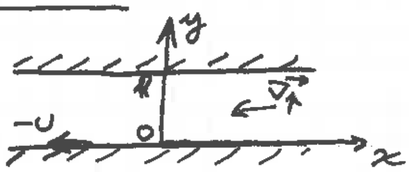
MATH 3620: Fluid Dynamics 2
Example sheet 2

Exercise 1



• 2D Navier–Stokes equation:
$$\begin{cases} \rho\partial_t u + \rho u\partial_x u + \rho v\partial_y u = -\partial_x p + \mu(\partial_x^2 u + \partial_y^2 u) \\ \rho\partial_t v + \rho u\partial_x v + \rho v\partial_y v = -\partial_y p + \mu(\partial_x^2 v + \partial_y^2 v) \end{cases}$$

Let us write $\vec{u} = u(y)\vec{e_x}$:
$$\begin{cases} 0 = -\partial_x p + \mu\partial_y^2 u \\ 0 = -\partial_y p \end{cases}$$

$$\Rightarrow \partial_y^2 u = -\frac{G}{\mu}$$
$$\Rightarrow u = -\frac{G}{2\mu}y^2 + k_1 y + k_2$$

$u(y=0) = -U \Rightarrow -U = k_2$

$u(y=h) = 0 \Rightarrow 0 = -\frac{Gh^2}{2\mu} + k_1 h + k_2 \Rightarrow k_1 = \frac{U}{h} + \frac{Gh}{2\mu}$

$$\hookrightarrow u = -\frac{G}{2\mu}y^2 + \left(\frac{U}{h} + \frac{Gh}{2\mu}\right)y - U$$
$$\Rightarrow u = \frac{G}{2\mu}(h-y)y - \frac{U}{h}(h-y)$$

• $Q = \int_0^h u(y)\,dy \Rightarrow Q = \int_0^h \frac{G}{2\mu}y(h-y)\,dy - \frac{U}{h}\int_0^h (h-y)\,dy$
$$\Rightarrow Q = \frac{Gh}{2\mu}\int_0^h y\,dy - \frac{G}{2\mu}\int_0^h y^2\,dy - Uh + \frac{Uh}{2}$$
$$\Rightarrow Q = \frac{Gh^3}{4\mu} - \frac{Gh^3}{6\mu} - \frac{Uh}{2}$$
$$\Rightarrow Q = \frac{Gh^3}{12\mu} - \frac{Uh}{2}$$

• $Q = \frac{6\mu U}{h^2}\frac{h^3}{12\mu} - \frac{Uh}{2}$
$= \frac{Uh}{2} - \frac{Uh}{2}$
$= 0$

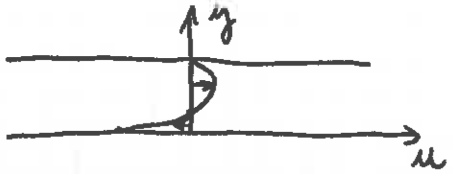
• $u = \frac{3U}{h^2}(h-y)y - \frac{U}{h}(h-y)$

$\partial_y u = \frac{3U}{h} - \frac{6Uy}{h^2} + \frac{U}{h}$
$= \frac{4U}{h} - \frac{6U}{h^2}y$

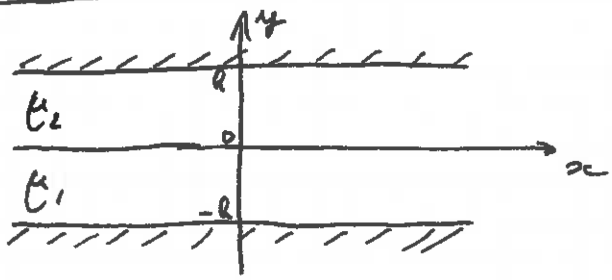
$\partial_y u = 0 \Rightarrow y = \frac{4U}{h}\frac{h^2}{6U}$
$= \frac{2}{3}h$

