MATH 3620 Fluid Dynamics 2
Example sheet 1

Exercise 1

(a) $$\begin{cases} u = \dfrac{dx}{dt} = 4x + 2z & \Rightarrow \quad \dfrac{d^2x}{dt^2} = 4\dfrac{dx}{dt} + 2\dfrac{dz}{dt} = 4\dfrac{dx}{dt} - 4x \\ v = \dfrac{dy}{dt} = -4y & \Rightarrow \quad y = y_0 e^{-4t} \\ w = \dfrac{dz}{dt} = -2x & \end{cases}$$

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0 \; : \; x \sim e^{\alpha t} \quad \Rightarrow \quad \alpha^2 - 4\alpha + 4 = 0$$
$$\Rightarrow \quad \alpha = 2 \quad \text{(multiplicity 2)}$$

$$\Rightarrow \quad x = (k_1 + k_2 t)\, e^{2t} = (x_0 + k_2 t)\, e^{2t}$$

$$\frac{dz}{dt} = -2x = (-2x_0 - 2k_2 t)\, e^{2t} \quad \text{or} \quad z = \frac{1}{2}\frac{dx}{dt} - 2x$$

$$\Rightarrow \quad z = (x_0 + k_2 t)\, e^{2t} + \frac{1}{2} k_2 e^{2t} - 2(x_0 + k_2 t)\, e^{2t}$$

$$\Rightarrow \quad z = -(x_0 + k_2 t)\, e^{2t} + \frac{k_2}{2} e^{2t}$$

$$t = 0 \Rightarrow z = z_0 \; : \; z_0 = -x_0 + \frac{k_2}{2} \quad \Rightarrow \quad k_2 = 2(x_0 + z_0)$$

$$\Rightarrow \quad z = \left(z_0 - 2(x_0 + z_0)t\right) e^{2t}$$

$$\text{and} \quad x = \left(x_0 + 2(x_0 + z_0)t\right) e^{2t}$$

$$xyz = \left(x_0 + 2(x_0 + z_0)t\right) y_0 \left(z_0 - 2(x_0 + z_0)t\right)$$

$$\Rightarrow \quad \boxed{\frac{xyz}{x_0 y_0 z_0} = \left[1 + 2\left(1 + \frac{z_0}{x_0}\right)t\right]\left[1 - 2\left(1 + \frac{x_0}{z_0}\right)t\right]}$$

(b) $$\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 4 - 4 + 0 = 0$$

$$\nabla \times \vec{u} = \begin{pmatrix} \partial w/\partial y - \partial v/\partial z \\ \partial u/\partial z - \partial w/\partial x \\ \partial v/\partial x - \partial u/\partial y \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ 2 + 2 \\ 0 - 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

$$(\vec{u} \cdot \nabla)\vec{u} = \begin{pmatrix} u\, \partial u/\partial x + v\, \partial u/\partial y + w\, \partial u/\partial z \\ u\, \partial v/\partial x + v\, \partial v/\partial y + w\, \partial v/\partial z \\ u\, \partial w/\partial x + v\, \partial w/\partial y + w\, \partial w/\partial z \end{pmatrix} = \begin{pmatrix} 4(4x+2z) + 0 + 2(-2x) \\ 0 - 4(-4y) + 0 \\ -2(4x+2z) + 0 + 0 \end{pmatrix}$$

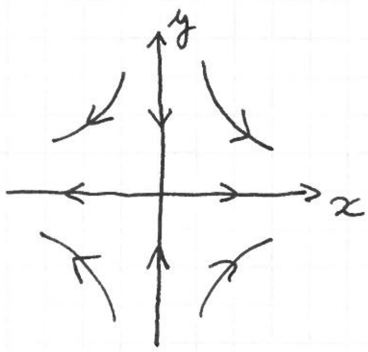
$$= \begin{pmatrix} 12x + 8z \\ 16y \\ -8x - 4z \end{pmatrix}$$

(c) $$\nabla \vec{u} = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{pmatrix} = \begin{pmatrix} 4 & 0 & 2 \\ 0 & -4 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

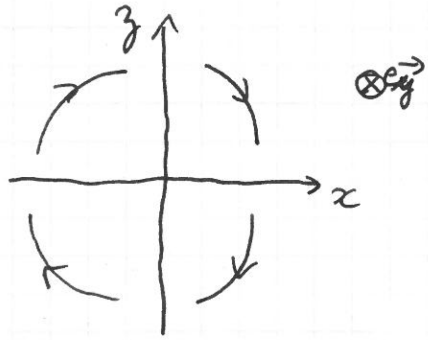
$$\bar{\bar{E}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \bar{\bar{\Omega}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

$$\Omega_{ij} = -\frac{1}{2}\varepsilon_{ijk}\,\omega_k$$
$$= \frac{1}{2}\varepsilon_{ikj}\,\omega_k$$
$$= \frac{1}{2}\varepsilon_{kji}\,\omega_k$$
$$= \frac{1}{2}\varepsilon_{kji}\,\varepsilon_{klm}\frac{\partial u_m}{\partial x_l} \qquad \text{because } \omega_k = \varepsilon_{klm}\frac{\partial u_m}{\partial x_l} \qquad (\vec{\omega} = \nabla \times \vec{u})$$
$$= \frac{1}{2}\left(\delta_{jl}\,\delta_{im} - \delta_{jm}\,\delta_{il}\right)\frac{\partial u_m}{\partial x_l}$$
$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \quad \text{which is the definition.}$$

