MATH3620 Fluid Dynamics 2
Example sheet 4

## Exercise 1

(a) $$\begin{cases} \rho\,\partial_t u + \rho u\,\partial_x u + \rho v\,\partial_y u = -\partial_x p + \mu\,\partial_x^2 u + \mu\,\partial_y^2 u \\ \rho\,\partial_t v + \rho u\,\partial_x v + \rho v\,\partial_y v = -\partial_y p + \mu\,\partial_x^2 v + \mu\,\partial_y^2 v \\ \partial_x u + \partial_y v = 0 \quad \Rightarrow \quad \partial_y v = 0 \quad \Rightarrow \quad v \equiv 0 \end{cases}$$

$$\Rightarrow \quad \rho\,\partial_t u = \mu\,\partial_y^2 u$$

$$\Rightarrow \quad \partial_t u = \nu\,\partial_y^2 u \,, \quad \nu = \frac{\mu}{\rho}$$

(b) $u(y,t) = f(\eta)$ with $\eta = y t^a \Rightarrow \begin{cases} \partial_t u = f' a y t^{a-1} \\ \partial_y u = f' t^a \quad \Rightarrow \partial_y^2 u = f'' t^{2a} \end{cases}$

$$\Rightarrow a y t^{a-1} f' = \nu t^{2a} f''$$

$$\Rightarrow f'' - \frac{a y t^{-a-1}}{\nu} f' = 0$$

Similarity solution exists if $\quad y t^{-a-1} = \eta = y t^a \quad \Rightarrow a = -\frac{1}{2}$

$$\Rightarrow \eta = y t^{-1/2} \text{ and } f'' + \frac{\eta}{2\nu} f' = 0$$

Boundary conditions: $u \to U$ at $y \to +\infty \Rightarrow f \to U$ at $y \to +\infty$

$u \to -U$ at $y \to -\infty \Rightarrow f \to -U$ at $y \to -\infty$

(c) Let $F = f'$ : $\quad F' + \frac{\eta}{2\nu} F = 0 \quad \Rightarrow \frac{dF}{d\eta} = -\frac{\eta}{2\nu} F$

$$\Rightarrow F = k_1 e^{-\eta^2/4\nu}$$

$$\Rightarrow f = k_1 \int_0^\eta e^{-\eta^2/4\nu}\, d\eta + k_2$$

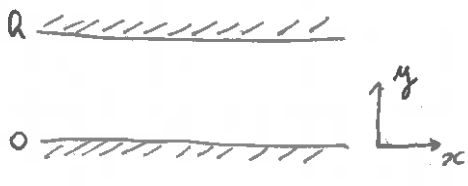
$$\Rightarrow f = k_1 \sqrt{\pi\nu}\, \operatorname{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) + k_2$$

$$\Rightarrow u = k_1 \sqrt{\pi\nu}\, \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) + k_2$$

$$\left.\begin{array}{l} \lim\limits_{y\to\infty} u = k_1\sqrt{\pi\nu} + k_2 = U \\ \lim\limits_{y\to-\infty} \mu = -k_1\sqrt{\pi\nu} + k_2 = -U \end{array}\right\} \Rightarrow k_2 = 0 \,; \; k_1 = \frac{U}{\sqrt{\nu\pi}}$$

$$\Rightarrow u = U \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)$$

## Exercise 2



(a) $\rho \partial_t u + \rho u \partial_x u + \rho v \partial_y u = -\partial_x p + \mu \partial_x^2 u + \mu \partial_y^2 u$

$u = u(y,t)\,;\ v = 0 \Rightarrow \rho \partial_t u = -\partial_x p + \mu \partial_y^2 u$

$\Rightarrow \partial_t u = +\frac{G}{\rho} + \nu \partial_y^2 u$

(b) $\partial_t u_{ss} = 0$

$\partial_y u_{ss} = \frac{G(h-2y)}{2\rho\nu} \Rightarrow \partial_{yy} u_{ss} = -\frac{G}{\rho\nu}$

$\Rightarrow \frac{G}{\rho} + \nu \partial_y^2 u = \frac{G}{\rho} - \nu \frac{G}{\rho\nu} = 0 = \partial_t u$

(c) $\partial_t u = \frac{G}{\rho} + \nu \partial_y^2 u \Rightarrow \partial_t u_{ss} + \partial_t v = \frac{G}{\rho} + \nu \partial_y^2 u_{ss} + \nu \partial_y^2 v$

$\Rightarrow \partial_t v = \nu \partial_y^2 v$

Boundary conditions: $u = 0$ on $y = 0, h \Rightarrow v = 0$ on $y = 0, h$.

(d) $v = Y(y)T(t) \Rightarrow YT' = \nu T Y''$

$\Rightarrow \frac{T'}{\nu T} = \frac{Y''}{Y} = k_1$

$T' = \nu k_1 T \Rightarrow T = k_2 \exp(\nu k_1 t)$

$Y'' = k_1 Y \Rightarrow Y = k_3 \exp(\sqrt{k_1}\, y) + k_4 \exp(-\sqrt{k_1}\, y)$

$\Rightarrow v = \exp(\nu k_1 t)\left[k_5 \exp(\sqrt{k_1}\, y) + k_6 \exp(-\sqrt{k_1}\, y)\right], \quad k_5 = k_2 k_3,\ k_6 = k_4 k_6$

At $y = 0$, $v = 0 \Rightarrow k_5 + k_6 = 0 \Rightarrow k_6 = -k_5$

$\Rightarrow v = k_5 \exp(\nu k_1 t)\left[\exp(\sqrt{k_1}\, y) - \exp(-\sqrt{k_1}\, y)\right]$