$u\left(y = \frac{2}{3}h\right) = \frac{3U}{h^2}\left(\frac{1}{3}h\right)\frac{2}{3}h - \frac{U}{h}\left(\frac{1}{3}h\right)$
$= \frac{2}{3}U - \frac{1}{3}U$
$= \frac{1}{3}U$

$$u\left(y=\frac{h}{3}\right)=\frac{3U}{h^2}\left(\frac{2}{3}h\right)\frac{1}{3}h-\frac{U}{h}\frac{2}{3}h$$
$$=\frac{2}{3}U-\frac{2}{3}U$$
$$=0$$



Exercise 2



(a) at $y=0$: $\begin{cases} u_- = u_+ & \text{continuous velocity} \\ \mu_1 \dfrac{\partial u_-}{\partial y} = \mu_2 \dfrac{\partial u_+}{\partial y} & \text{continuous stress} \end{cases}$

(b) Lower subdomain: $\vec{u}=u_-(y)\,\vec{e}_x \Rightarrow 0=-\partial_x p+\mu\,\partial_y^2 u_-$

$$\Rightarrow \partial_y^2 u_- = 0$$
$$\Rightarrow u_- = a\,y + b$$

$$u_-(0)=u_0 \Rightarrow b=u_0$$
$$u_-(-h)=-U \Rightarrow -U=a(-h)+u_0$$
$$\Rightarrow a=\frac{u_0+U}{h}$$
$$\Rightarrow u_-=\frac{u_0}{h}(y+h)+\frac{Uy}{h}$$

Upper subdomain: $\vec{u}=u_+(y)\,\vec{e}_x \Rightarrow \partial_y^2 u_+ = 0$

$$\Rightarrow u_+ = c\,y + d$$

$$u_+(0)=u_0 \Rightarrow d=u_0$$
$$u_+(+h)=U \Rightarrow c=\frac{U-u_0}{h}$$
$$\Rightarrow u_+=\frac{u_0}{h}(h-y)+\frac{U}{h}y$$

Boundary condition: $\mu_1 \left.\dfrac{\partial u_-}{\partial y}\right|_0 = \mu_2 \left.\dfrac{\partial u_+}{\partial y}\right|_0 \Rightarrow \mu_1 \dfrac{u_0+U}{h}=\mu_2\dfrac{U-u_0}{h}$

$$\Rightarrow (\mu_1+\mu_2)\,u_0=(\mu_2-\mu_1)\,U$$
$$\Rightarrow u_0=\frac{\mu_2-\mu_1}{\mu_1+\mu_2}\,U$$

$$\Rightarrow \begin{cases} u_-=\dfrac{\mu_2-\mu_1}{\mu_1+\mu_2}\dfrac{U}{h}(y+h)+\dfrac{U}{h}y \\[2ex] u_+=\dfrac{\mu_2-\mu_1}{\mu_1+\mu_2}\dfrac{U}{h}(h-y)+\dfrac{U}{h}y \end{cases}$$

(c) Lower subdomain: $0 = G + \mu \partial_y^2 u_- \Rightarrow u_- = -\frac{G}{2\mu_1} y^2 + k_1 y + k_2$

$$u_-(0) = u_0 \Rightarrow k_2 = u_0$$

$$u_-(y=-h) = 0 \Rightarrow -\frac{Gh^2}{2\mu_1} - k_1 h + u_0 = 0$$

$$\Rightarrow k_1 = \frac{u_0}{h} - \frac{Gh}{2\mu_1}$$

$$\Rightarrow u_- = -\frac{G}{2\mu_1} y^2 + \left(\frac{u_0}{h} - \frac{Gh}{2\mu_1}\right) y + u_0$$

Upper subdomain: $0 = G + \mu \partial_y^2 u_+ \Rightarrow u_+ = -\frac{G}{2\mu_2} y^2 + k_3 y + k_4$

$$u_+(0) = u_0 \Rightarrow k_4 = u_0$$

$$u_+(y=h) = 0 \Rightarrow -\frac{Gh^2}{2\mu_2} + k_3 h + u_0 = 0$$

$$\Rightarrow k_3 = -\frac{u_0}{h} + \frac{Gh}{2\mu_2}$$

$$\Rightarrow u_+ = -\frac{G}{2\mu_2} y^2 + \left(\frac{Gh}{2\mu_2} - \frac{u_0}{h}\right) y + u_0$$

