MATH5453M Foundations of Fluid Dynamics

Mathematical Basics: Vector Calculus and Cartesian Tensors needed for Fluid Dynamics

Here are some brief notes on the main things you need to know about vector calculus and Cartesian tensors. It is covered in more detail in Chapter 2 of Kundu, Cohen and Dowling, available online from the library. You can also look in a vector calculus text book such as Matthews, "Vector Calculus".

1 Grad, div and curl

(i) Grad. If $\phi(x,y,z)$ is any scalar field,

$$\operatorname{grad}\phi = \nabla\phi = (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}).$$

Grad (or, in full, gradient) acts on a scalar, result is a vector. This definition is in Cartesian coordinates. For the definition of grad in polar coordinates (cylindrical or spherical) see the curvilinear coordinates section.

Some uses in fluid dynamics:

(a) in irrotational flow, i.e. when there is no vorticity, you can define a velocity potential ϕ such that $\mathbf{u} = \nabla\phi$. Sometimes ϕ is a function of time also, but to find $\nabla\phi$ you assume t is constant and just take the spatial derivatives. Sadly most flows are not irrotational, but some are irrotational in some regions, and others may be 'almost' irrotational.

(b) If the temperature is $T(x,y,z)$, the conductive heat flux is $-k\nabla T$ where k is the thermal conductivity (Fourier's law of conduction).

Useful fact: $\int_A^B \nabla\phi \cdot \mathbf{dl} = \phi(B) - \phi(A)$, where the line integral is taken along any path joining A to B.

(ii) Divergence. If $\mathbf{u}(x,y,z)$ is any vector field,

$$\operatorname{div}\mathbf{u} = \nabla\cdot\mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}.$$

When Div acts on a vector the result is a scalar. Again, Div in polar coordinates is more complicated: see the curvilinear coordinates section.

Some uses in fluid dynamics:

(a) in incompressible flow $\nabla\cdot\mathbf{u} = 0$.

(b) In a thermally conducting fluid, the rate at which heat flows into a fluid element is $\operatorname{div}(k\nabla T) = \nabla\cdot k\nabla T$, where T is the temperature. So $\rho c_v \partial T/\partial t = \nabla\cdot k\nabla T$, where ρ is the density and c_v is the specific heat at constant volume (for incompressible fluid).

Useful facts:

(a) the divergence theorem, sometimes called Gauss's theorem:

$$\int_V \nabla\cdot\mathbf{u}\, dv = \int_S \mathbf{u}\cdot\mathbf{n}\, dS$$

Here V is any closed volume, S is the surface enclosing it, and $\mathbf{n}$ is the outward pointing normal unit vector to the surface S. This theorem also tells you what the divergence is: if $\nabla\cdot\mathbf{u} > 0$ inside V, then the fluid is diverging there. It must on average being flowing out through S therefore $\mathbf{u}$ must be mainly in the direction of $\mathbf{n}$ to get a positive surface integral. Conversely, if $\nabla\cdot\mathbf{u} < 0$ the fluid is contracting, or imploding on itself. Incompressible fluids can't do this so they have $\nabla\cdot\mathbf{u} = 0$.

(b) div curl $\mathbf{v} = 0$ for any vector field $\mathbf{v}$.

(c) div grad is written ∇^2, pronounced del-squared.

(iii) Curl

$$\operatorname{curl}\mathbf{u} = \nabla\times\mathbf{u} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix}.$$

The curl acts on a vector and returns a vector. As usual, in polar coordinate systems the curl is given on the curvilinear coordinates sheet. There is little point in trying to memorize the curvilinear coordinate formulae for div and curl in the polar coordinate systems. It is safer to look them up.

Key use in fluid dynamics: the vorticity vector is $\nabla \times \mathbf{u}$.

Some useful facts about curl:

(a) Stokes' theorem.

$$\int_S \nabla \times \mathbf{u} \cdot \mathbf{n}\, dS = \int_C \mathbf{u} \cdot \mathbf{dl}$$

Here S is any closed surface, $\mathbf{n}$ is the unit normal vector to S, C is the curve enclosing it, and $\int_C$ means the line integral taken round the closed curve. $1/2\pi$ times the line integral is called the circulation because it is non-zero if the flow is circling round the curve C. Stokes' theorem relates the circulation round C to the surface integral of the vorticity. The curl of $\mathbf{u}$ is therefore a measure of the local rotation of the fluid. Be careful though: fluid uniformly rotating with angular velocity vector Ω has uniform vorticity 2Ω.

(b) curl grad $\phi = 0$ for any scalar ϕ. To see this, just put $\nabla\phi$ in the formula for curl, and all the terms cancel. Since the pressure force is ∇p in the Navier-Stokes equation, taking the curl of the Navier-Stokes equation gets rid of the pressure.