At $y = h$, $v = 0 \Rightarrow \exp(\sqrt{k_1}\, y)\big|_h - \exp(-\sqrt{k_1}\, y)\big|_h = 0$

$\Rightarrow \exp(2\sqrt{k_1}\, h) = 1$

$\Rightarrow 2\sqrt{k_1}\, h = 2 i n \pi$

$\Rightarrow k_1 = -\frac{n^2\pi^2}{h^2}$

$\Rightarrow v_n = k_{5_n} \exp\left(-\frac{\nu n^2 \pi^2}{h^2} t\right)\left[\exp\left(i \frac{n\pi}{h} y\right) - \exp\left(-i\frac{n\pi}{h} y\right)\right]$

$\Rightarrow v_n = a_n \exp\left(-\frac{\nu n^2\pi^2}{h^2} t\right) \sin\left(\frac{n\pi y}{h}\right)$ with $a_n = 2 i k_{5_n}$

$\Rightarrow u = \frac{G y (h-y)}{2\rho\nu} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{\nu n^2 \pi^2}{h^2} t\right) \sin\left(\frac{n\pi y}{h}\right)$

(e) $u(t=0) = 0 \Rightarrow \frac{G y (h-y)}{2\rho\nu} + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{h}\right) = 0$

$\Rightarrow \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{h}\right) = \frac{G y (y-h)}{2\rho\nu}$

$\Rightarrow a_n = \frac{2}{h} \int_0^h \frac{G y (y-h)}{2\rho\nu} \sin\left(\frac{n\pi y}{h}\right) dy$

$\Rightarrow a_n = \frac{G}{\rho\nu h}\left[\frac{h}{n\pi} \cos\left(\frac{n\pi y}{h}\right) y (h-y)\right]_0^h - \frac{G}{\rho\nu h}\frac{h}{n\pi}\int_0^h \cos\left(\frac{n\pi y}{h}\right)(h-2y)\, dy$

$\Rightarrow a_n = -\frac{G h}{\rho\nu}\frac{1}{n^2\pi^2}\left[\sin\left(\frac{n\pi y}{h}\right)(h-2y)\right]_0^h - \frac{2Gh}{\rho\nu}\frac{1}{n^2\pi^2}\int_0^h \sin\left(\frac{n\pi y}{h}\right) dy$

$\Rightarrow a_n = -\frac{2Gh^2}{\rho\nu}\frac{1}{n^3\pi^3}\left[-\cos\left(\frac{n\pi y}{h}\right)\right]_0^h$

$$\Rightarrow a_n = -\frac{2Gh^2}{\rho\nu n^3\pi^3}\left(1-(-1)^n\right)$$

$$\Rightarrow u(y,t) = \frac{-Gy(y-h)}{2\rho\nu} - \frac{4Gh^2}{\rho\nu\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\exp\left(-\frac{\nu(2k+1)^2\pi^2}{h^2}t\right)\sin\left(\frac{(2k+1)\pi}{h}y\right)$$

## Exercise 3

(a) $\rho\,\partial_t u + \rho u\,\partial_x u + \rho v\,\partial_y u = -\partial_x p + \mu\,\partial_x^2 u + \mu\,\partial_y^2 u$

$$\left.\begin{array}{l} u = u(y,t),\ v = 0 \\ \partial_x p = 0 \end{array}\right\} \Rightarrow \quad \partial_t u = \nu\,\partial_y^2 u$$

(b) $u(y,t) = f(y)\exp(i\Omega t) \Rightarrow \begin{cases} \partial_t u = i\Omega f(y)\exp(i\Omega t) \\ \partial_y u = f(y)\exp(i\Omega t) \end{cases}$

$$\Rightarrow \partial_y^2 u = f''(y)\exp(i\Omega t)$$

$$\Rightarrow i\Omega f = \nu f''$$

$$\Rightarrow f'' = \frac{i\Omega}{\nu}f$$

$$\Rightarrow f(y) = k\exp\left(\sqrt{\frac{i\Omega}{\nu}}\,y\right) + k_2\exp\left(-\sqrt{\frac{i\Omega}{\nu}}\,y\right)$$

Recall $i = \frac{(i+1)^2}{2} \Rightarrow \sqrt{i} = \pm\frac{\sqrt{2}}{2}(i+1)$

$$\Rightarrow f(y) = k_1\exp\left((i+1)\sqrt{\frac{\Omega}{2\nu}}\,y\right) + k_2\exp\left(-(i+1)\sqrt{\frac{\Omega}{2\nu}}\,y\right)$$

$$\Rightarrow u(y,t) = \exp(i\Omega t)\left[k_1\exp\left[(i+1)\sqrt{\frac{\Omega}{2\nu}}\,y\right] + k_2\exp\left[-(i+1)\sqrt{\frac{\Omega}{2\nu}}\,y\right]\right]$$

At $y=0$, $u = U\exp(i\Omega t) \Rightarrow k_1 + k_2 = U$

At $y\to\infty$, $u = 0 \Rightarrow k_1 = 0 \quad \Rightarrow k_2 = U$

$$\Rightarrow u(y,t) = U\exp(i\Omega t)\exp\left[-(i+1)\sqrt{\frac{\Omega}{2\nu}}\,y\right]$$

$$\Rightarrow u(y,t) = U\exp\left[i\Omega t - i\sqrt{\frac{\Omega}{2\nu}}\,y - \sqrt{\frac{\Omega}{2\nu}}\,y\right]$$

$$\Rightarrow u(y,t) = U\exp(-ky)\exp(i\Omega t - iky) \quad \text{with } k = \sqrt{\frac{\Omega}{2\nu}}$$