MATH5453M Foundations of Fluid Dynamics

Lecture 19: High Reynolds number flow and boundary layers

See: Acheson, chapter 8; Tritton, chapter 11; Kundu, chapter 10.

19.1 High Reynolds Number Flow

When the Reynolds number is large, we can no longer neglect advection. By choosing appropriate scales for the fluid velocity, the distances and the pressure the Navier–Stokes equation can be written in the following non-dimensional form:

$$\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}. \tag{1}$$

It follows naively that the viscous term $\mu\nabla^2\mathbf{u}$ can be neglected for sufficiently high Reynolds numbers so that the flow is given by solutions to the inviscid Euler equations. However, in neglecting this term, we are removing the highest spatial derivative from the equation and are no longer able to satisfy all the boundary conditions. In particular, the no-slip boundary condition at the surface derives from a viscous effect and cannot be imposed on solutions of the Euler equations. The solution to this apparent paradox is that, near boundaries, the relevant lengthscale for the viscous term is not L (the lengthscale of the flow geometry) but a smaller length δ such that the viscous term is comparable in magnitude with the other terms in the equation. Balancing the viscous term with advection requires that

$$|\mu\nabla^2\mathbf{u}| \sim |(\mathbf{u}\cdot\nabla)\mathbf{u}| \tag{2}$$

$$\Rightarrow \quad \mu\frac{U}{\delta^2} \sim \frac{\rho U^2}{L}, \tag{3}$$

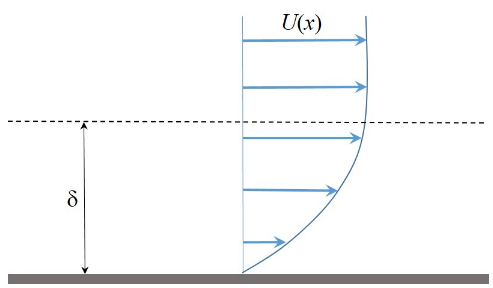
and, hence, that

$$\frac{\delta}{L} \sim \left(\frac{\mu}{\rho U L}\right)^{1/2} = Re^{-1/2}. \tag{4}$$

Hence, for high Reynold number flows in the presence of a rigid boundary, there are two different lengthscales governing the flow: L and δ. Away from boundary, the fluid is effectively inviscid and can be approximated by solutions of the Euler equations. Close to the boundary, there exists a thin layer of fluid near the boundary, where viscous effects remain important.

19.2 The Prandtl boundary layer

We consider the developing boundary layer sketched below.



Such a two-dimensional stationary flow is governed by the following Navier–Stokes equations:

$$\rho\left(u\partial_x u + v\partial_y u\right) = -\partial_x p + \mu\left(\partial_x^2 u + \partial_y^2 u\right), \tag{5}$$

$$\rho\left(u\partial_x v + v\partial_y v\right) = -\partial_y p + \mu\left(\partial_x^2 v + \partial_y^2 v\right), \tag{6}$$

where ρ is the fluid density, u is the streamwise (x-) velocity, v the wall-normal (y-) velocity, p is the pressure and μ is the fluid dynamic viscosity. The flow is incompressible, so we pose:

$$\partial_x u + \partial_y v = 0. \tag{7}$$

Lastly, the boundary conditions are:

$$u = v = 0 \qquad \text{at } y = 0, \tag{8}$$

$$(u, v) \to (U, 0) \qquad \text{at } y \to \infty. \tag{9}$$

Dynamics occur on a(n arbitrary) length scale L in the streamwise direction and δ, the boundary layer thickness, in the wall-normal direction. The spatial derivatives then follow the scalings:

$$\partial_x \sim \frac{1}{L}, \qquad \partial_y \sim \frac{1}{\delta}. \tag{10}$$

In addition, these length scales are not comparable: $L \gg \delta$. We can thus introduce a small parameter $\epsilon \ll 1$ such that:

$$\frac{\delta}{L} = \epsilon. \tag{11}$$

The streamwise velocity u is of the same order of magnitude as the velocity infinitely far away from the plate U but the order of magnitude of the wall-normal velocity v is yet to be determined. To that end, we assume that both terms in the incompressibility constraint are of the same order of magnitude:

$$\frac{U}{L} \sim \frac{v}{\delta}, \tag{12}$$

which gives:

$$v \sim \frac{U\delta}{L} \tag{13}$$

$$\Rightarrow \qquad v \sim \epsilon\, U, \tag{14}$$

implying that the wall-normal velocity is smaller than the streamwise one.