(c) curl curl $\mathbf{v}$ = grad div $\mathbf{v}$ - del-squared $\mathbf{v}$, acting on any vector field $\mathbf{v}$. Here del-squared is the vector version of the operator, $\nabla^{\mathbf{2}}$, which in Cartesians is $(\nabla^2 u_x, \nabla^2 u_y, \nabla^2 u_z)$. This curl curl vector identity works for any coordinate system, but the definition of the vector del-squared operator is complicated in polar coordinate systems. The formulae are in the section on curvilinear coordinates.

(iv) **Some useful vector identities**

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$
$$(\mathbf{a} \cdot \nabla)\mathbf{a} = \nabla(a^2/2) - \mathbf{a} \times (\nabla \times \mathbf{a})$$
$$\nabla \cdot (\phi\mathbf{a}) = \phi\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla\phi$$
$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$$
$$\nabla \times (\phi\mathbf{a}) = \phi\nabla \times \mathbf{a} + \nabla\phi \times \mathbf{a}$$
$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}\nabla \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{a}$$

2 Cartesian tensors and suffix notation

(i) Suffix notation

A vector field such as the fluid velocity, $\mathbf{u}$, can be represented by its coefficients (u_1, u_2, u_3) with respect to a set of Cartesian axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as a column vector,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

This can be represented more compactly as u_i where the index i is understood to take the values 1,2 and 3.

The gradient of the pressure ∇p is another vector quantity and is given in matrix form by

$$\nabla p = \begin{pmatrix} \frac{\partial p}{\partial x_1} \\ \\ \frac{\partial p}{\partial x_2} \\ \\ \frac{\partial p}{\partial x_3} \end{pmatrix}.$$

or more compactly in index notation as $\frac{\partial p}{\partial x_j}$ where j is now the index.

The scalar product of these two vectors $\mathbf{u} \cdot \nabla p$ is given by

$$\mathbf{u} \cdot \nabla p = u_1 \frac{\partial p}{\partial x_1} + u_2 \frac{\partial p}{\partial x_2} + u_3 \frac{\partial p}{\partial x_3}$$

and in suffix notation is written in the compact format

$$\mathbf{u} \cdot \nabla p = u_i \frac{\partial p}{\partial x_i}.$$

Here, we are using the Einstein convention that a repeated suffix denotes summation over $i = 1, 2, 3$.

The basic rules of suffix notation are

(a) A suffix that appears once is called a *free* index. The number free indices denote the type of quantity in question. A scalar quantity has no free indices, a vector one and in general an nth rank tensor has n. Terms that are added or equated must have the same free indices.

(b) If a suffix appears twice it is called a *dummy* index. Since we sum over dummy indices the number of pairs of dummy indices does not affect the type of the quantity being described. It is also possible to change the index letter, without affecting the result. However, it is important not to use a letter already in use as a free index.

It is conventional to use the letters $i, j, k, l, m, n \ldots$ as indices, but you can use whatever letters you like.

As we have already seen, taking the gradient of a scalar produces a vector quantity and so taking the gradient of vector produces a quantity with two associated directions, called a second rank tensor. Since there are three components of velocity and three coordinate directions $\nabla \mathbf{u}$ has 9 components. It can be represented in the form of a matrix as

$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}.$$

but it is much more convenient to write this in suffix notation as

$$\frac{\partial u_i}{\partial x_j}$$

where there are now two indices i and j each of which can take values $1, 2, 3$. In terms of the matrix representation i denotes the row and j the column of the entry.

The Kronecker delta δ_{ij} is another example of a second rank tensor

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1}$$

which in matrix representation is the identity matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(iii) **Scalar and Vector Products**

We have already seen that the scalar product of two vectors is equivalent to summing over a pair of suffices, so that

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i.$$

This operation is equivalent to the action of the Kronecker delta on the two vectors $\mathbf{a}$ and $\mathbf{b}$ since

$$\mathbf{a} \cdot \mathbf{b} = a_i \delta_{ij} b_j = a_i b_i \qquad \text{since } \delta_{ij} b_j = b_i.$$

The vector product can also be represented in Einstein notation by introducing the alternating tensor ϵ_{ijk}

$$\epsilon_{ijk} = \begin{cases} 1 & ijk = \text{even, } i.e. \text{ 123, 231 or 312} \\ -1 & ijk = \text{odd, } i.e. \text{ 132, 213, 321} \\ 0 & i = j,\ j = k, \text{ or } k = i, \end{cases} \tag{2}$$

which is a third rank tensor. Since ϵ_{ijk} has 3 free indices the resulting quantity $\epsilon_{ijk}a_jb_k$ is a vector with index i. (j and k are repeated indices and so are summed over). Using the properites of ϵ_{ijk} we find that

$$c_i = \epsilon_{ijk}a_jb_k$$

has the following components

$$c_1 = a_2b_3 - a_3b_2, \qquad c_2 = a_3b_1 - a_1b_3, \qquad c_3 = a_1b_2 - a_2b_1$$

and so represents the vector product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

