# MATH5453M Foundations of Fluid Dynamics

## Lecture 20. From laminar to turbulence

*http://www.cbeaume.com/en/research_turbulence.html*

When a fluid flows, it can be subject to an instability due to the simple presence of internal shear. This instability, transition to turbulence, links two important flow regimes: laminar flows and turbulent flows. Laminar flows are flows in which the fluid (macro-)particles flow along locally parallel undisturbed layers. As a result, they do not yield macroscopic mixing and appear smooth. On the other hand, the definition of turbulence is not established ("you know it when you see it") but suffices to say that it is characeterized by its ability to produce mixing on the macroscopic scale. It looks messy and is difficult to describe and to predict.

While it is true that low Reynolds numbers yield laminar flows and large Reynolds numbers yield turbulent flows, the approximate value at which transition to turbulence is observed depends on the system investigated. Nonetheless, turbulence is ubiquitous in practical applications (a fluid like water has kinematic viscosity of $O(10^{-6})\mathrm{m}^2{\cdot}\mathrm{s}^{-1}$ so would yield large Reynolds numbers unless the flow is very slow or has very small characteristic length-scales). This, and the problem's difficulty, is why understanding and controlling turbulence is one of the great challenges of fluid dynamics.

## 20.1 Plane Couette flow

Let us consider one of the simplest example of a fluid flow: plane Couette flow, i.e., the flow created by two parallel plates moving in opposite directions. This flow is governed by the Navier–Stokes equation and the incompressibility condition:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2}$$

together with boundary conditions $\mathbf{u} = y\mathbf{e_x}$ at $y = \pm 1$ and assuming that the domain is infinite in the $x$ and $z$ directions. In writing these, we introduced the velocity field $\mathbf{u}$, the pressure $p$, the time $t$ and the Reynolds number $Re$. The directions are commonly known as streamwise $x$, wall-normal $y$ and spanwise $z$.

This configuration admits the laminar flow: $\mathbf{u} = y\,\mathbf{e_x}$, shown in Figure 1. This flow is known to be linearly stable

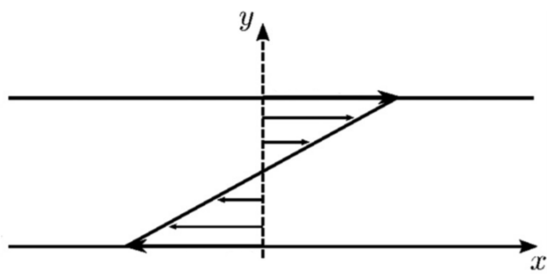


Figure 1: Sketch of plane Couette flow illustrating the laminar flow.

for all values of the Reynolds number, which means that any infinitesimal perturbation from it will eventually decay. This is, however, not necessarily the case for larger perturbations, which may transition to turbulence.

## 20.2 Towards turbulence

As turbulence is a completely nonlinear behaviour, the natural way to investigate it is the use numerics. Numerical simulations invariably display laminar flow for Reynolds numbers up to $Re = O(100)$. The dynamics substantially changes around $Re = 400$ to display time-dependent dynamics.

To understand the dynamics involved at these values of the Reynolds number, a simulation is run at $Re = 500$ and for a spatially periodic domain with periods $4\pi$ in the streamwise ($x$) and $2\pi$ in the spanwise ($z$) directions. The flow velocity is dominated by its streamwise component by approximately an order of magnitude, so we will focus on it for representation purposes but can deduce the structure of the other components where needed. The resulting turbulent flow is thus shown in terms of the streamwise ($x$) velocity at a constant streamwise ($x$) location in the left panel of Figure 2 and in the mid-plane between the walls ($y = 0$) in the right panel of Figure 2. The sheets of constant streamwise velocity visible on the left panel are undulated in the spanwise direction to produce *streaks*. They are maintained by *rolls* which advect some of the negative (resp. positive) streamwise velocity up (resp. down) from the bottom (resp. top) wall. These structures have a non-trivial dependence on the streamwise direction, as shown on the right panel.

