MATH5453M Foundations of Fluid Dynamics

Lecture 1: Continuum Hypothesis and Kinematics

1.1 Continuum hypothesis: Tritton, p.48–51; Kundu, p.5; Paterson, p.32–34

Liquids and gases are made up of large numbers of frequently colliding molecules. A key approximation in fluid mechanics is the continuum hypothesis. This assumes a fluid can be represented as a continuous medium, which has a density, a pressure and a velocity defined at every point inside the domain of interest. In almost all circumstances, these quantities are assumed continuous and differentiable. When is assumption valid?

(i) The continuum hypothesis applies when the shortest length of interest in a fluid problem (the **macroscopic** length a) is much greater than either the interparticle spacing or the mean free path, ℓ, which is the average distance between collisions. Usually these distances are smaller than 10^{-8} m which is normally much smaller than a. As a result, there are billions of molecules in even the smallest volume of interest to us. We can then average over that small volume to get well-defined flow characteristics.

(ii) We define the fluid velocity at a point to be the average the vector velocity of all the molecules in all small volume around that point. This is typically much slower in magnitude than the speed of an individual particles. Note that we can get zero mean velocity if the particles are moving at similar speeds in different directions, as the vector sum can be zero. This random motion is governed by the temperature of the fluid and exists for a fluid at rest. A non-zero fluid velocity means the molecules locally have a bias towards one direction. When this bias is small (at subsonic speeds) we can neglect the consequence of fluid motion on the temperature. However for objects smaller than one micron (10^{-6} m) the effect of collisions with individual molecules results in a diffusive motion superimposed on the translation with the fluid velocity, called Brownian motion.

(iii) If the density is very low, e.g. the uppermost regions of the Earth's atmosphere, the mean free path ℓ can become quite long. If the Knudsen number $Kn = \ell/a$ is no longer small, the continuum approximation breaks down. When this happens we are in the realm of Rarefied Gas Dynamics (RGD), an active area of research out of the scope of this course. In RGD, people typically use Monte Carlo methods, computing particle trajectories with random initial conditions, and averaging to get results to compare with observation.

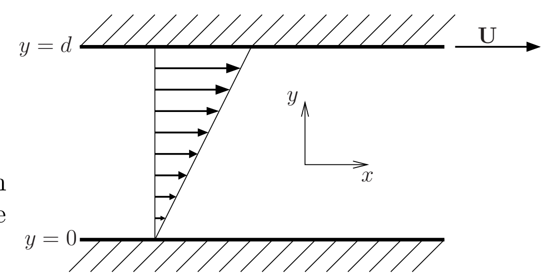
1.2 Two simple flows

(i) **Plane Couette flow**

Plane Couette flow is the simple flow with

$$\mathbf{u} = \frac{U_0 y}{d}\mathbf{e_x}, \quad \text{or} \quad \mathbf{u} = \left(\frac{U_0 y}{d}, 0, 0\right) \quad \text{in component form,}$$

where $\mathbf{e_x}$ is the unit vector in the x-direction. This flow can be generated by two parallel plates separated by a distance d and sliding relative to each other.

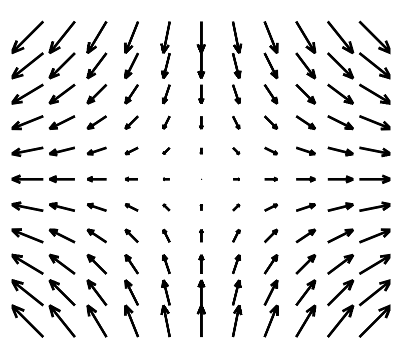


(ii) **Stagnation point flow**

This flow corresponds to

$$\mathbf{u} = (Ex, -Ey, 0),$$

where E is a constant. The point where $\mathbf{u} = 0$ is called a stagnation point.



Sections 1.3,1.5 and 1.6 are covered in: Paterson, p.34–39; Kundu, p.67–76 (see example 3.1 on page 73); and Tritton, p.73–80.

1.3 Particle paths

One method for visualising fluid motion is to follow the motion of a "tracer" particle in the flow.

Let a particle be released at time t_0 and at position $\mathbf{x}_0$ within the velocity field. Since the particle moves with the fluid velocity

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}, t) \quad \text{such that} \quad \mathbf{x} = \mathbf{x}_0 \text{ at } t = t_0. \tag{1}$$

Example using the stagnation point flow:

$$\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix} Ex \\ -Ey \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = Ex \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -Ey \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x(t) = x_0 \mathrm{e}^{Et} \\ y(t) = y_0 \mathrm{e}^{-Et} \\ z(t) = z_0 \end{cases} .$$

These formulae give the path of the tracer particle as a function of time. Note that particles at the stagnation point $x_0 = y_0 = z_0 = 0$ do not move since $\mathbf{u} = 0$.

In this example, we can go a bit further and eliminate the time variable, t. By proceeding this way, we can show that particle paths are hyperbolae of equation $xy = x_0 y_0$.

