MATH3620: Fluid Dynamics 2
Example sheet 3

Exercise 1

(a) $Re = \frac{\rho U L}{\mu} = 1625$

(b) $Re = 10^{-3}$

(c) $Re = 25$

(d) $Re = \frac{\rho U L}{\mu} = \frac{1.3 \cdot 10^3 \times 10^{-2} \times 10^{-2}}{10} = 0{,}013$

(e) $Re = \frac{\rho U L}{\mu} = \frac{4 \cdot 10^3 \times [2 \cdot 10^{-3} / (365{,}25 \times 24 \times 60 \times 60)] \times 7 \cdot 10^5}{10^{22}} = 1.8 \cdot 10^{-23}$

Exercise 2

- Stokes equation: $\vec{0} = -\nabla p + \mu \nabla^2 \vec{u} \Rightarrow \vec{0} = -\nabla \times \nabla p + \mu \nabla \times \nabla^2 \vec{u}$

$$\Rightarrow \vec{0} = \vec{0} + \mu \nabla^2 (\nabla \times \vec{u})$$
$$\Rightarrow \nabla^2 \vec{\omega} = \vec{0}$$

- Planar flow: $\vec{\omega} = \omega \vec{e_z} = (\partial_x v - \partial_y u) \vec{e_z}$

$$\Rightarrow \omega = \partial_x(-\partial_x \psi) - \partial_y(\partial_y \psi)$$
$$\Rightarrow -\omega = \nabla^2 \psi$$

- $\nabla^2 \omega = 0 \Rightarrow \nabla^2(-\nabla^2 \psi) = 0$

$$\Rightarrow \nabla^4 \psi = 0$$

Exercise 3

- For a steady flow: $\rho \frac{D\vec{u}}{Dt} = \rho (\vec{u} \cdot \nabla) \vec{u}$

u has size U
y has size $\bar{h}$
x has size $\bar{h}/h'$ with $|h'| \ll 1$
v has size V with $\partial_x u + \partial_y v = 0 \Rightarrow V = U h'$

$$\Rightarrow \begin{cases} \rho u \partial_x u \sim \rho U^2 h' / \bar{h} \\ \rho v \partial_y u \sim \rho U^2 h' / \bar{h} \\ \rho u \partial_x v \sim \rho U^2 h'^2 / \bar{h} \\ \rho v \partial_y v \sim \rho U^2 h'^2 / \bar{h} \end{cases}$$

$\Rightarrow$ the largest advection term is of the size $\rho \frac{U^2 h'}{\bar{h}}$

- $\mu \nabla^2 \vec{u}$ is of size $\frac{\mu U}{\bar{h}^2} \Rightarrow$ inertia is negligible if: $\rho \frac{U^2 h'}{\bar{h}} \ll \frac{\mu U}{\bar{h}^2}$

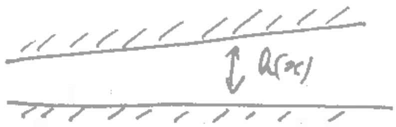
$$\Rightarrow U \ll \frac{\mu}{\rho h' \bar{h}}$$

Recall: $h' = \dfrac{d_2-d_1}{L}$, $\bar{h} = \dfrac{d_1+d_2}{2}$ $\Big\} \Rightarrow U \ll \dfrac{2\mu L}{\rho(d_2^2-d_1^2)}$

- $$Re = \frac{\rho U (d_1+d_2)}{2\mu} \ll \frac{\rho(d_1+d_2)}{2\mu}\,\frac{2\mu L}{\rho(d_2^2-d_1^2)} \ll \frac{d_1+d_2}{d_2^2-d_1^2}\,L \ll \frac{L}{d_2-d_1}$$

If $\dfrac{L}{d_2-d_1} \gg 1$, Re can be large.

Exercise 4

-  $h' \ll 1$

$$\left.\begin{aligned} \frac{\partial}{\partial y} &\sim \frac{1}{\bar h} \\ \frac{\partial}{\partial x} &\sim \frac{h'}{\bar h} \end{aligned}\right\} \Rightarrow \partial_x u + \partial_y v = 0 \Rightarrow \frac{h' U}{\bar h} \sim \frac{V}{\bar h} \Rightarrow V \sim U h'$$

Stokes equations:
$$\partial_x p = \mu\,\partial_x^2 u + \mu\,\partial_y^2 u \simeq \mu\,\partial_y^2 u \Rightarrow p \sim \frac{\mu U}{\bar h h'}$$
$$\partial_y p = \mu\,\partial_x^2 v + \mu\,\partial_y^2 v \simeq \mu\,\partial_y^2 v \Rightarrow p \sim \frac{\mu U h'}{\bar h}$$
$$\Rightarrow p \sim \frac{\mu U}{\bar h h'} \quad \text{thus} \quad \underbrace{\partial_y p}_{\frac{\mu U}{\bar h^2 h'}} \simeq \underbrace{\mu\,\partial_y^2 v}_{\frac{\mu U h'}{\bar h^2}}$$
$$\Rightarrow \partial_y p \simeq 0$$

- Using the above scaling quantities: $x = \bar h / h'\, x^*$, $y = \bar h\, y^*$ $\quad (h = \bar h\, h^*)$
$u = U u^*$, $v = U h' v^*$
$p = \dfrac{\mu U}{\bar h h'}\, p^*$

$$\Rightarrow \begin{cases} \dfrac{\partial p^*}{\partial x^*} = \dfrac{\partial^2 u^*}{\partial y^{*2}} \\ \dfrac{\partial p^*}{\partial y^*} = 0 \\ \dfrac{\partial u^*}{\partial x^*} + \dfrac{\partial v^*}{\partial y^*} = 0 \end{cases} \Rightarrow u^* = \frac{1}{2}\frac{\partial p^*}{\partial x^*} y^{*2} + k_1 y^* + k_2$$