We can now rescale the wall-normal quantities according to the streamwise quantities. We define the rescaled quantities by:

$$x^* = \frac{x}{L}, \tag{15}$$

$$y^* = \frac{y}{\delta} = \frac{y}{\epsilon\, L}, \tag{16}$$

$$u^* = \frac{u}{U}, \tag{17}$$

$$v^* = \frac{v}{\epsilon\, U}, \tag{18}$$

$$p^* = \frac{p}{\rho U^2}, \tag{19}$$

where the pressure is nondimensionalised in such a way that it remains of the same order of magnitude as the other terms.

Remark: in the case of an incompressible flow, the pressure can be thought of as a mathematical function whose role is to ensure incompressibility.

Using the dimensionless variables, the incompressibility constraint reads:

$$\frac{U}{L}\partial_{x^*} u^* + \frac{\epsilon U}{\epsilon L}\partial_{y^*} v^* = 0, \tag{20}$$

yielding:

$$\partial_{x^*} u^* + \partial_{y^*} v^* = 0. \tag{21}$$

Similarly, the Navier–Stokes equation becomes:

$$\rho\left(\tfrac{U^2}{L} u^* \partial_{x^*} u^* + \tfrac{U^2}{L} v^* \partial_{y^*} u^*\right) = -\frac{\rho U^2}{L}\partial_{x^*} p^* + \mu\left(\frac{U}{L^2}\partial^2_{x^*} u^* + \frac{U}{\epsilon^2 L^2}\partial^2_{y^*} u^*\right), \tag{22}$$

$$\rho\left(\tfrac{\epsilon U^2}{L} u^* \partial_{x^*} v^* + \tfrac{\epsilon U^2}{L} v^* \partial_{y^*} v^*\right) = -\frac{\rho U^2}{\epsilon L}\partial_{y^*} p^* + \mu\left(\frac{\epsilon U}{L^2}\partial^2_{x^*} v^* + \frac{U}{\epsilon L^2}\partial^2_{y^*} v^*\right), \tag{23}$$

and simplifies into:

$$u^* \partial_{x^*} u^* + v^* \partial_{y^*} u^* = -\partial_{x^*} p^* + \frac{1}{Re_L}\partial^2_{x^*} u^* + \frac{1}{\epsilon^2 Re_L}\partial^2_{y^*} u^*, \tag{24}$$

$$u^* \partial_{x^*} v^* + v^* \partial_{y^*} v^* = -\frac{1}{\epsilon^2}\partial_{y^*} p^* + \frac{1}{Re_L}\partial^2_{x^*} v^* + \frac{1}{\epsilon^2 Re_L}\partial^2_{y^*} v^*, \tag{25}$$

where we have introduced the Reynolds number $Re_L = \rho U L/\mu$.

We are interested in $Re_L \gg 1$. As $\epsilon \ll 1$, the term $\partial^2_{x^*} u^*/Re_L$ is the smallest term in equation (24) and $\partial^2_{x^*} v^*/Re_L$ is the smallest term in equation (25). We drop them.

To keep a balance between advection (left-hand-side) and diffusion (right-hand-side), and therefore retain sensible physics, we impose $\epsilon^2 Re_L = 1$. As a result, the small quantity we have introduced is now related to the Reynolds number:

$$\epsilon = \frac{\delta}{L} = Re_L^{-1/2}. \tag{26}$$

Consequently, the leading order of system (24), (25) is:

$$u^* \partial_{x^*} u^* + v^* \partial_{y^*} u^* = -\partial_{x^*} p^* + \partial^2_{y^*} u^*, \tag{27}$$

$$\partial_{y^*} p^* = 0, \tag{28}$$

where equation (28) implies that the pressure does not vary across the boundary layer, but only along it.

We can then express the pressure at any point in the critical layer by applying the Bernoulli equation on a streamline away from the boundary layer:

$$p + \frac{\rho u^2}{2} = \text{cst}, \tag{29}$$

which gives in dimensionless form:

$$p^* + \frac{{u^*}^2}{2} = \text{cst}. \tag{30}$$

Outside the boundary layer, the velocity is $u^* = 1$, so we have:

$$p^* + \frac{1}{2} = \text{cst}, \tag{31}$$

$$\Rightarrow \quad \partial_{x^*} p^* = 0, \tag{32}$$

a relation valid for all y^*.

The resulting set of reduced equations for the boundary layer writes:

$$\partial_{x^*} u^* + \partial_{y^*} v^* = 0, \tag{33}$$

$$u^* \partial_{x^*} u^* + v^* \partial_{y^*} u^* = \partial^2_{y^*} u^*, \tag{34}$$

and is to be solved together with the following boundary conditions:

$$u^* = v^* = 0 \qquad \text{at } y^* = 0, \tag{35}$$

$$u^* \to 1 \qquad \text{at } y^* \to \infty. \tag{36}$$

A second boundary condition for v is not necessary as v is only derived once with respect to y in the above equations.