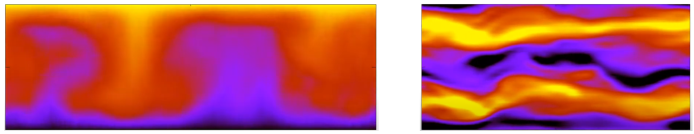

Figure 2: Snapshots of simulations of plane Couette flow at $Re = 500$ for a spatially periodic domain of periods $4\pi$ in the streamwise direction and $2\pi$ in the spanwise direction represented by the streamwise velocity. The left panel shows the plane $x = 0$ with the spanwise direction being represented horizontally (full video: https://www.youtube.com/watch?v=Ie7yGWQm8iY) while the right panel shows the plane $y = 0$ with the streamwise direction being represented horizontally (full video: https://www.youtube.com/watch?v=T7aqoFJOCd0). In both cases, the colour scheme is evenly distributed around 0 with yellow indicating positive values and black negative values.

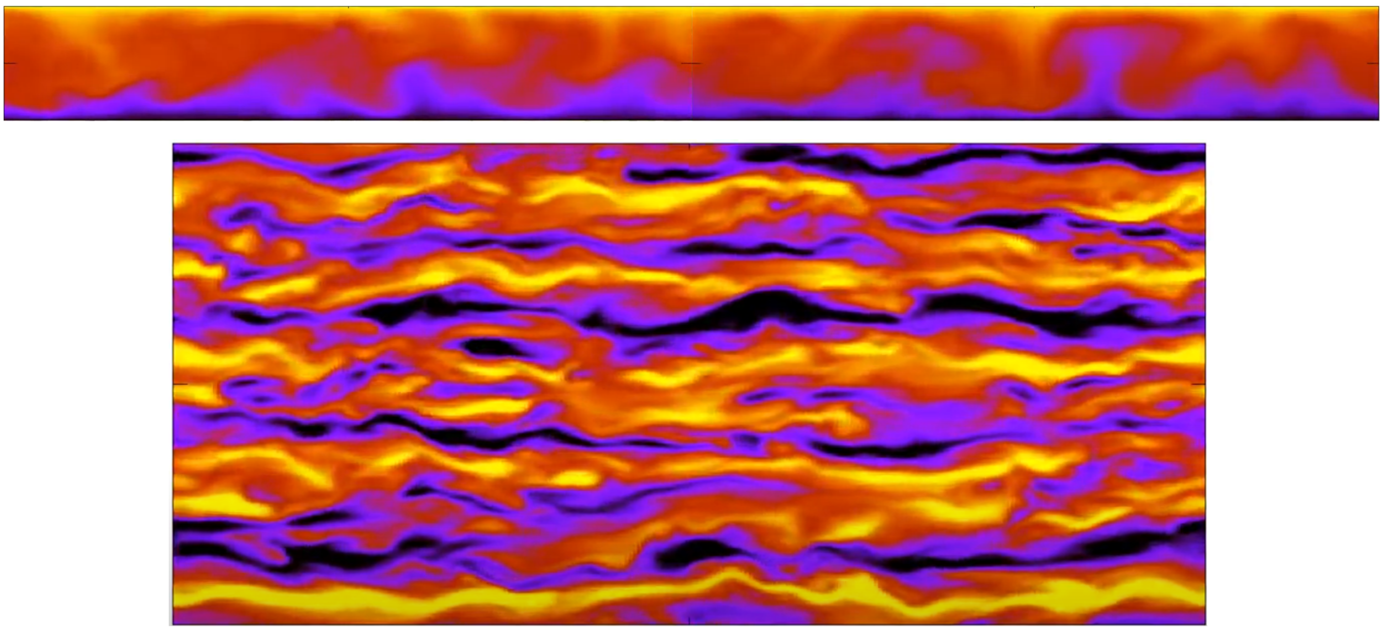

Figure 3: Snapshots of simulations of plane Couette flow at $Re = 500$ for a spatially periodic domain of periods $16\pi$ in the streamwise direction and $8\pi$ in the spanwise direction represented by the streamwise velocity. The top panel shows the plane $x = 0$ with the spanwise direction being represented horizontally (full video: https://www.youtube.com/watch?v=eOrtDJhvWE4) while the bottom panel shows the plane $y = 0$ with the streamwise direction being represented horizontally (full video: https://www.youtube.com/watch?v=KVrIe6PxQ68). In both cases, the colour scheme is evenly distributed around 0 with yellow indicating positive values and black negative values.

