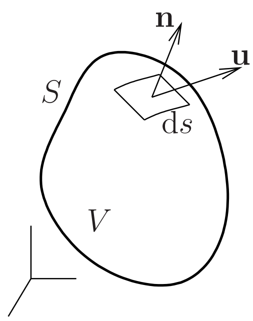
MATH5453M Foundations of Fluid Dynamics

Lecture 2: Mass conservation and streamfunctions

2.1 Mass conservation and the Continuity Equation: Paterson, p.45–46; Kundu, p.97–99

In any situation, the mass of a fluid must be conserved. For a continuous medium, such as a fluid, this principle is expressed in the form of a *continuity equation*.

Consider a fixed volume V of surface S and unit outward pointing normal vector $\hat{\mathbf{n}}$.



The total mass in V is

$$M_V = \int_V \rho \, \mathrm{d}V,$$

where ρ is the density of mass (mass per unit volume). This quantity can only change if mass is transported into or out of the volume.

The mass flowing through the surface per unit time (i.e. the mass flux) is

$$\frac{\mathrm{d}M_V}{\mathrm{d}t} = -\int_S \rho \mathbf{u}\cdot\hat{\mathbf{n}} \, \mathrm{d}S,$$

so that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \, \mathrm{d}V &= -\int_S \rho\mathbf{u}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\ \Rightarrow \qquad \int_V \frac{\partial \rho}{\partial t}\,\mathrm{d}V &= -\int_S \rho\mathbf{u}\cdot\hat{\mathbf{n}}\,\mathrm{d}S, \quad \text{since } V \text{ is fixed.} \end{aligned}$$

Applying the divergence theorem,

$$\begin{aligned} \int_V \frac{\partial \rho}{\partial t}\,\mathrm{d}V &= -\int_V \nabla\cdot(\rho\mathbf{u})\,\mathrm{d}V \\ \Leftrightarrow \qquad \int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right]\mathrm{d}V &= 0. \end{aligned}$$

Since V is arbitrary, this equation must hold for all volume V. Thus, the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \qquad (1)$$

holds at all points in the fluid. Expanding the divergence as $\nabla\cdot(\rho\mathbf{u}) = \rho\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\rho$ we obtain the Lagrangian form of the continuity equation:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} = 0. \qquad (2)$$

The density of a fluid particle moving with the fluid only changes if there is an expansion or a contraction of the flow.

Expansions and contractions are associated with sound waves. The associated speed of the these sound waves c is typically much faster than the flow speed $U = |\mathbf{u}|$ and so for sub-sonic flows, where $U \ll c$, the density remains constant. One exception is the case of large scale atmospheric flows, where the density varies vertically due to gravity. This phenomenon is called stratification.

2.2 Incompressible fluids and streamfunctions

In an *incompressible* fluid, the density of each fluid particle remains constant, and the continuity equation (2) reduces to

$$\begin{aligned} & \frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \\ \Leftrightarrow \qquad & \rho \nabla \cdot \mathbf{u} = 0. \end{aligned}$$

So, for an incompressible flow, we obtain the incompressibility constraint:

$$\nabla \cdot \mathbf{u} = 0. \tag{3}$$

This places a restriction on the form of the fluid velocity, which can be expressed as

$$\mathbf{u} = \nabla \times \mathbf{\Psi}, \tag{4}$$

for some vector field $\mathbf{\Psi}$. Since $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any vector field $\mathbf{F}$, writing the velocity as the curl of a vector field makes the incompressibility constraint automatically satisfied.

2.3 Two-dimensional planar flows: Paterson, p.48–58

If the flow is confined to a plane, so that $\mathbf{u} = u(x, y)\,\hat{\boldsymbol{x}} + v(x, y)\,\hat{\boldsymbol{y}}$, then $\mathbf{\Psi} = \psi(x, y)\hat{\boldsymbol{z}}$, where $\psi(x, y)$ is a scalar function defined by:

$$u = \frac{\partial \psi}{\partial y}, \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \tag{5}$$

The function $\psi(x, y)$ is referred to as the streamfunction and has a number of important properties.

