MATH 3620 Fluid Dynamics 2
Example sheet 5

Exercise 1

(a) $\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\psi}{\partial x\partial y}$ and $\frac{\partial^2\phi}{\partial y^2} = -\frac{\partial^2\psi}{\partial y\partial x}$ $\Rightarrow$ $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\psi}{\partial y^2}$ and $-\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\psi}{\partial x^2}$ $\Rightarrow$ $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$

(b)
$$\begin{aligned}\nabla\psi\cdot\nabla\phi &= \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial y}\\ &= \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\partial\psi}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right)\\ &= 0\end{aligned}$$

Exercise 2

$$\begin{aligned}\mathrm{Re}\left[\int_{\mathcal{C}} \frac{dw}{dz}\,dz\right] &= \mathrm{Re}\left[\int_{\mathcal{C}} (u-iv)(dx+idy)\right]\\ &= \mathrm{Re}\left[\int_{\mathcal{C}} u\,dx + \int_{\mathcal{C}} v\,dy + i\int_{\mathcal{C}} u\,dy - i\int_{\mathcal{C}} v\,dx\right]\\ &= \int_{\mathcal{C}} u\,dx + \int_{\mathcal{C}} v\,dy\\ &= \int_{\mathcal{C}} \vec{u}\cdot d\vec{x}\\ &= \Gamma\end{aligned}$$

Exercise 3

(a)
$$\begin{aligned}w(a\cos\theta + ib\sin\theta) &= \frac{U}{a-b}\left[a(a\cos\theta+ib\sin\theta) - b\left((a\cos\theta+ib\sin\theta)^2 - a^2+b^2\right)^{1/2}\right]\\ &= \frac{U}{a-b}\left[a^2\cos\theta + iab\sin\theta - b\left(a^2\cos^2\theta + 2iab\cos\theta\sin\theta - b^2\sin^2\theta - a^2+b^2\right)^{1/2}\right]\\ &= \frac{U}{a-b}\left[a^2\cos\theta + iab\sin\theta - b\left(a^2\sin^2\theta + 2iab\cos\theta\sin\theta + b^2\cos^2\theta\right)^{1/2}\right]\\ &= \frac{U}{a-b}\left[a^2\cos\theta + iab\sin\theta - b(b\cos\theta + ia\sin\theta)\right]\\ &= \frac{U}{a-b}(a^2-b^2)\cos\theta\\ &= U(a+b)\cos\theta \in \mathbb{R}\end{aligned}$$

$\Rightarrow \psi = 0$ on $z = a\cos\theta + ib\sin\theta$, that is, on $x = a\cos\theta$, $y = b\sin\theta$
or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow$ the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a streamline.

(b) $\lim_{|z|\to\infty} (z^2 - a^2 + b^2)^{1/2} = z$ so that:

$$\lim_{|z|\to\infty} \frac{U}{a-b}\left[az - b(z^2-a^2+b^2)^{1/2}\right] = \lim_{|z|\to\infty} \frac{U}{a-b}(az - bz) \simeq Uz$$

Away from the ellipse, the flow is uniform. It is thus the flow past an ellipse.

Exercise 4

- The imaginary axis corresponds to $z = iy \Rightarrow w(iy) = f(iy) + \overline{f(-\overline{iy})}$
$= f(iy) + \overline{f(iy)} \in \mathbb{R}$

$\Rightarrow \psi = 0$ on the imaginary axis which is thus a streamline.

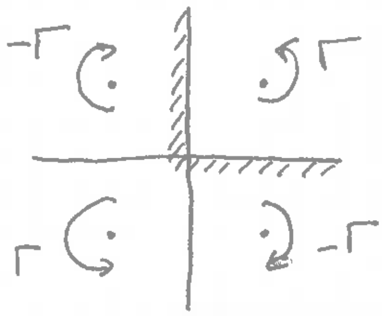
- $f(z) = -\frac{i\Gamma}{2\pi}\log(z - z_0)$ with $z_0 = a + ib$

$$\Rightarrow w(z) = f(z) + \overline{f(-\bar{z})}$$
$$= -\frac{i\Gamma}{2\pi}\log(z - z_0) + \frac{i\Gamma}{2\pi}\log\left(\overline{-\bar{z} - z_0}\right)$$
$$= \frac{i\Gamma}{2\pi}\left[-\log(z - z_0) + \log(-z - \bar{z}_0)\right]$$
$$= \frac{i\Gamma}{2\pi}\left[\log(-1(z + \bar{z}_0)) - \log(z - z_0)\right]$$
$$= \frac{i\Gamma}{2\pi}\left[i\pi + \log\frac{z + \bar{z}_0}{z - z_0}\right]$$
$$= \frac{i\Gamma}{2\pi}\log\frac{z + \bar{z}_0}{z - z_0} - \frac{\Gamma}{2}$$
$$\Rightarrow w(z) = \frac{i\Gamma}{2\pi}\log\frac{z + \bar{z}_0}{z - z_0}$$ as $w(z)$ defined up to a constant