Importantly, the flow never seems to repeat itself and seems unpredictable, as the "plume dance" in the video associated with the left panel and the "billow highway" in the video associated with the right panel indicate. It is this temporal complexity, or chaos, that leads to macroscopic mixing.

The small domain simulation presented above allows to make two critical observations: (i) the flow displays temporal complexity and (ii) it is supported by spatial structures, namely streaks and rolls. Figure 3 shows similar flow conditions to Figure 2 but within a larger domain: the streamwise and spanwise periods of the flow are here each 4 times larger. This less constraining domain allows complex spatial dynamics to develop. Many plumes of various sizes coexist and interact with each other (top panel), while billows propagate along irregular paths and give rise to coalescence and splitting events (bottom panel). The flow is thus constituted of a multitude of structures of different size and amplitude, each of which describing a behavior similar to that identified in Figure 2. While a given structure may be out of phase with its neighbours, interaction between neighbouring structures is a strong dynamical mechanism that enhances flow complexity. The resulting spatio-temporal dynamics is a feature shared by most turbulent flows and is what makes them instantly recognizable.

The turbulence presented above is typical of the transitional regime, i.e., of Reynolds numbers which are barely large enough to sustain turbulence. We can gain more insight into turbulence by increasing the Reynolds number. A similar simulation to that in Figure 3, except for a Reynolds number three times larger, is reported in Figure 4. The video associated with the top panel shows that, for $Re = 1500$, the dynamics occurs over shorter time-scales. The plumes are finer and do not extend as far into the domain as they did for $Re = 500$ (top panel of Figure 3). As shear builds at the