We can extend these products to tensors, so for example $\mathbf{a}\cdot\mathbf{A} = a_iA_{ij}$ is a vector formed from the scalar product of the vector $\mathbf{a}$ with the first index of the tensor, A_{ij}. By convention dot signifies scalar product of the two neighbouring indices. This product may be performed using the matrix notation by writing $\mathbf{a}$ as a row vector and then multiplying it by the matrix $\mathbf{A}$,

$$\mathbf{a}\cdot\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Similarly the $\mathbf{A}\cdot\mathbf{a} = A_{ij}a_j$ may be performed in matrix notation as

$$\mathbf{A}\cdot\mathbf{a} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Note that these two scalar products give different results unless $\mathbf{A}$ is symmetric ($i.e. A_{ij} = A_{ji}$). For example, if we use $K_{ij} = \dfrac{\partial u_i}{\partial x_j}$ to denote the velocity gradient then

$$[\mathbf{K}\cdot\mathbf{u}]_j = K_{ji}u_i = u_iK_{ji} = u_i\frac{\partial u_j}{\partial x_i} = [\mathbf{u}\cdot\nabla\mathbf{u}]_j$$

whereas

$$[\mathbf{u}\cdot\mathbf{K}]_j = u_iK_{ij} = u_i\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\frac{\partial}{\partial x_j}(u_iu_i) = \frac{1}{2}[\nabla u^2]_j.$$

We can also form products between two indices on the same tensor. For example

$$\delta_{ij}A_{ij} = A_{ii} = A_{11} + A_{22} + A_{33} = \mathrm{Tr}\mathbf{A}$$

the trace of matrix $\mathbf{A}$, which is a scalar quantity.

The scalar product of two second rank tensors $\mathbf{A}$ and $\mathbf{B}$ is another second rank tensor $\mathbf{C} = \mathbf{A}\cdot\mathbf{B}$ where

$$C_{ij} = A_{ik}B_{kj}.$$

This is equivalent to matrix multiplication. We can also form the double dot product $\mathbf{A}:\mathbf{B}$, which is the scalar formed by contracting i with j.

$$\mathbf{A}:\mathbf{B} = \delta_{ij}A_{ik}B_{kj} = A_{ik}B_{ki},$$

which is equal to the trace of the $\mathbf{C}$. Note that suffix notation removes any ambiguity over which components are contracted with each other.

We can also apply cross-products between components of a tensor, for example

$$c_i = \epsilon_{ijk}A_{jk}$$

is a vector with components

$$c_1 = A_{23} - A_{32}, \qquad c_2 = A_{31} - A_{13}, \qquad c_3 = A_{12} - A_{21}.$$

Finally we have the triple product rule,

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$

which results from the following relationship between ϵ_{ijk} and δ_{ij}

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{3}$$

One way to remember this rule is (second-with-second $\times$ third-with-third - alternative pairings).

(iii) **Grad, Div and Curl in suffix notation**

We have already seen that we can write the gradient of a scalar of a vector as

$$\frac{\partial p}{\partial x_j} \quad \text{and} \quad \frac{\partial u_i}{\partial x_j} \quad \text{respectively.}$$

Taking the gradient increases the rank of the quantity by one, from scalar $\rightarrow$ vector, vector $\rightarrow$ second rank tensor, etc.

The divergence is obtained by form the dot product between the derivative and one of the indices of the tensor. For a vector $\mathbf{u}$,

$$\nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i}$$

Note that this is simply the gradient operator together with δ_{ij}. Similarly we can define the divergence of a tensor A_{ij} as

$$\frac{\partial}{\partial x_i} A_{ij},$$

and is a vector quantity. Note that the summation can be over either of the two indices, so we can obtain a second vector using the second index,

$$\frac{\partial}{\partial x_j} A_{ij}.$$

By convection the notation, $\nabla \cdot \mathbf{A}$ is taken to mean summation over the first index (the one closest to the dot), but the potential for ambiguities in this formulation means that it is better to stick to suffix notation when dealing with tensors.