2.3.1 Streamfunctions and streamlines

The gradient of the streamfunction is orthogonal to the velocity field:

$$\begin{aligned} \mathbf{u} \cdot \nabla \psi &= u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y} \\ &= \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} \\ &= 0. \end{aligned}$$

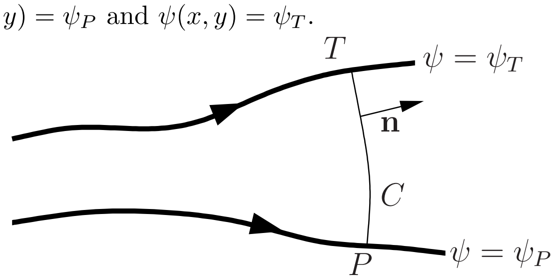
This means that ψ remains constant along streamlines, i.e. the streamlines are just the curves $\psi = cst$. Conversely:

$$\begin{aligned} \mathrm{d}\psi = 0 \quad &\Rightarrow \quad \frac{\partial \psi}{\partial x}\mathrm{d}x + \frac{\partial \psi}{\partial y}\mathrm{d}y = 0 \\ &\Rightarrow \quad -v\mathrm{d}x + u\mathrm{d}y = 0 \\ &\Rightarrow \quad \mathbf{u} \times \mathbf{dl} = 0, \end{aligned}$$

where $\mathbf{dl} = (\mathrm{d}x, \mathrm{d}y)$ is an element of displacement along the streamline. This result implies that the streamlines are locally parallel to the velocity field and, therefore, that the curve where ψ remains constant is a streamline.

2.3.2 Flux between streamlines

Consider the two streamlines, $\psi(x, y) = \psi_P$ and $\psi(x, y) = \psi_T$.



The fluid flow or the *volume flux* through $C : \{x(s), y(s)\}$, an arbitrary curve connecting P and T, is

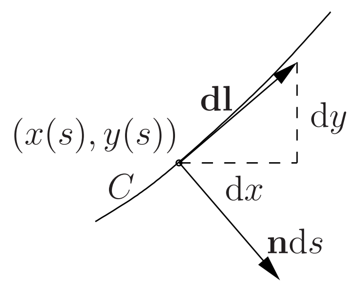
$$Q = \int_P^T \mathbf{u} \cdot \hat{\mathbf{n}} \, \mathrm{d}s. \tag{6}$$

Let

$$\begin{aligned} \mathbf{dl} &= \mathrm{d}x \hat{\boldsymbol{x}} + \mathrm{d}y \hat{\boldsymbol{y}} \\ &= \mathrm{d}s \left(\frac{\mathrm{d}x}{\mathrm{d}s} \hat{\boldsymbol{x}} + \frac{\mathrm{d}y}{\mathrm{d}s} \hat{\boldsymbol{y}} \right) \end{aligned}$$

be an infinitesimal displacement along the curve C. We define the infinitesimal vector normal to $\mathbf{dl}$:

$$\begin{aligned} \hat{\mathbf{n}} \, \mathrm{d}s &= \mathrm{d}y \hat{\boldsymbol{x}} - \mathrm{d}x \hat{\boldsymbol{y}} \\ &= \left(\frac{\mathrm{d}y}{\mathrm{d}s} \hat{\boldsymbol{x}} - \frac{\mathrm{d}x}{\mathrm{d}s} \hat{\boldsymbol{y}} \right) \mathrm{d}s. \end{aligned}$$



We can then express the volume flux in the following way:

$$\begin{aligned} Q &= \int_P^T \left(\frac{\partial \psi}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial \psi}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}s} \right) \mathrm{d}s \\ &= \int_P^T \frac{\mathrm{d}\psi}{\mathrm{d}s} \mathrm{d}s \\ &= \int_{\psi_P}^{\psi_T} \mathrm{d}\psi \\ &= \psi_T - \psi_P. \end{aligned}$$

The flux of fluid flowing between two streamlines is therefore equal to the difference between the values taken by the streamfunction on these two streamlines. Consequently, if streamlines are close together, the flow is fast. This statement is supported by the equality $\|\mathbf{u}\| = \|\nabla\psi\|$.

2.3.3 Example on a 2D streamfunction

The flow defined as $\mathbf{u} = y/(x^2 + y^2)\mathbf{e_x} - x/(x^2 + y^2)\mathbf{e_y}$ is incompressible since

$$\begin{aligned} \nabla \cdot \mathbf{u} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\ &= \frac{-2xy}{x^2 + y^2} + \frac{(-2y)(-x)}{x^2 + y^2} \\ &= 0. \end{aligned}$$

We can then express the streamfunction:

$$(u =) \frac{\partial \psi}{\partial y} = \frac{y}{x^2 + y^2} \Rightarrow \psi(x, y) = \frac{1}{2} \ln \left(x^2 + y^2 \right) + \alpha(x).$$