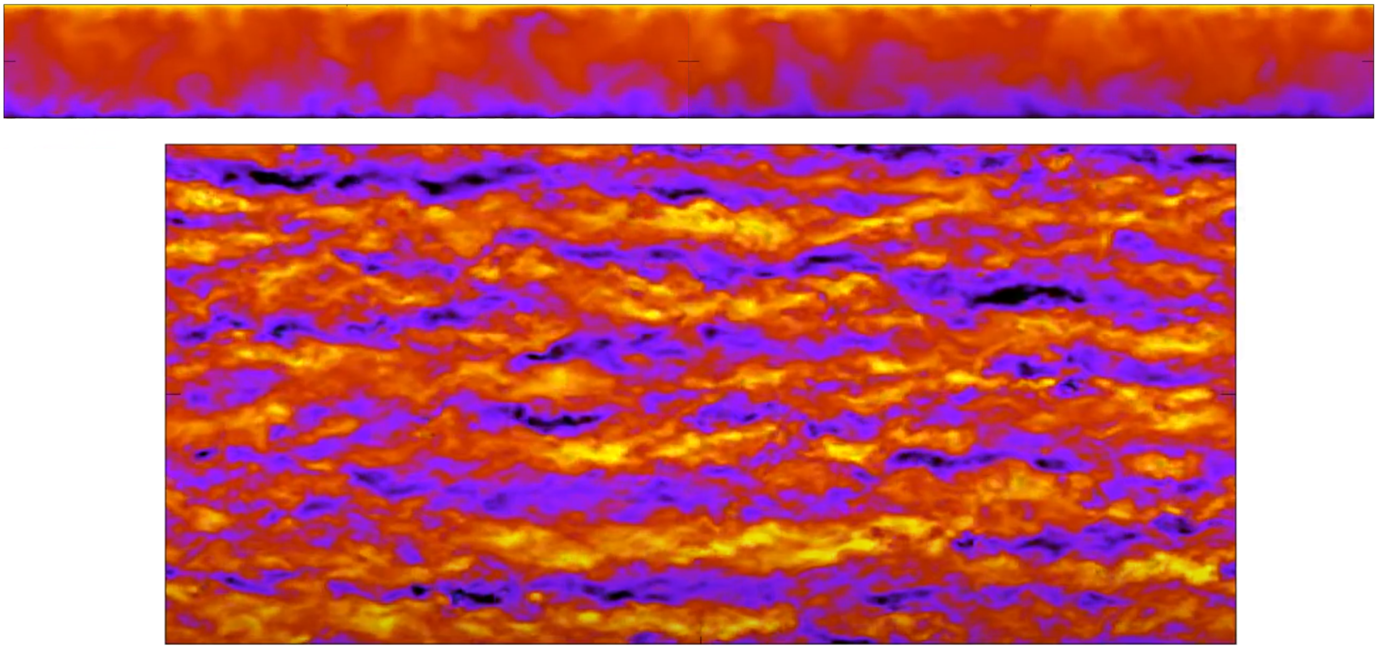

Figure 4: Snapshots of simulations of plane Couette flow at $Re = 1500$ for a spatially periodic domain of periods $16\pi$ in the streamwise direction and $8\pi$ in the spanwise direction represented by the streamwise velocity. The top panel shows the plane $x = 0$ with the spanwise direction being represented horizontally (full video: ) while the bottom panel shows the plane $y = 0$ with the streamwise direction being represented horizontally (full video: ). In both cases, the colour scheme is evenly distributed around 0 with yellow indicating positive values and black negative values.

wall, the boundary layers become thinner and momentum becomes better mixed over a larger fraction of the domain. The bottom panel highlights the fact that the structures constituting the flow become finer as the Reynolds number is increased. As a consequence, larger Reynolds number flows display a larger range of spatial scales, each of which undergoing dynamics on its own time-scale: the smaller the structure, the faster the dynamics. This accounts for the ever-growing complexity of turbulent flows as the Reynolds number is increased.

## 20.3 Transition to turbulence

To understand the instability leading to turbulence in such a flow, we need to remember that the laminar flow is linearly stable and coexist with turbulence. The linear stability of the laminar flow implies that any infinitesimal perturbation of it will eventually decay. It therefore takes a finite-amplitude pertubation for transition to turbulence to be triggered.

### 20.3.1 Toy problem

Let us study the dynamical equation:

$$\dot{u} = u\left[1 - \frac{1}{r} - (u-1)^2\right], \tag{3}$$

where $u$ is the scalar (living on a one-dimensional space) solution and $r > 0$ is a parameter akin to the Reynolds number. Solution $u_0 = 0$ is linearly stable for all values of $r$. For $r < 1$ it is the only stable solution so that all initial conditions eventually lead to it. Beyond $r = 1$, a second stable solution exists, $u_+ = 1 + \sqrt{1 - 1/r}$, so that there are two families of initial conditions: those that converge to $u_0$ and those that converge to $u_+$. There exists an initial condition that separates these two families: $u_- = 1 - \sqrt{1 - 1/r}$. This initial condition is unique (it separates a one-dimensional space in two, hence is zero-dimensional, or a point) and unstable: any small perturbation from it either leads to $u_0$ or $u_+$. The corresponding bifurcation diagram is shown in Figure 5.

### 20.3.2 The edge of chaos

In the toy problem above, $u_0$ is a surrogate for the laminar flow in plane Couette flow and $u_+$ plays the role of turbulence. As both the laminar and turbulent flows coexist, there exists perturbations that neither laminarize nor transition to turbulence. Fluids being continuous media, a fluid flow like plane Couette flow lives in infinite dimensions, as opposed to the previous toy problem which one one-dimensional. It follows that the separatrix between both types of perturbations

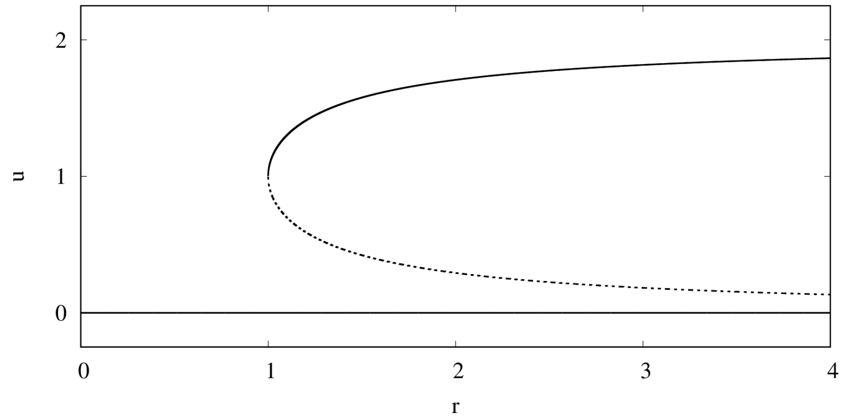


Figure 5: Bifurcation diagram showing the solutions of equation (3), $u$, as a function of parameter $r$. Stable solutions are represented using solid lines while unstable solutions are shown using dashed lines.

is no longer a point but rather a topological surface of infinite dimension (one less than the space in which the flow lives: the missing dimension being associated with the direction of laminarization/transition). It is called the *edge of chaos* due to the chaotic character of turbulent flows. In practice, both the amplitude (one degree of freedom) and the shape (infinitely many degrees of freedom) of the perturbation play an active role in determining whether transition is triggered or not.