Finally we can obtain the curl of a vector or tensor by the operation of ϵ_{ijk} on the gradient, so for example

$$[\nabla \times \mathbf{u}]_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}.$$

3 Velocity Gradient, Strain-rate and Vorticity

Let us now examine the velocity gradient $\frac{\partial u_i}{\partial x_j}$. For an incompressible $\nabla \cdot \mathbf{u} = 0$ and so this tensor has zero trace, but there are still 8 remaining components. A useful simplification is to divide into the sum of a symmetric and antisymmetric tensor, as

$$\frac{\partial u_i}{\partial x_j} = E_{ij} + \Omega_{ij}, \quad \text{where} \quad E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{and} \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{4}$$

It is easily verified that $E_{ij} = E_{ji}$ and $\Omega_{ij} = -\Omega_{ji}$.

The symmetric tensor, $\mathbf{E}$, is called the strain-rate tensor and the antisymmetric tensor, $\boldsymbol{\Omega}$ is called the vorticity tensor, because its non-zero elements are related to the elements of the vorticity, ω. Recall that the vorticity ω is defined as the $\nabla \times \mathbf{u}$ and so in index notation it is defined as

$$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}. \tag{5}$$

Multiplying this equation by ϵ_{ilm} and using the triple product rule, we obtain

$$\epsilon_{ilm}\omega_i = \epsilon_{ijk}\epsilon_{ilm}\frac{\partial u_k}{\partial x_j} = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\frac{\partial u_k}{\partial x_j} = \frac{\partial u_m}{\partial x_l} - \frac{\partial u_l}{\partial x_m} = 2\Omega_{ml},$$

so that

$$\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k. \tag{6}$$

This result is clear if we write $\boldsymbol{\Omega}$ in matrix notation

$$\boldsymbol{\Omega} = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} & \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial u_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} & 0 & \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial u_2} \\ \frac{\partial u_3}{\partial x_1} - \frac{\partial u_2}{\partial u_3} & \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

4 Div grad and curl in curvilinear coordinates

4.1 General curvilinear coordinates q_1, q_2, q_3

Let $\mathbf{e}_i$ be a unit vector along the q_i-axis, with

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2 \, .$$

$$\text{Then} \quad \nabla\Phi = \sum_i \frac{1}{h_i}\frac{\partial \Phi}{\partial q_i}\mathbf{e}_i$$

$$\nabla\cdot\mathbf{F} = \frac{1}{h_1 h_2 h_3}\sum_{i,j,k}\frac{\partial}{\partial q_i}(h_j h_k F_i)$$

(where here and subsequently the sum is over i with j and k selected cyclically, so that for $i = 1$ j and k are 2 and 3 respectively, for $i = 2$ they are 3 and 1, for $i = 3$ they are 1 and 2)

$$\nabla\times\mathbf{F} = \sum_{i,j,k}\frac{1}{h_j h_k}\Big[\frac{\partial}{\partial q_j}(h_k F_k) - \frac{\partial}{\partial q_k}(h_j F_j)\Big]\mathbf{e}_i$$

$$\nabla^2\Phi = \frac{1}{h_1 h_2 h_3}\sum_{i,j,k}\frac{\partial}{\partial q_i}\Big(\frac{h_j h_k}{h_i}\frac{\partial\Phi}{\partial q_i}\Big)$$

$$\nabla^2\mathbf{F} = \nabla(\nabla\cdot\mathbf{F}) - \nabla\times(\nabla\times\mathbf{F})$$

$$(\mathbf{B}\cdot\nabla)\mathbf{A} = \sum_{i,j,k}\Big\{\mathbf{B}\cdot\nabla A_i + \frac{A_j}{h_i h_j}\Big(B_i\frac{\partial h_i}{\partial q_j} - B_j\frac{\partial h_j}{\partial q_i}\Big) + \frac{A_k}{h_i h_k}\Big(B_i\frac{\partial h_i}{\partial q_k} - B_k\frac{\partial h_k}{\partial q_i}\Big)\Big\}\mathbf{e}_i$$

4.2 Cylindrical polar coordinates (R, ϕ, z)

In these coordinates $h_R = 1$, $h_\phi = R$ and $h_z = 1$.

$$\nabla\Phi = \frac{\partial\Phi}{\partial R}\widehat{\mathbf{R}} + \frac{1}{R}\frac{\partial\Phi}{\partial\phi}\widehat{\boldsymbol{\phi}} + \frac{\partial\Phi}{\partial z}\widehat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{R}\frac{\partial}{\partial R}(RF_R) + \frac{1}{R}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}$$

$$\nabla\times\mathbf{F} = \Big[\frac{1}{R}\frac{\partial F_z}{\partial\phi} - \frac{\partial F_\phi}{\partial z}\Big]\widehat{\mathbf{R}} + \Big[\frac{\partial F_R}{\partial z} - \frac{\partial F_z}{\partial R}\Big]\widehat{\boldsymbol{\phi}} + \frac{1}{R}\Big[\frac{\partial}{\partial R}(RF_\phi) - \frac{\partial F_R}{\partial\phi}\Big]\widehat{\mathbf{z}}$$