We can also find that

$$(v =) - \frac{\partial \psi}{\partial x} = -\frac{x}{x^2 + y^2} + \frac{\mathrm{d}\alpha}{\mathrm{d}x} \Rightarrow \frac{\mathrm{d}\alpha}{\mathrm{d}x} = 0.$$

So, α is constant, which we can choose to be 0 owing to the fact that streamfunctions are defined up to a constant. We finally obtain:

$$\psi(x, y) = \frac{1}{2} \ln\left(x^2 + y^2\right).$$

This flow is more easily visualised using polar coordinates. The streamfunction becomes $\psi = \ln r$ and is independent of θ which shows that the streamlines are circles about the origin. The velocity in polar coordinates follows:

$$\mathbf{u} = \nabla\times\left(\psi(r,\theta)\hat{\boldsymbol{z}}\right) \quad \Rightarrow \quad \begin{cases} u_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \\ u_\theta = -\dfrac{\partial\psi}{\partial r} \end{cases} \tag{7}$$

$$\Rightarrow \quad \begin{cases} u_r = 0 \\ u_\theta = -\frac{1}{r} \end{cases}. \tag{8}$$

This flow is an approximate model of bath-plug vortex.

2.4 Stokes streamfunction – axisymmetric flows with no swirl: Paterson, p.62–64, Kundu, p.231–232

The axisymmetric flows of interest here are those of the form $\mathbf{u} = u(R, z)\,\hat{\boldsymbol{R}} + w(R, z)\,\hat{\boldsymbol{z}}$ with (R, ϕ, z) the cylindrical polar coordinates. A flow is said to be axisymmetric if it depends only on R and z and not on ϕ. Such a flow can have a non-zero ϕ component, the swirl $v(R, z)$, but, for now, we consider only axisymmetric flows with $v = 0$.

The incompressibility constraint yields:

$$\begin{aligned} & \nabla\cdot\mathbf{u} = 0 \\ \Rightarrow \qquad & \frac{1}{R}\frac{\partial}{\partial R}(Ru) + \frac{\partial w}{\partial z} = 0. \end{aligned}$$

We can define the Stokes streamfunction $\Psi(R, z)$ by

$$w(R, z) = \frac{1}{R}\frac{\partial\Psi}{\partial R} \quad \text{and} \quad u(R, z) = -\frac{1}{R}\frac{\partial\Psi}{\partial z}. \tag{9}$$

Note the fluid velocity can be written as a curl again

$$\mathbf{u} = \nabla\times\left(\frac{1}{R}\Psi(R, z)\,\hat{\boldsymbol{\phi}}\right), \tag{10}$$

so that

$$\boldsymbol{\Psi} = \frac{1}{R}\Psi(R, z)\,\hat{\boldsymbol{\phi}}.$$

2.4.1 Streamlines

As in the case of planar flows, Ψ is constant on streamlines:

$$\begin{aligned} (\mathbf{u}\cdot\nabla)\,\Psi &= u\frac{\partial\Psi}{\partial R} + w\frac{\partial\Psi}{\partial z} \\ &= \frac{1}{R}\left(Ru\frac{\partial\Psi}{\partial R} + Rw\frac{\partial\Psi}{\partial z}\right) \\ &= \frac{1}{R}\left(-\frac{\partial\Psi}{\partial z}\frac{\partial\Psi}{\partial R} + \frac{\partial\Psi}{\partial R}\frac{\partial\Psi}{\partial z}\right) \\ &= 0. \end{aligned}$$

Thus, Ψ is constant along streamlines.

2.4.2 Volume flux

For axisymmetric flows, it is useful to think of *streamtubes*, i.e., surfaces of revolution spanned by all the streamlines through a circle about the axis of symmetry. The volume flux, or fluid flow, between two streamtubes with $\Psi = \Psi_i$ and $\Psi = \Psi_o$ is

$$\begin{aligned} Q &= \int_S \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S & (11) \\ &= 2\pi(\Psi_0 - \Psi_i). & (12) \end{aligned}$$