Flow associated with $\bar{\bar{E}}$:



Flow associated with $\bar{\bar{\Omega}}$:



Exercise 2

Navier–Stokes equation:
$$\begin{cases} \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 u}{\partial z^2} \\ \rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \mu\frac{\partial^2 v}{\partial x^2} + \mu\frac{\partial^2 v}{\partial y^2} + \mu\frac{\partial^2 v}{\partial z^2} \\ \rho\frac{\partial w}{\partial t} + \rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \rho w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \mu\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial z^2} \end{cases}$$

$$\Rightarrow \begin{cases} 0 + \rho\alpha^2 x + 0 + 0 = -\frac{\partial p}{\partial x} + 0 + 0 + 0 \\ 0 + 0 + \rho\alpha^2 y + 0 = -\frac{\partial p}{\partial y} + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 = -\frac{\partial p}{\partial z} + 0 + 0 + 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial p}{\partial x} = -\rho\alpha^2 x \\ \frac{\partial p}{\partial y} = -\rho\alpha^2 y \\ \frac{\partial p}{\partial z} = 0 \end{cases} \qquad \Rightarrow \quad p = p(x,y)$$

$$\Rightarrow \begin{cases} p = -\frac{\rho\alpha^2 x^2}{2} + f(y) \\ p = -\frac{\rho\alpha^2 y^2}{2} + f(x) \end{cases}$$

$$\Rightarrow \boxed{p = p_0 - \frac{1}{2}\rho\alpha^2\left(x^2 + y^2\right)}$$

Exercise 3

- $\vec{u} \times (\nabla \times \vec{u}) = \varepsilon_{ijk}\, u_j \left(\varepsilon_{klm} \frac{\partial u_m}{\partial x_l}\right)$

$$= \varepsilon_{ijk}\, \varepsilon_{klm}\, u_j \frac{\partial u_m}{\partial x_l}$$

$$= \varepsilon_{kij}\, \varepsilon_{klm}\, u_j \frac{\partial u_m}{\partial x_l}$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, u_j \frac{\partial u_m}{\partial x_l}$$

$$= u_j \frac{\partial u_j}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j}$$

$$= \frac{1}{2}\frac{\partial u_j^2}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j}$$

$$= \frac{1}{2}\nabla(\vec{u}^2) - (\vec{u}\cdot\nabla)\vec{u}$$

$$\Rightarrow \boxed{(\vec{u}\cdot\nabla)\vec{u} = \frac{1}{2}\nabla(\vec{u}^2) - \vec{u}\times(\nabla\times\vec{u})}$$

- $\nabla \times (\nabla \times \vec{u}) = \varepsilon_{ijk} \frac{\partial}{\partial x_j}\left(\varepsilon_{klm}\frac{\partial u_m}{\partial x_l}\right)$

$$= \varepsilon_{ijk}\, \varepsilon_{klm} \frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

$$= \varepsilon_{kij}\, \varepsilon_{klm} \frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

$$= \frac{\partial^2 u_j}{\partial x_i \partial x_j} - \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$= \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} - \frac{\partial^2 u_i}{\partial x_j^2}$$

$$= \nabla(\nabla\cdot\vec{u}) - \nabla^2\vec{u}$$

$$\Rightarrow \boxed{\nabla^2\vec{u} = \nabla(\nabla\cdot\vec{u}) - \nabla\times(\nabla\times\vec{u})}$$

- $\rho \frac{D\vec{u}}{Dt} = -\nabla P + \rho\vec{g} + \mu\nabla^2\vec{u} \Rightarrow \rho\frac{\partial\vec{u}}{\partial t} + \rho(\vec{u}\cdot\nabla)\vec{u} = -\nabla P + \rho\vec{g} + \mu\nabla^2\vec{u}$

$$\Rightarrow \rho\frac{\partial\vec{u}}{\partial t} + \frac{\rho}{2}\nabla(\vec{u})^2 - \rho\,\vec{u}\times(\nabla\times\vec{u}) = -\nabla P + \rho\vec{g} + \mu\nabla(\nabla\cdot\vec{u}) - \mu\nabla\times(\nabla\times\vec{u})$$

Let us write $\vec{\omega} = \nabla\times\vec{u}$ and recall that $\nabla\cdot\vec{u} = 0$ for an incompressible flow:

$$\rho\frac{\partial\vec{u}}{\partial t} + \frac{\rho}{2}\nabla(\vec{u}^2) - \rho\,\vec{u}\times\vec{\omega} = -\nabla P + \rho\vec{g} - \mu\nabla\times\vec{\omega}$$

$$\Rightarrow \rho\frac{\partial\vec{u}}{\partial t} - \rho\,\vec{u}\times\vec{\omega} = -\nabla\left(P + \frac{1}{2}\rho\vec{u}^2 - \rho\vec{g}\cdot\vec{x}\right) - \mu\nabla\times\vec{\omega}$$

$$\Rightarrow \boxed{\frac{\partial\vec{u}}{\partial t} - \vec{u}\times\vec{\omega} = -\nabla\left(\frac{P}{\rho} + \frac{1}{2}\vec{u}^2 - \vec{g}\cdot\vec{x}\right) - \nu\nabla\times\vec{\omega}} \quad \text{with } \nu = \frac{\mu}{\rho}.$$