roughness is due to trees or buildings). In this case, the logarithmic law still applies but the viscous lengthscale ν/u_* is replaced by the surface roughness.

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