$$\nabla^2\Phi = \frac{1}{R}\frac{\partial}{\partial R}\Big(R\frac{\partial\Phi}{\partial R}\Big) + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\begin{aligned}\nabla^2\mathbf{F} \quad &= \quad \Big[\nabla^2 F_R - \frac{1}{R^2}F_R - \frac{2}{R^2}\frac{\partial F_\phi}{\partial\phi}\Big]\widehat{\mathbf{R}} \\ &+ \quad \Big[\nabla^2 F_\phi - \frac{1}{R^2}F_\phi + \frac{2}{R^2}\frac{\partial F_R}{\partial\phi}\Big]\widehat{\boldsymbol{\phi}} + \nabla^2 F_z\,\widehat{\mathbf{z}}\end{aligned}$$

$$\begin{aligned}(\mathbf{B}\cdot\nabla)\mathbf{A} \quad &= \quad \big[\mathbf{B}\cdot\nabla A_R - B_\phi A_\phi/R\big]\widehat{\mathbf{R}} \\ &+ \quad \big[\mathbf{B}\cdot\nabla A_\phi + B_\phi A_R/R\big]\widehat{\boldsymbol{\phi}} + \mathbf{B}\cdot\nabla A_z\,\widehat{\mathbf{z}}\end{aligned}$$

4.3 Spherical polar coordinates (r, θ, ϕ)

In these coordinates $h_r = 1$, $h_\theta = r$ and $h_\phi = r\sin\theta$.

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\widehat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\widehat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\widehat{\boldsymbol{\phi}}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2 F_r\Big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta F_\theta\Big) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \frac{1}{r\sin\theta}\Big[\frac{\partial}{\partial\theta}(\sin\theta F_\phi) - \frac{\partial F_\theta}{\partial\phi}\Big]\widehat{\mathbf{r}} \\
&+ \frac{1}{r}\Big[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi)\Big]\widehat{\boldsymbol{\theta}} + \frac{1}{r}\Big[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\Big]\widehat{\boldsymbol{\phi}}
\end{aligned}
$$

$$
\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial\Phi}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial\Phi}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
$$

$$
\begin{aligned}
\nabla^2\mathbf{F} &= \Big[\nabla^2 F_r - \frac{2}{r^2}F_r - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta F_\theta\Big) - \frac{2}{r^2\sin\theta}\frac{\partial F_\phi}{\partial\phi}\Big]\widehat{\mathbf{r}} \\
&+ \Big[\nabla^2 F_\theta - \frac{1}{r^2\sin^2\theta}F_\theta + \frac{2}{r^2}\frac{\partial F_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\phi}{\partial\phi}\Big]\widehat{\boldsymbol{\theta}} \\
&+ \Big[\nabla^2 F_\phi - \frac{1}{r^2\sin^2\theta}F_\phi + \frac{2}{r^2\sin\theta}\frac{\partial F_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\theta}{\partial\phi}\Big]\widehat{\boldsymbol{\phi}}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{B}\cdot\nabla)\mathbf{A} &= \big[\mathbf{B}\cdot\nabla A_r - (B_\theta A_\theta + B_\phi A_\phi)/r\big]\widehat{\mathbf{r}} \\
&+ \big[\mathbf{B}\cdot\nabla A_\theta + (B_\theta A_r - \cot\theta B_\phi A_\phi)/r\big]\widehat{\boldsymbol{\theta}} \\
&+ \big[\mathbf{B}\cdot\nabla A_\phi + (B_\phi A_r + \cot\theta B_\phi A_\theta)/r\big]\widehat{\boldsymbol{\phi}}
\end{aligned}
$$

4.4 Unit vectors in spherical polar coordinates (r, θ, ϕ)

$$
x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta
$$

$$
\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}
$$

$$
\widehat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}
$$

$$
\widehat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}
$$

$$
\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\widehat{\boldsymbol{\theta}} - \sin\phi\,\widehat{\boldsymbol{\phi}}
$$

$$
\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\widehat{\boldsymbol{\theta}} + \cos\phi\,\widehat{\boldsymbol{\phi}}
$$

$$
\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\widehat{\boldsymbol{\theta}}
$$

$$
B_x = \sin\theta\cos\phi\,B_r + \cos\theta\cos\phi\,B_\theta - \sin\phi\,B_\phi
$$

$$
B_y = \sin\theta\sin\phi\,B_r + \cos\theta\sin\phi\,B_\theta + \cos\phi\,B_\phi
$$