19.3 The Blasius boundary layer

While studying the dimensional boundary layer equations:

$$\partial_x u + \partial_y v = 0, \tag{37}$$

$$u \partial_x u + v \partial_y u = \nu \partial^2_y u, \tag{38}$$

where $\nu = \mu/\rho$, Blasius conjectured that the boundary layer is self-similar, i.e., that at any given point along the boundary layer, the velocity profile is the same to a stretching factor on the spatial dimension. He wrote:

$$\eta = y \left(\frac{U}{\nu x} \right)^{1/2}, \qquad \frac{u(x,y)}{U} = f'(\eta), \tag{39}$$

where $f(\eta)$ is a dimensionless quantity and $f'(\eta)$ denotes its derivative with respect to the dimensionless wall-normal coordinate η.

One of the virtues of this rescaling is that the wall-normal direction is rescaled by a quantity proportional to the laminar boundary layer thickness. In other words, the laminar boundary layer is mapped onto a rectangle.

As the flow is incompressible and two-dimensional, we introduce a streamfunction ψ such that:

$$u = \partial_y \psi, \qquad v = -\partial_x \psi. \tag{40}$$

The incompressibility constraint is automatically verified:

$$\partial_x u + \partial_y v = \partial_x(\partial_y \psi) + \partial_y(-\partial_x \psi) = 0. \tag{41}$$

Using the streamfunction, equation (38) reduces down to:

$$\partial_y \psi \, \partial_x \partial_y \psi - \partial_x \psi \, \partial_y^2 \psi = \nu \partial_y^3 \psi. \tag{42}$$

Equation (42) has yet to be written in terms of Blasius's variables (39). To do so, we note that the definition of the streamfunction (40) implies:

$$U f'(\eta) = \partial_\eta \psi \partial_y \eta \tag{43}$$

$$\Rightarrow \quad \psi = U \int_0^\eta \frac{1}{\partial_y \eta} f'(\eta) d\eta, \tag{44}$$

providing the new definitions:

$$\psi = U\gamma(x) f(\eta), \qquad \gamma(x) = (\nu x/U)^{1/2}. \tag{45}$$

With these new variables, the spatial derivatives of ψ become:

$$\begin{aligned} \partial_x \psi &= U\left(\partial_x \gamma f + \gamma \partial_\eta f \partial_x \eta\right) && (46)\\ &= U\left(\gamma' f - \gamma f' \frac{y\gamma'}{\gamma^2}\right) && (47)\\ &= U\gamma' f - U f' \frac{y\gamma'}{\gamma}, && (48) \end{aligned}$$

and:

$$\begin{aligned} \partial_y \psi &= U\gamma \partial_\eta f \partial_y \eta && (49)\\ &= U\gamma f' \frac{1}{\gamma} && (50)\\ &= U f', && (51) \end{aligned}$$

where $\gamma' = \partial_x \gamma$ and $f' = \partial_\eta f$.

The terms of equation (42) therefore become:

$$\begin{aligned} \partial_y \psi \, \partial_x \partial_y \psi &= (U f') \, \partial_x (U f') && (52)\\ &= (U f') \, U f'' \left(-\frac{y\gamma'}{\gamma^2}\right) && (53)\\ &= -U^2 \frac{y\gamma'}{\gamma^2} f' f'', && (54) \end{aligned}$$

$$\begin{aligned} \partial_x \psi \, \partial_y^2 \psi &= \left(U\gamma' f - U f' \frac{y\gamma'}{\gamma}\right) \partial_y (U f') && (55)\\ &= \left(U\gamma' f - U f' \frac{y\gamma'}{\gamma}\right) U \frac{1}{\gamma} f'' && (56)\\ &= U^2 \frac{\gamma'}{\gamma} f f'' - U^2 \frac{y\gamma'}{\gamma^2} f' f'', && (57) \end{aligned}$$

$$\begin{aligned} \nu \partial_y^3 \psi &= \nu \partial_y^2 (U f') && (58)\\ &= \nu U \partial_y \left(\frac{1}{\gamma} f''\right) && (59)\\ &= \nu U \frac{1}{\gamma^2} f'''. && (60) \end{aligned}$$

In the end, equation (42) simplifies into:

$$-U^2 \frac{y\gamma'}{\gamma^2} f' f'' - \left(U^2 \frac{\gamma'}{\gamma} f f'' - U^2 \frac{y\gamma'}{\gamma^2} f' f''\right) = \nu U \frac{1}{\gamma^2} f''', \tag{61}$$

$$\nu U \frac{1}{\gamma^2} f''' + U^2 \frac{\gamma'}{\gamma} f f'' = 0, \tag{62}$$