Boundary conditions: $\mu_1 \frac{\partial u_-}{\partial y}\Big|_0 = \mu_2 \frac{\partial u_+}{\partial y}\Big|_0 \Rightarrow \mu_1\left(\frac{u_0}{h} - \frac{Gh}{2\mu_1}\right) = \mu_2\left(\frac{Gh}{2\mu_2} - \frac{u_0}{h}\right)$

$$\Rightarrow \mu_1 \frac{u_0}{h} - \frac{Gh}{2} = -\mu_2 \frac{u_0}{h} + \frac{Gh}{2}$$

$$\Rightarrow (\mu_1 + \mu_2) u_0 = Gh^2$$

$$\Rightarrow u_0 = \frac{Gh^2}{\mu_1 + \mu_2}$$

$$\Rightarrow \begin{cases} u_- = -\frac{G}{2\mu_1} y^2 + \left[\frac{Gh}{\mu_1+\mu_2} - \frac{Gh}{2\mu_1}\right] y + \frac{Gh^2}{\mu_1+\mu_2} \\ u_+ = -\frac{G}{2\mu_2} y^2 + \left[\frac{Gh}{2\mu_2} - \frac{Gh}{\mu_1+\mu_2}\right] y + \frac{Gh^2}{\mu_1+\mu_2} \end{cases}$$

$$Q_1 = \int_{-h}^{0} u_- \, dy = \int_{-h}^{0} -\frac{G}{2\mu_1} y^2 dy + \left(\frac{Gh}{\mu_1+\mu_2} - \frac{Gh}{2\mu_1}\right) \int_{-h}^{0} y \, dy + \frac{Gh^2}{\mu_1+\mu_2} \int_{-h}^{0} dy$$

$$= -\frac{Gh^3}{6\mu_1} + -\frac{Gh^3}{2(\mu_1+\mu_2)} + \frac{Gh^3}{4\mu_1} + \frac{Gh^3}{\mu_1+\mu_2}$$

$$= \frac{Gh^3}{12\mu_1} + \frac{Gh^3}{2(\mu_1+\mu_2)}$$

$$Q_2 = \int_0^h u_+ \, dy = \int_0^h -\frac{G}{2\mu_2} y^2 dy + \left(\frac{Gh}{2\mu_2} - \frac{Gh}{\mu_1+\mu_2}\right) \int_0^h y \, dy + \frac{Gh^2}{\mu_1+\mu_2} \int_0^h dy$$

$$= -\frac{Gh^3}{6\mu_2} + \frac{Gh^3}{4\mu_2} - \frac{Gh^3}{2(\mu_1+\mu_2)} + \frac{Gh^3}{\mu_1+\mu_2}$$

$$= \frac{Gh^3}{2(\mu_1+\mu_2)} + \frac{Gh^3}{12\mu_2}$$

$$\frac{Q_1}{Q_2} = \frac{\frac{1}{6\mu_1} + \frac{1}{\mu_1+\mu_2}}{\frac{1}{\mu_1+\mu_2} + \frac{1}{6\mu_2}}$$

$$= \frac{7\mu_1+\mu_2}{6\mu_1(\mu_1+\mu_2)} \cdot \frac{6\mu_2(\mu_1+\mu_2)}{7\mu_2+\mu_1}$$

$$= \frac{\mu_2(7\mu_1+\mu_2)}{\mu_1(7\mu_2+\mu_1)}$$

Exercise 3

- $\vec{u} = w(r)\vec{e_z} \Rightarrow \begin{cases} 0 = -\partial_r p \\ 0 = -\frac{1}{r}\partial_\theta p \\ 0 = -\partial_z p + \mu\left[\frac{1}{r}\partial_r(r\partial_r w)\right] \end{cases}$

$$\Rightarrow \frac{1}{r}\partial_r(r\partial_r w) = -\frac{G}{\mu}$$
$$\Rightarrow r\partial_r w = -\frac{Gr^2}{2\mu} + k_1$$
$$\Rightarrow w = -\frac{Gr^2}{4\mu} + k_1 \ln r + k_2$$