$$
B_z = \cos\theta\,B_r - \sin\theta\,B_\theta
$$

$$
B_r = \sin\theta\cos\phi\,B_x + \sin\theta\sin\phi\,B_y + \cos\theta\,B_z
$$

$$
B_\theta = \cos\theta\cos\phi\,B_x + \cos\theta\sin\phi\,B_y - \sin\theta\,B_z
$$

$$
B_\phi = -\sin\phi\,B_x + \cos\phi\,B_y
$$

4.5 Incompressible Navier-Stokes equation with no body force in cylindrical polar coordinates

$$
\begin{aligned}
\frac{\partial u_R}{\partial t} + \mathbf{u}\cdot\nabla u_R - \frac{u_\phi^2}{R} &= -\frac{1}{\rho}\frac{\partial p}{\partial R} + \nu\left[\nabla^2 u_R - \frac{u_R}{R^2} - \frac{2}{R^2}\frac{\partial u_\phi}{\partial\phi}\right] \\
\frac{\partial u_\phi}{\partial t} + \mathbf{u}\cdot\nabla u_\phi + \frac{u_R u_\phi}{R} &= -\frac{1}{\rho R}\frac{\partial p}{\partial\phi} + \nu\left[\nabla^2 u_\phi - \frac{u_\phi}{R^2} + \frac{2}{R^2}\frac{\partial u_R}{\partial\phi}\right] \\
\frac{\partial u_z}{\partial t} + \mathbf{u}\cdot\nabla u_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z
\end{aligned}
$$

4.6 Incompressible Navier-Stokes equation with no body force in spherical polar coordinates

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + \mathbf{u}\cdot\nabla u_r - \frac{u_\theta^2}{r} - \frac{u_\phi^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right] \\
\frac{\partial u_\theta}{\partial t} + \mathbf{u}\cdot\nabla u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cot\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left[\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial u_\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right] \\
\frac{\partial u_\phi}{\partial t} + \mathbf{u}\cdot\nabla u_\phi + \frac{u_r u_\phi}{r} + \frac{u_\theta u_\phi\cot\theta}{r} &= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi} + \nu\left[\nabla^2 u_\phi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi} - \frac{u_\phi}{r^2\sin^2\theta}\right]
\end{aligned}$$

4.7 Velocity gradient Tensor in cylindrical and spherical polar coordinates

In curvilinear coordinates systems the velocity gradient tensor $K_{ij} = \frac{\partial u_i}{\partial u_j}$ has additional terms due to changes in the coordinate directions.

In cylindrical polar coordinates:

$$\mathbf{K} = \begin{pmatrix} \dfrac{\partial u_R}{\partial R} & \dfrac{1}{R}\dfrac{\partial u_R}{\partial\phi} - \dfrac{u_\phi}{R} & \dfrac{\partial u_R}{\partial z} \\[2ex] \dfrac{\partial u_\phi}{\partial R} & \dfrac{1}{R}\dfrac{\partial u_\phi}{\partial\phi} + \dfrac{u_R}{R} & \dfrac{\partial u_\phi}{\partial z} \\[2ex] \dfrac{\partial u_z}{\partial R} & \dfrac{1}{R}\dfrac{\partial u_z}{\partial\phi} & \dfrac{\partial u_z}{\partial z} \end{pmatrix}$$

In spherical polar coordinates:

$$\mathbf{K} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta} - \dfrac{u_\theta}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial\phi} - \dfrac{u_\phi}{r} \\[2ex] \dfrac{\partial u_\theta}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_r}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial\phi} - \dfrac{u_\phi}{r}\cot\theta \\[2ex] \dfrac{\partial u_\phi}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_\phi}{\partial\theta} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial\phi} + \dfrac{u_r}{r} + \dfrac{u_\theta}{r}\cot\theta \end{pmatrix}$$

5 Solutions of $\nabla^2 V = 0$

(i) Solutions of $\nabla^2 V = 0$ in two dimensions.

The most useful solutions here are in polar coordinates, (r, θ). Then

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} = 0.$$

Look for separable solutions $f(r)\sin n\theta$ where n is an integer. Then

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2 f}{r^2} = 0.$$

Look for solutions of the form $f = r^p$. Then $p^2 r^{p-2} - n^2 r^{p-2} = 0$, so $p = n$ or $p = -n$. So we have solutions

$$V = \frac{A}{r^n}\sin n\theta + Br^n\sin n\theta.$$

Clearly

$$V = \frac{C}{r^n}\cos n\theta + Dr^n\cos n\theta$$

works just as well. Since the problem is linear we can add together any of these solutions and its also a solution. Can also add together different n solutions.

If we want solutions to decay at infinity (often used for flow round obstacles where we match on to some uniform flow at infinity) then its the A and C solutions we need, though the $n = 1$ case $V = Br\sin\theta = By$ which has $\mathbf{u} = \nabla V = B\hat{\mathbf{y}}$ i.e. a uniform flow in the y direction.