$$f''' + \frac{U\gamma'\gamma}{\nu} f f'' = 0. \tag{63}$$

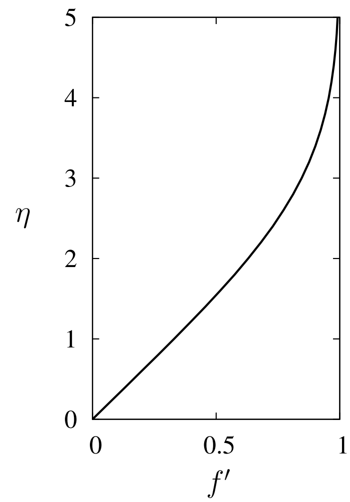
The variable γ is easily obtained from definition (45) and yields $U\gamma'\gamma/\nu = 1/2$. The equation Blasius obtained for the dimensionless quantity f is then:

$$f''' + \frac{1}{2} f f'' = 0. \tag{64}$$

This equation is complemented with the boundary conditions (35) and (36) that now write:

$$f' = f = 0, \qquad \eta = 0, \tag{65}$$
$$f' \to 1, \qquad \eta \to \infty. \tag{66}$$

Equation (64) is generally solved numerically, as below.



This laminar boundary layer solution is self-similar: the same profile holds at any given position x along the boundary layer, the only change being a stretching coefficient in the wall-normal direction as η is a linear function of y and depends on x.

The boundary layer equation (64) and this solution are valid for $x \in [0; L_{max}]$ and $\eta \in \mathcal{R}^+$, where L_{max} represents the point at which a change in dynamics occurs that violates one of the hypotheses made. This point can arise due to a transition where the boundary layer becomes turbulent. There, wall-normal velocities become of the same order as streamwise velocities because of the creation and advection of eddies and the whole analysis carried out here breaks down.

Since $u/U \to 1$ as $\eta \to \infty$, we can define the boundary layer thickness as the region in which $u/U \leq 0.99$, or in other words $f' \leq 0.99$. The data from the figure above provides a boundary layer thickness of $\eta \approx 5.0$. Therefore:

$$\delta \left(\frac{U}{\nu x}\right)^{1/2} \approx 5.0, \tag{67}$$
$$\Rightarrow \qquad \delta \approx 5.0 \left(\frac{\nu x}{U}\right)^{1/2}, \tag{68}$$
$$\Rightarrow \qquad \frac{\delta}{x} \approx 5.0 \left(\frac{\nu}{U x}\right)^{1/2}, \tag{69}$$
$$\Rightarrow \qquad \frac{\delta}{x} \approx 5.0 Re_x^{-1/2}. \tag{70}$$

19.4 Pressure gradient and boundary layer separation

Since the velocity is zero on $y = 0$, the x-component of the Navier–Stokes equation (**??**) reduces to

$$\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2} \big|_{y=0} .$$

Moreover, since $\partial v/\partial x = 0$ at $y = 0$, the vorticity becomes $\omega = -\partial u/\partial y$ so that

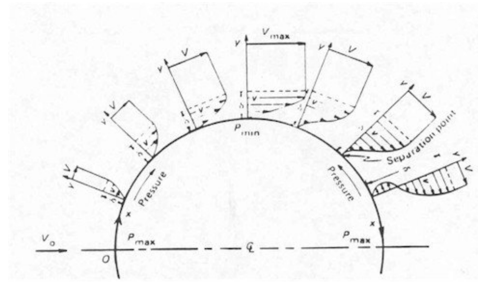
$$\frac{dp}{dx} = -\mu \frac{\partial \omega}{\partial y} \big|_{y=0} .$$

The sign of the vorticity gradient at the wall is thus determined by the external pressure gradient. Considering an external flow in the positive x direction, we expect the vorticity in the boundary layer is negative on average. If the pressure gradient becomes positive in the boundary layer (this can happen as the flow slows down), the vorticity gradient at the wall will be in the wrong direction. A large enough pressure gradient will drive a boundary layer flow in the opposite direction to the external flow, leading to separation of the boundary layer from the wall.

Classical boundary layer theory is unable to analyse the point of separation because the equations become singular at a point where $\partial u/\partial y = 0$. Instead, a more complex three layer or "triple deck" structure is required, but we will not discuss this here.

Instead, let us consider the case of flow past a cylinder. Potential flow calculations give the pressure distribution at the surface of the cylinder:

$$p(a,\theta) = \frac{1}{2}\rho U^2 \left(2\cos(2\theta) - 1\right) + p_\infty.$$



Hence moving from the front stagnation point to the rear the pressure gradient is given by

$$-\frac{1}{a}\frac{\partial p}{\partial \theta} = \frac{2\rho U^2}{a}\sin(2\theta).$$

The pressure gradient is therefore negative from $\theta = \pi$ to $\theta = \pi/2$ but then becomes positive, leading to separation of the boundary layer and generating a wake behind the cylinder.

Turbulent boundary layers tend to separate further downstream, yielding smaller wakes and, hence, a lower drag. This is the reason why golf balls have dimples and cricket balls swing.

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