1.4 Time derivatives: Kundu, p.69–71; Paterson p.46–48

The time derivative $\partial\mathbf{u}/\partial t$ measures the rate of change of velocity at the fixed position $\mathbf{x}$. This is referred to as the *Eulerian* time-derivative. A flow characterized by $\partial\mathbf{u}/\partial t = 0$ is called a "steady flow". However, this does not give the acceleration of a fluid particle at this point, since the particle is moving through this point along its particle path. Instead we require the *convective* derivative (also called Lagrangian derivative or material derivative) $\mathrm{D}\mathbf{u}(\mathbf{x}, t)/\mathrm{D}t$, which is the rate of change of $\mathbf{u}$ along the particle path and $\mathbf{x}(t)$ is the position of a fluid particle (i.e. moving with the fluid). There is no need to go through particle paths calculations to evaluate $\mathrm{D}/\mathrm{D}t$ of a quantity $f(\mathbf{x}, t)$:

$$\frac{\mathrm{D}f}{\mathrm{D}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t), t) \tag{2}$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \tag{3}$$

$$= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} \tag{4}$$

$$= \frac{\partial f}{\partial t} + (\mathbf{u} \cdot \nabla) f. \tag{5}$$

Hence the acceleration of a fluid particle is

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u}. \tag{6}$$

1.5 Streamlines

A streamline is a line everywhere tangent to the local fluid velocity. If the line is parametrised by a parameter s ("distance" along the streamline), then

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{u}(\mathbf{x}(s), t) \tag{7}$$

or, equivalently,

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w} \quad (= \mathrm{d}s), \tag{8}$$

since $\mathbf{u}(\mathbf{x}, t)$ is not explicitly function of s. If the flow is steady ($\partial\mathbf{u}/\partial t = 0$), then the streamlines are the same as the particle paths. Note that the converse is not necessarily true.

1.6 Streaklines

An experimental method for visualising a flow is to introduce a dye source at a fixed point (x_0, y_0, z_0) at time t_0 so that the streak created can be photographed at a later time. The streakline is therefore the set of all points that the given fluid particle occupied during the time interval (t_0, t).

For a steady flow, particle paths, streamlines and streaklines are all the same. For unsteady flows, the particle paths, streamlines and streaklines are usually different.

1.7 Example

Consider the flow given in Cartesian coordinates by

$$\mathbf{u} = (U\cos\omega t, U\sin\omega t, 0).$$

(i) Particle paths:

$$\begin{cases} \dfrac{dx}{dt} = U\cos\omega t \\ \dfrac{dy}{dt} = U\sin\omega t \\ \dfrac{dz}{dt} = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{U}{\omega}\sin\omega t + x_0, \\ y = -\frac{U}{\omega}\cos\omega t + \frac{U}{\omega} + y_0, \\ z = z_0 \end{cases}, \tag{9}$$

where (x_0, y_0, z_0) is the location of the particle at $t = 0$. If we eliminate t, we see that the particle paths are circles:

$$(x - x_0)^2 + (y - \frac{U}{\omega} - y_0)^2 = \frac{U^2}{\omega^2}. \tag{10}$$

This flow is generated when a wave passes over deep water. The fluid particles do indeed describe circular trajectories. We shall see later that, in shallow water, particle trajectories describe ellipses.

(ii) Streamlines:

$$\begin{cases} \dfrac{dx}{ds} = U\cos\omega t \\ \dfrac{dy}{ds} = U\sin\omega t \end{cases} \Rightarrow \begin{cases} x = Us\cos\omega t + x_0 \\ y = Us\sin\omega t + y_0 \end{cases}. \tag{11}$$

Eliminating s, we see that the streamlines are straight lines, not circles. At any one time, all the particles are moving in the same direction, but that direction changes with time.

(iii) Streaklines: Suppose a dye source is at $(0, 0, 0)$ and is activated at time $t = 0$. The streakline consists of the set of points occupied by particles whose path went through $(0, 0, 0)$ at time $\bar{t}$ such that $0 \le \bar{t} \le t$:

$$\begin{cases} \frac{U}{\omega}\sin\omega\bar{t} + x_0 = 0 \\ -\frac{U}{\omega}\cos\omega\bar{t} + \frac{U}{\omega} + y_0 = 0 \\ z_0 = 0 \end{cases}, \tag{12}$$

from which we can deduce (x_0, y_0, z_0). Now at time t this particle is at

$$\begin{cases} x = \frac{U}{\omega}\sin\omega t + x_0 = \frac{U}{\omega}(\sin\omega t - \sin\omega\bar{t}) \\ y = -\frac{U}{\omega}\cos\omega t + \frac{U}{\omega} + y_0 = \frac{U}{\omega}(\cos\omega\bar{t} - \cos\omega t) \\ z = 0 \end{cases}. \tag{13}$$

If we eliminate $\bar{t}$, we get:

$$(x - \frac{U}{\omega}\sin\omega t)^2 + (y + \frac{U}{\omega}\cos\omega t)^2 = \frac{U^2}{\omega^2} \tag{14}$$

which are circles of radius U/ω like the particle paths, but the centre of the streakline circles is located at a different position from that of the streakline circles! Be very careful when interpreting what you see in a streakline photograph if the flow is not steady!