- Image system when wall along $y = 0$: $W(z) = F(z) + \overline{F(\bar{z})}$

Image system when wall along $x = 0$: $F(z) = f(z) + \overline{f(-\bar{z})}$

$$\Rightarrow W(z) = f(z) + \overline{f(-\bar{z})} + \overline{f(\bar{z}) + \overline{f(-\bar{\bar{z}})}}$$
$$= f(z) + \overline{f(-\bar{z})} + \overline{f(\bar{z})} + f(-z)$$
$$= f(z) + f(-z) + \overline{f(\bar{z}) + f(-\bar{z})}$$
$$= -\frac{i\Gamma}{2\pi}\left[\log(z - z_0) + \log(-z - z_0) - \log(z - \bar{z}_0) - \log(-z - \bar{z}_0)\right]$$
$$= -\frac{i\Gamma}{2\pi}\left[\log\frac{(z - z_0)(z + z_0)}{(z - \bar{z}_0)(z + \bar{z}_0)}\right]$$



Exercise 5

- Consider the $y = 0$, $x \leq 0$ axis: $z = e^{i\pi}$ or $z = e^{-i\pi}$

Applying the mapping, we get: $Z = e^{i3\pi/4}$ or $Z = e^{-i3\pi/4}$. As a result, the line $y = 0$, $x \leq 0$ splits into two lines:

- $w(z) = -Uz \Rightarrow W(Z) = -UZ^{4/3}$

<u>Exercise 6</u>

- We need to show that $\oint_{\mathcal{C}} Z\left(\frac{dW}{dZ}\right)^2 dZ = \oint_{\mathcal{C}_z} Z(z)\left(\frac{dw}{dz}\right)^2\left(\frac{dZ}{dz}\right)^{-1} dz$

  We know that $\frac{dW}{dZ} = \frac{dw}{dz}\frac{dz}{dZ}$ and $dZ = \frac{dZ}{dz}dz$

$$\Rightarrow \oint_{\mathcal{C}} Z\left(\frac{dW}{dZ}\right)^2 dZ = \oint_{\mathcal{C}_z} Z(z)\left(\frac{dw}{dz}\right)^2\left(\frac{dz}{dZ}\right)^2 \frac{dZ}{dz}\,dz$$
$$= \oint_{\mathcal{C}_z} Z(z)\left(\frac{dw}{dz}\right)^2\left(\frac{dZ}{dz}\right)^{-1}dz$$

- Flow past a cylinder: $w(z) = U_0 z e^{-i\alpha} + U_0 e^{i\alpha}\frac{a^2}{z} - \frac{i\Gamma}{2\pi}\log(z)$

$$\Rightarrow \frac{dw}{dz} = U_0 e^{-i\alpha} - U_0 e^{i\alpha}\frac{a^2}{z^2} - \frac{i\Gamma}{2\pi z}$$

Joukowski transformation: $Z = z + \frac{c^2}{z} \Rightarrow \frac{dZ}{dz} = 1 - \frac{c^2}{z^2}$

Since $c < a$: $\left(\frac{dZ}{dz}\right)^{-1} \simeq 1 + \frac{c^2}{z^2} + \dots$

$$\Rightarrow T = -\frac{\rho}{2}\mathrm{Re}\left[\oint_{\mathcal{C}_z} Z(z)\left(\frac{dw}{dz}\right)^2\left(\frac{dZ}{dz}\right)^{-1}dz\right]$$
$$= -\frac{\rho}{2}\mathrm{Re}\left[\oint_{|z|=a}\left(z+\frac{c^2}{z}\right)\left(U_0e^{-i\alpha} - U_0e^{+i\alpha}\frac{a^2}{z^2} - \frac{i\Gamma}{2\pi z}\right)^2\left(1+\frac{c^2}{z^2}+\dots\right)dz\right]$$

Remember: $\oint_{|z|=a} z^{-n}dz = \begin{cases}2\pi i & \text{for } n=1\\ 0 & \text{otherwise}\end{cases}$

$$\Rightarrow T = -\frac{\rho}{2}\mathrm{Re}\Big[\oint_{|z|=a}\left(z+\frac{c^2}{z}\right)\Big(U_0^2 e^{-2i\alpha} - 2U_0^2\frac{a^2}{z^2} - \frac{2i\Gamma U_0 e^{-i\alpha}}{2\pi z} - \frac{\Gamma^2}{4\pi^2 z^2}$$
$$+ U_0^2 e^{2i\alpha}\frac{a^4}{z^4} + \frac{2iU_0\Gamma e^{i\alpha}a^2}{2\pi z^3}\Big)\left(1+\frac{c^2}{z^2}+\dots\right)dz\Big]$$
$$= -\frac{\rho}{2}\mathrm{Re}\Big[\oint_{|z|=a}\left(z+\frac{c^2}{z}\right)\Big(U_0^2e^{-2i\alpha} + U_0^2 e^{-2i\alpha}\frac{c^2}{z^2} - 2U_0^2\frac{a^2}{z^2}$$
$$- \frac{\Gamma^2}{4\pi^2 z^2} + \dots\Big)dz\Big]$$
$$= -\frac{\rho}{2}\mathrm{Re}\left[\oint_{|z|=a}\left(\left(U_0^2c^2e^{-2i\alpha} + U_0^2e^{-2i\alpha}c^2 - 2U_0^2a^2 - \frac{\Gamma^2}{4\pi^2}\right)\frac{1}{z}+\dots\right)dz\right]$$
$$= -\rho\pi\,\mathrm{Re}\left[2iU_0^2c^2e^{-2i\alpha} - 2iU_0^2a^2 - \frac{i\Gamma^2}{4\pi^2}\right]$$
$$= -\rho\pi\left(-2U_0^2c^2\sin(-2\alpha)\right)$$
$$= -\rho\pi\, 2U_0^2 c^2\sin 2\alpha$$