$w(r=a) = 0 \Rightarrow -\frac{Ga^2}{4\mu} + k_1 \ln a + k_2 = 0 \quad (1)$

$w(r=b) = 0 \Rightarrow -\frac{Gb^2}{4\mu} + k_1 \ln b + k_2 = 0 \quad (2)$

$(1)-(2): \frac{G}{4\mu}(b^2 - a^2) + k_1(\ln a - \ln b) = 0 \Rightarrow k_1 = \frac{G}{4\mu}\frac{a^2-b^2}{\ln(a/b)}$

$$\Rightarrow -\frac{Ga^2}{4\mu} + \frac{G}{4\mu}\frac{a^2-b^2}{\ln(a/b)}\ln a + k_2 = 0$$
$$\Rightarrow k_2 = \frac{Ga^2}{4\mu} - \frac{G}{4\mu}\frac{a^2-b^2}{\ln(a/b)}\ln a$$

$\hookrightarrow w(r) = \frac{G}{4\mu}\left[(a^2 - r^2) + \frac{a^2-b^2}{\ln(a/b)}(\ln r - \ln a)\right]$

$$\Rightarrow w(r) = \frac{G}{4\mu}\left[a^2 - r^2 + \frac{a^2-b^2}{\ln a/b}\ln(r/a)\right]$$

- $\tau_{rz} = \mu\frac{\partial w}{\partial r} = \frac{G}{4}\left(-2r + \frac{a^2-b^2}{\ln(a/b)}\frac{1}{r}\right)$

at $r=a$: $\tau_{rz}(a) = \frac{G}{4}\left(\frac{a^2-b^2}{a\ln(a/b)} - 2a\right)$

at $r=b$: $\tau_{rz}(b) = \frac{G}{4}\left(\frac{a^2-b^2}{b\ln(a/b)} - 2b\right)$

Exercise 4

(a) $\vec{u} = w(x,y)\vec{e_z} \Rightarrow \begin{cases} 0 = -\partial_x p \\ 0 = -\partial_y p \\ 0 = -\partial_z p + \mu(\partial_x^2 w + \partial_y^2 w) \end{cases}$

$$\Rightarrow \partial_x^2 w + \partial_y^2 w = -\frac{G}{\mu}$$

No-slip boundary conditions: $w = 0$ on ∂S.

(b) $w(x,y) = A\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \Rightarrow \frac{2A}{a^2} + \frac{2A}{b^2} = -\frac{G}{\mu}$

$$\Rightarrow A = -\frac{G}{2\mu}\frac{a^2b^2}{a^2+b^2}$$
$$\Rightarrow w = \frac{G}{2\mu}\frac{a^2b^2}{a^2+b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$a = b \Rightarrow w = \frac{G}{4\mu}(a^2 - x^2 - y^2)$

Let $r^2 = x^2+y^2$: $w = \frac{G}{4\mu}(a^2 - r^2)$ $\Rightarrow$ Poiseuille flow

(c) $w = Ay\left[(y-\sqrt{3}a)^2 - 3x^2\right]$: $y = 0 \Rightarrow w = A \times 0 \times \ldots = 0$

$y = \sqrt{3}(a-x) \Rightarrow w = A\sqrt{3}(a-x)\left[3x^2 - 3x^2\right] = 0$

$y = \sqrt{3}(a+x) \Rightarrow w = A\sqrt{3}(a+x)\left[3x^2 - 3x^2\right] = 0$

$w = A(y^3 - 2\sqrt{3}ay^2 + 3a^2y - 3x^2y)$

$\Rightarrow \frac{\partial^2 w}{\partial x^2} = -6Ay$; $\frac{\partial^2 w}{\partial y^2} = 6Ay - 4\sqrt{3}aA$

$\Rightarrow A\left[6y - 4\sqrt{3}a - 6y\right] = -\frac{G}{\mu}$

$\Rightarrow A = \frac{G}{4\sqrt{3}a\mu}$

$\Rightarrow w = \frac{G}{4\sqrt{3}\mu a}\, y\left[(y-\sqrt{3}a)^2 - 3x^2\right]$