On the other hand, if we want solutions finite at the origin, then we want the B and D solutions only.

Example: irrotational flow past a cylinder

Here $n = 1$, $C = U_0$, $D = U_0 a^2$ where a is the radius of the cylinder, so $V = U_0 r\cos\theta + U_o a^2\cos\theta/r$, so

$$\nabla V = \left(U_0\cos\theta - \frac{U_0 a^2\cos\theta}{r^2}\right)\hat{\mathbf{r}} - \left(U_0\sin\theta + \frac{U_0 a^2\sin\theta}{r^2}\right)\hat{\boldsymbol{\theta}},$$

which gives uniform flow in the x-direction ($\theta = 0$ direction in 2D polars) and no flow through the cylinder at $r = a$.

(ii) Axisymmetric solutions of $\nabla^2 V = 0$.

These are usually discussed in spherical polar coordinates (r, θ, ϕ) but if we are axisymmetric, the flow is independent of ϕ. In these circumstances

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0.$$

We seek separable solutions of the form $V = r^p y(\theta)$. Plugging in the r dependence, we get

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right) + p(p+1)y = 0$$

which is Legendre's equation with variable $\cos\theta$. If we put $x = \cos\theta$ we get

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + Ky = 0, \quad K = p(p+1),$$

which is the more standard form of Legendre's equation with x as the independent variable. We are only interested in solutions which are finite at the poles, $\theta = 0$ and $\theta = \pi$ which correspond to $x = \pm 1$. These regular solutions only exist when $K = n(n+1)$ for some non-negative integer n. The most useful solutions are

$$n = 0,\ y = 1; \quad n = 1,\ y = x; \quad n = 2,\ y = \frac{1}{2}(3x^2 - 1),$$

which can be verified by direct substitution. There is a polynomial solution of degree n for every integer n: they are known as the Legendre polynomials. They have lots of nice mathematical properties: properties are listed in the *NIST Handbook of Mathematical Functions* by Olver, Lozier, Boisvert and Clark. This text book also has all the properties of Bessel functions and all the other Mathematical functions that turn up in Applied Mathematics.

Since $K = n(n+1) = p(p+1)$ an obvious soution is $n = p$, but $p = -(n+1)$ also satisfies $n(n+1) = p(p+1)$. So there are two types of solution, those that are finite at $r = 0$,

$$V = 1,\ (n = 0), \quad V = r\cos\theta,\ (n = 1), \quad V = \frac{r^2}{2}(3\cos^2\theta - 1),\ (n = 2),\ \text{etc.}$$

and those that go to zero at large r (the $p = -n-1$ solutions)

$$V = \frac{1}{r},\ (n = 0), \quad V = \frac{\cos\theta}{r^2},\ (n = 1) \quad V = \frac{1}{2r^3}(3\cos^2\theta - 1),\ (n = 2),\ \text{etc.}$$

Example: spherically symmetric bubble

The time-dependent solution of the flow surrounding a collapsing or expanding spherically symmetric bubble is $A(t)/r$ because this is the only spherically symmetric potential flow that vanishes at infinity. Another familiar example is the gravitational potential outside a sphere GM/r which has to have this form as its the only spherically symmetric potential that vanishes at infinity.

Example: irrotational flow past a sphere

Sphere has radius a and flow at infinity is $U_0\hat{\mathbf{z}}$.

$$V = U_0 r\cos\theta + \frac{U_0 a^3\cos\theta}{2r^2},$$

which combines the two $n = 1$ solutions. Note $\partial V/\partial r = u_r = 0$ on $r = a$ the surface of the sphere, and $u_\theta = -(1/r)\partial V/\partial\theta = -U_0\sin\theta - U_0 a^3\sin\theta/2r^3$, and since $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$ the flow at infinity has the correct form. Note this is the potential flow, not the Stokes flow round the sphere, so the no-slip boundary at $r = a$ is not satisfied by this potential flow solution.

Some non-fluids uses of solutions of Laplace's equations

Gravitational potential: the moon raises tides on the surface of the Earth. The moon's gravitation adds a quadrupole term to the Earth's gravitational field

$$V = \frac{GM}{r} + \frac{J_2}{2r^3}(3\cos^2\theta - 1)$$

where the axis of coordinates here is pointing towards the moon.

Magnetic fields often obey $\nabla^2 V = 0$, with $\mathbf{B} = \nabla V$. A bar magnet parallel to the z axis gives rise to a dipole magnetic potential $m\cos\theta/r^2$. $B_r = \partial V/\partial r$ and $B_\theta = (1/r)\partial V/\partial\theta$ then give the dipolar magnetic field familiar from iron filings experiments.