Given that the edge of chaos provides a threshold for transition to turbulence, it is of practical interest to characterize it. To do so, we create a family of initial condition spanning the interpolating line between an arbitrary snapshot of a turbulent flow and the laminar flow. Along this line, initial conditions that are too close to the laminar flow decay and those too close to turbulence transition. These two families are separated by (at least) one initial condition that lies on the edge of chaos. To compute (some of) the edge of chaos, we first look for the transitioning initial condition that is the closest to the laminar flow. Very close to this initial condition, one can find a laminarizing initial condition and these two initial conditions are on different sides of the edge of chaos. Computing the edge of chaos, or *edge tracking* consists in evolving these two initial conditions until their evolution diverges. The trajectory described until then lies on the edge of chaos and the divergence is due to the edge unstable eigendirection (laminarization/transition). One can continue to track the edge for longer periods of time. This is done by deciding on a point of divergence between the two simulations and interpolating between the two instantaneous solutions obtained to generate a new family of initial conditions. Within this new family, we look again for the transitioning initial condition closest to the laminarizing simulation and pair it with the closest laminarizing initial condition. Both of these new initial conditions are computed further until we notice divergence. This process can be continued as long as one desires and leads to the computation of a local attractor on the edge of chaos, usually called *edge state*. Such a computation for plane Couette flow is shown in Figure 6. In this specific example, tracking the edge for extended periods of time leads to a steady edge state as

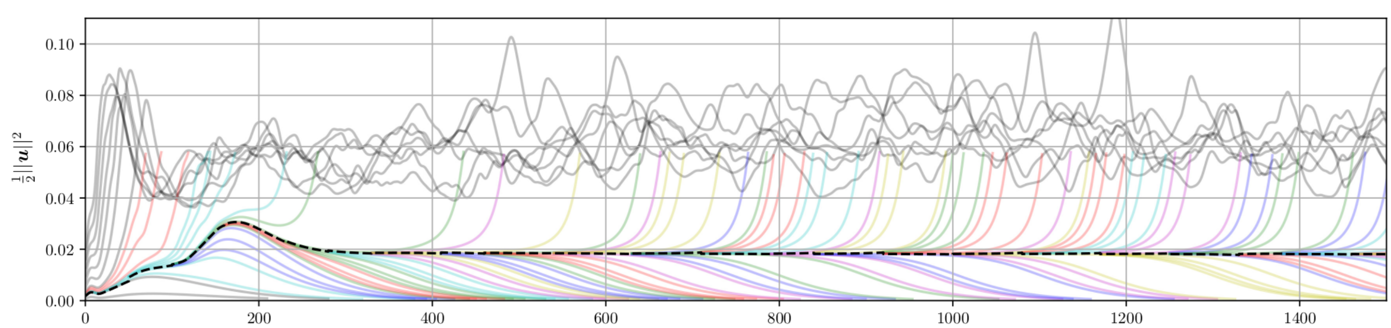


Figure 6: Evolution of the kinetic energy with time during edge tracking in plane Couette flow within a $4\pi \times 2 \times 32\pi/15$ domain at $Re = 500$. Gray simulations show turbulence, coloured simulations represent the diverging simulations during edge-tracking and the thick dashed line represents the edge of chaos.

the one shown in Figure 7. The simplicity of this situation is serendipitous and attractors along the edge of chaos can be time-dependent. In particular, in large domains, where the dynamics is less restricted, similar computations have yielded chaotic attractors that display spatial localization in both streamwise and spanwise directions, as shown in Figure 8.