Proof

$$\begin{aligned} Q &= \int_S \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S \\ &= \int_S \nabla \times \left(\frac{1}{R}\Psi\,\hat{\boldsymbol{\phi}}\right) \cdot \mathbf{n}\, \mathrm{d}S \quad \text{(definition of } \Psi) \\ &= \oint_{C_o} \frac{1}{R}\Psi\,\hat{\boldsymbol{\phi}} \cdot \mathrm{d}\mathbf{l} + \oint_{C_i} \frac{1}{r}\Psi\,\hat{\boldsymbol{\phi}} \cdot \mathrm{d}\mathbf{l} \quad \text{(Stokes' theorem)} \\ &= \Psi_o \oint_{C_o} \frac{1}{R}\,\hat{\boldsymbol{\phi}} \cdot \mathrm{d}\mathbf{l} + \Psi_i \oint_{C_i} \frac{1}{R}\,\hat{\boldsymbol{\phi}} \cdot \mathrm{d}\mathbf{l} \quad (\Psi \equiv \Psi_{\{o,i\}} \text{ onto } C_{\{o,i\}}) \\ &= \Psi_o \int_0^{2\pi} \mathrm{d}\phi + \Psi_i \int_{2\pi}^{0} \mathrm{d}\phi \\ &= 2\pi(\Psi_o - \Psi_i) \quad \text{(Note, } \mathrm{d}\mathbf{l} = \mathrm{d}R\,\hat{\boldsymbol{R}} + R\mathrm{d}\phi\,\hat{\boldsymbol{\phi}}). \end{aligned}$$

2.4.3 Examples of axisymmetric flows

For a uniform flow parallel to the axis, $u = 0$ and $w = U$,

$$\begin{cases} \dfrac{\partial \Psi}{\partial R} = RU \\ \dfrac{\partial \Psi}{\partial z} = 0 \end{cases} \Rightarrow \Psi(R) = \frac{1}{2}UR^2.$$

(We choose the integration constant such that $\Psi = 0$ on the axis, at $r = 0$).

Consider the streamtube or radius a. The volume flux is given by

$$\begin{aligned} Q &= \int_S \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S \\ &= \int_S \mathbf{u} \cdot \hat{\mathbf{e}}_{\boldsymbol{z}}\, \mathrm{d}S \\ &= \int_S w\, \mathrm{d}S = U \int_S \mathrm{d}S \\ &= \pi U a^2. \end{aligned}$$

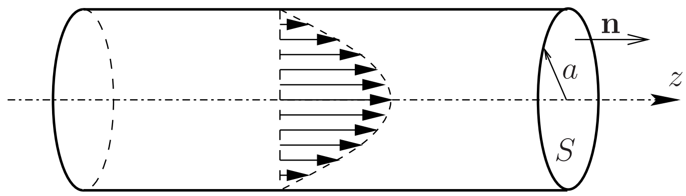
Also:

$$\begin{aligned} 2\pi(\Psi_o - \Psi_i) &= 2\pi(\Psi(a) - \Psi(0)) \\ &= 2\pi\left(\frac{1}{2}Ua^2 - 0)\right) \\ &= \pi U a^2, \end{aligned}$$

as required.

As a second example, consider the steady flow in a long pipe a radius a, with velocity given by

$$\begin{cases} u = 0 \\ w = \frac{U}{a^2}(a^2 - R^2) \end{cases} \quad \text{with} \quad \begin{cases} w = 0 \text{ on } R = a, \\ w = U \text{ on } R = 0. \end{cases}$$



We can calculate the streamfunction:

$$\begin{aligned} \frac{\partial \Psi}{\partial z} &= -Ru \\ &= 0, \\ \text{and } \frac{\mathrm{d}\Psi}{\mathrm{d}R} &= Rw \\ &= \frac{U}{a^2}(a^2R - R^3) \end{aligned}$$

provide enough information to deduce that:

$$\Psi(R) = \frac{UR^2}{4a^2}(2a^2 - R^2),$$

where we chose 0 for the constant of integration so that $\Psi(0) = 0$. The volume flux follows:

$$\begin{aligned} Q &= \int_S \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S \\ &= 2\pi\left(\Psi(a) - \Psi(0)\right) \\ &= \frac{\pi}{2}Ua^2, \end{aligned}$$

which is also found by calculating the integral:

$$\begin{aligned} \int_S \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S &= \int_0^{2\pi}\int_0^a wR\,\mathrm{d}R\mathrm{d}\phi \\ &= \frac{2\pi U}{a^2}\int_0^a (a^2R - R^3)\,\mathrm{d}R, \quad (\text{since } \mathrm{d}S \\ &= R\mathrm{d}\phi\mathrm{d}R) \\ &= \frac{2\pi U}{a^2}\left[\frac{a^2R^2}{2} - \frac{R^4}{4}\right]_0^a \\ &= \frac{\pi U a^2}{2}. \end{aligned}$$

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