Boundary condition: $u^* = 0$ at $y^* = 0$; h^* $\Rightarrow u^* = \dfrac{1}{2}\dfrac{\partial p^*}{\partial x^*}\, y^*(y^* - h^*)$

$$\Rightarrow Q^* = \int_0^{h^*} u^*\,dy^* = \frac{1}{2}\frac{\partial p^*}{\partial x^*}\int_0^{h^*} y^*(y^*-h^*)\,dy^* = -\frac{h^{*3}}{12}\frac{\partial p^*}{\partial x^*}$$

$$\Rightarrow Q^* = -\frac{1}{12}\,\frac{h^3}{\bar{h}^3}\,\frac{\bar{h}h'}{\mu U}\,\frac{\bar{h}}{h'}\,\frac{\partial p}{\partial x}$$
$$= -\frac{h^3}{12}\,\frac{1}{\mu U \bar{h}}\,\frac{\partial p}{\partial x}$$

However: $Q = Q^* U \bar{h}$

$$\Rightarrow Q = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x}$$

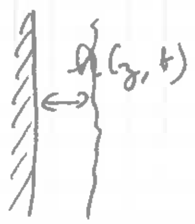
- $\frac{\partial p}{\partial x} = \frac{-12\mu Q}{h^3} \Rightarrow \Delta p = \int_0^L -\frac{12\mu Q}{h^3}\,dx$

$$= -12\mu Q \int_0^L \frac{1}{h^3}\,dx$$

$$\frac{dh}{dx} = \frac{d_2 - d_1}{L} \Rightarrow dx = \frac{L}{d_2 - d_1}\,dh$$

$$\Rightarrow \Delta p = -12\mu Q \int_{d_1}^{d_2} \frac{L}{d_2-d_1}\,\frac{1}{h^3}\,dh$$
$$= -\frac{12\mu Q L}{d_2 - d_1}\int_{d_1}^{d_2}\frac{1}{h^3}\,dh$$
$$= \frac{12\mu Q L}{d_2-d_1}\left(\frac{1}{2d_2^2} - \frac{1}{2d_1^2}\right)$$
$$= -\frac{6\mu Q L\,(d_1+d_2)}{d_1^2\,d_2^2}$$

Exercise 5



- Lubrication approximation valid if $u \ll w$.
- Since $h' \ll 1$, $\frac{\partial}{\partial z} \ll 1 \Rightarrow -\frac{\partial p}{\partial z} + \rho g + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) = 0$

$$\Rightarrow \rho g + \mu \frac{\partial^2 w}{\partial x^2} = 0$$
$$\Rightarrow w = -\frac{\rho g}{2\mu}x^2 + k_1 x + k_2$$

Boundary conditions: $w(x=0) = 0 \Rightarrow k_2 = 0$

$$\partial_x w(x=h) = 0 \Rightarrow k_1 = \frac{\rho g h}{\mu}$$

$$\Rightarrow w = \frac{\rho g}{2\mu}\left(2hx - x^2\right)$$

Incompressibility: $\frac{\partial u}{\partial x} = -\frac{\partial w}{\partial z} \Rightarrow \frac{\partial u}{\partial x} = -\frac{\rho g}{\mu}\,x\,\frac{\partial h}{\partial z}$

$$\Rightarrow u = -\frac{\rho g}{2\mu}\frac{\partial h}{\partial z}x^2$$

Boundary condition: at $x = h$: $\frac{\partial h}{\partial t} + w\frac{\partial h}{\partial z} = u$

$$\Rightarrow \frac{\partial h}{\partial t} + \frac{\rho g}{2\mu}h^2\frac{\partial h}{\partial z} = -\frac{\rho g}{2\mu}h^2\frac{\partial h}{\partial z}$$
$$\Rightarrow \frac{\partial h}{\partial t} + \frac{\rho g h^2}{\mu}\frac{\partial h}{\partial z} = 0$$

- $h(z,t) = f\left(z - \frac{\rho g h^2}{\mu} t\right) = f(\xi)$

$$\left.\begin{aligned} \frac{\partial h}{\partial t} &= \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial t} = -\frac{\partial f}{\partial \xi}\frac{\rho g h^2}{\mu} \\ \frac{\partial h}{\partial z} &= \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial f}{\partial \xi} \end{aligned}\right\} \Rightarrow \frac{\partial h}{\partial t} + \frac{\rho g h^2}{\mu}\frac{\partial h}{\partial z} = -\frac{\partial f}{\partial \xi}\frac{\rho g h^2}{\mu} + \frac{\rho g h^2}{\mu}\frac{\partial f}{\partial \xi} = 0$$

So $h(z,t) = f(\xi)$ is a solution of this problem.

- Propagation given by $z - \frac{\rho g h^2}{\mu} t = cst$

$$\Rightarrow z = cst + \underbrace{\frac{\rho g h^2}{\mu}}_{= \frac{dz}{dt} = \text{speed of propagation}} t$$

$\Rightarrow$ propagation speed increases with h.

bulge:

$\Rightarrow \frac{\partial h}{\partial z} \not< 1$