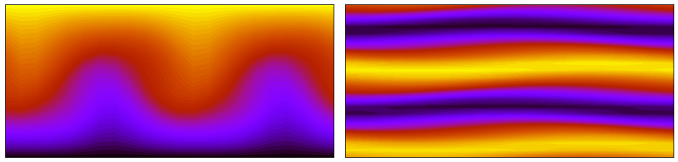

Figure 7: Edge state obtained by the simulations reported in Figure 6 and represented as in Figure 2.

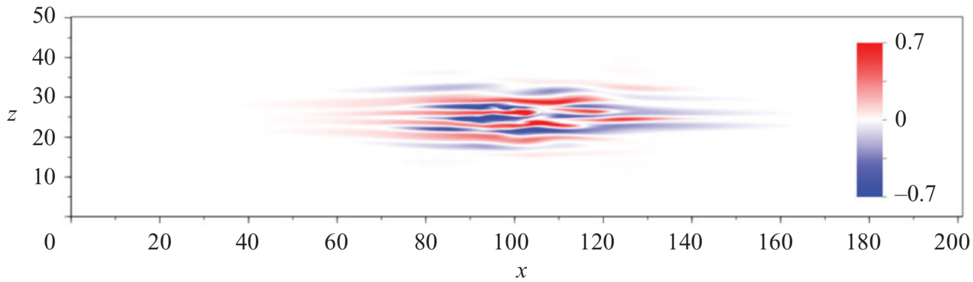


Figure 8: Instantaneous flow along the edge of chaos in plane Couette flow within a $64\pi \times 2 \times 16\pi$ domain at $Re = 400$ represented by the streamwise velocity at $y = 0$. After Schneider, Marinc & Eckhardt, *J. Fluid Mech.* **646**, 441–451 (2010).

A (very crude) sketch of the phase space associated with plane Couette flow is shown in Figure 9 and highlights all the phenomena discussed. The figure highlights the complexity of the edge of chaos, with the potential for multiple

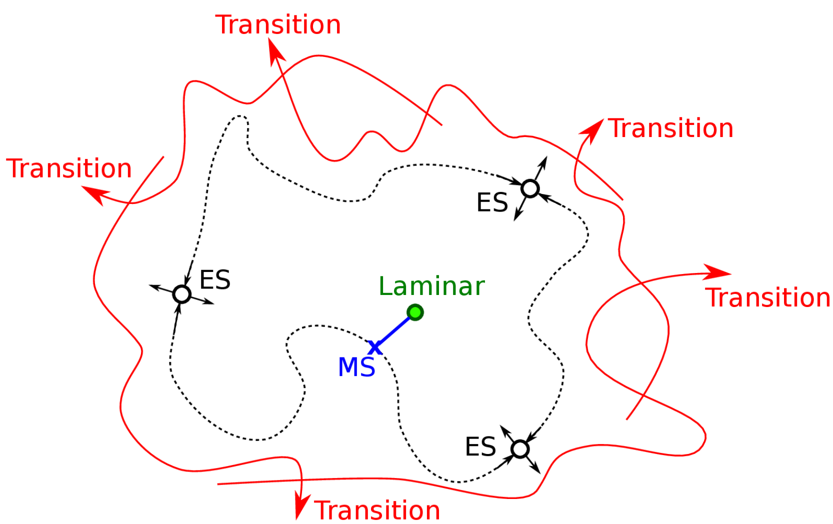


Figure 9: Sketch of the phase space associated with transition to turbulence in plane Couette flow. The laminar flow is represented by the green dot, the edge of chaos is represented by the dashed black line. Local attractors along the edge of chaos (edge states, ES) are represented by the black circles together with arrows indicating their stable and unstable manifolds. Initial conditions outside the edge of chaos transition to turbulence, as represented by the red trajectories. A special point, the minimal seed (MS) is shown in blue and corresponds to the perturbation of minimal energy from the laminar flow that crosses the edge.

attractors along it. Additionally, this complexity means that there exists a special point along the edge of chaos that is energetically (in terms of the kinetic energy for example) the closest to the laminar flow. This point, called the *minimal seed*, provides crucial information for the control of transition to turbulence. The value of its energy is an important quantity to know, since no perturbation of smaller energy can trigger transition to turbulence. Its shape is also interesting because it provides information on the flow structure (type of pattern, wavelength, size, etc.) that is susceptible to lead to instability.