Wacky versions of Laplace's equation

Sometimes equations turn up which look like Laplace's equation but are not Laplace's equation.

Example: the axisymmetric Stokes' stream function in spherical polars is

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r},$$

so the vorticity is given by

$$\omega_\phi = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial\theta} = -\frac{1}{r\sin\theta}\left[\frac{\partial^2\Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right)\right] = -\frac{1}{r\sin\theta}E^2\Psi,$$

the right-hand side operator in the square brackets looks similar to the Laplacian of Ψ, but is a different second order differential operator, sometimes called E^2. However, if we choose a new variable $\hat{\Psi}$ such that $\Psi = r\sin\theta\partial\hat{\Psi}/\partial\theta$ then

$$E^2(\Psi) = r\sin\theta\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\hat{\Psi}}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\hat{\Psi}}{\partial\theta}\right)\right] = r\sin\theta\frac{\partial}{\partial\theta}\nabla^2\hat{\Psi}.$$

So if the flow is irrotational, $\hat{\Psi}$ satisfies the standard Laplace equation with its standard solutions, to get the solutions for Ψ just evaluate $r\sin\theta\partial\hat{\Psi}/\partial\theta$. So using this transformation, we can write down the solutions using the solutions of Laplace's equation.

(iii) **Non-axisymmetric solutions of $\nabla^2 V = 0$.**

These involve spherical harmonics. They are used (i) in numerical methods for solving problems where spherical geometry is natural, (ii) to describe the gravitational field of the Earth or the magnetic field of the Earth or other planets, (iii) acoustic scattering from spherical particles, (iv) in quantum mechanics, seismology and lots of other scientific disciplines.

The full Laplace equation in spherical polars is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0.$$

We proceed in the same way as in the axisymmetric case, looking for solutions of the form

$$V = r^p y(\theta)\exp(im\phi).$$

We get an ODE for $y(\theta)$ which is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right)+p(p+1)y-\frac{m^2}{\sin^2\theta}y=0.$$

This is called the associated Legendre equation (also discussed extensively in chapter 14 of NIST Handbook). The Wikipedia entry on 'Spherical harmonics' is also helpful, and has nice pictures of what the simpler spherical harmonics look like. As before, the associated Legendre equation only has solutions that are finite at the poles if $p(p+1)=n(n+1)$ for some integer n. So again the r dependence corresponding to the n solution is either r^n or $1/r^{n+1}$. The solutions of

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right)+n(n+1)y-\frac{m^2}{\sin^2\theta}y=0$$

are called associated Legendre functions, written

$$y(\cos\theta)=P_n^m(\cos\theta)$$

where n is called the degree of the associated Legendre function and m is the order. m must be $\leq n$. If $m=0$, the associated Legendre function of degree n is the same as the Legendre polynomial of degree n.

The functions

$$Y_n^m(\theta,\phi)=AP_n^m(\cos\theta)\exp(im\phi)$$

are called the spherical harmonics.

As they are solutions of a linear differential equation, there is an arbitrary constant A multiplying the solution. To make the spherical harmonics definite we need to choose a definite constant for each spherical harmonic. This is called the normalization of the spherical harmonics. Unfortunately, because spherical harmonics are useful in so many different fields, e.g. quantum mechanics, magnetism, gravitation, seismology, fluid dynamics etc, and every different group of scientists normalized the spherical harmonics in a different way, there is now massive confusion in the scientific literature! For example, a large part of the Wikipedia article is devoted to the normalization issue. The definition of the Legendre polynomials is now completely standard as defined in the NIST Handbook. The Associated Legendre functions are now also standardly defined as

$$P_n^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^mP_n(x)}{dx^m}.$$

The complete solution of Laplace's equation that decays at infinity is

$$V=\sum_{n=0}^{n=\infty}\sum_{m=0}^{m=n}\frac{1}{r^{n+1}}P_n^m(\cos\theta)(A_{nm}\cos m\phi+B_{nm}\sin m\phi).$$

and the solution that is finite at the origin has the same form but with r^n replacing $1/r^{n+1}$.

Some of the simpler associated Legendre functions are (here $x=\cos\theta$)

$P_n^m(x)=$: (i) $n=0,m=0:$ $P_0^0=1$. (ii) $n=1,m=0:$ $P_1^0=x$. (iii) $n=1,m=1:$ $P_1^1=-(1-x^2)^{1/2}$.

(iv) $n=2,m=0:$ $P_2^0=\frac{1}{2}(3x^2-1)$. (v) $n=2,m=1:$ $P_2^1=-3x(1-x^2)^{1/2}$. (vi) $n=2,m=2:$ $P_2^2=3(1-x^2)$.