MATH 3620
Fluid Dynamics II

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Mondays 11am LT 11

Tuesdays 10 am LT18

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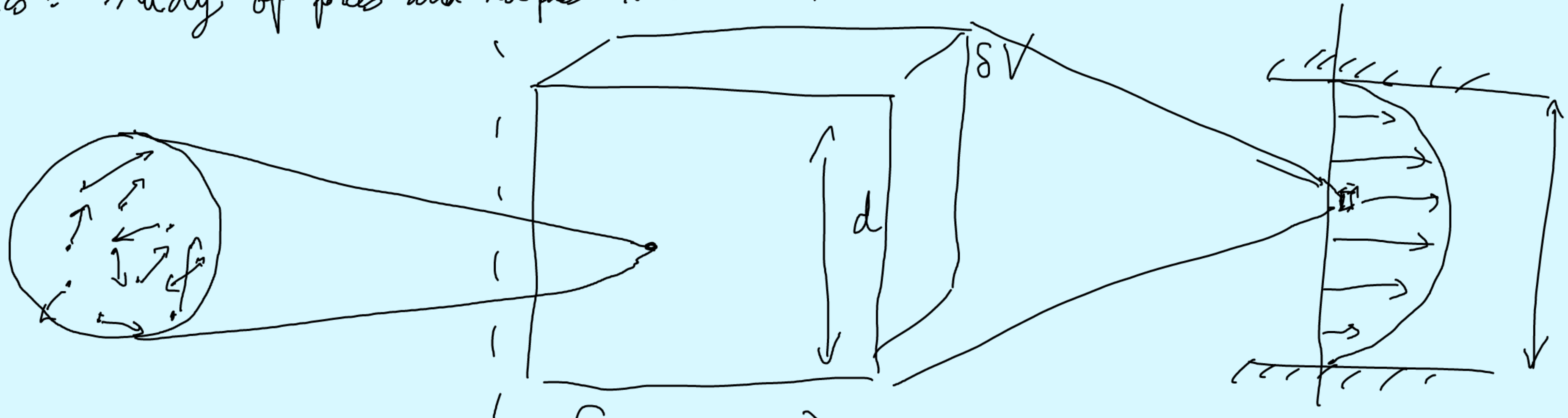
Tutorials: W3
on W5
Tuesdays W7
W9
W11

Fluid: substance that flows and takes the shape of its container.

Dynamics: study of forces and torques when there is motion



particle: $l_m = 1\,\text{Å} = 10^{-10}\,m$

(10^{23} water molecules per $1\,cm^3$ of water)

mean free path: $l_p = 10^{-8}\,m$

element of fluid
fluid "particle"
blob of fluid

$d \gg l_p$

$d \ll a$

MICRO $\xrightarrow[\text{(MESO)}]{\text{Continuum Hypothesis}}$ MACRO

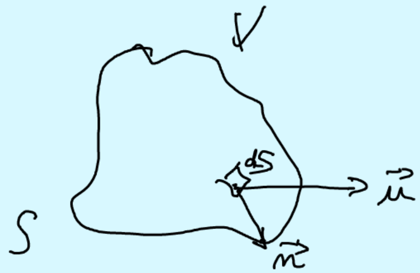
$$\vec{v}(P,t) = \frac{\sum_{\text{particles in } \delta V} \vec{v}_{\text{particles}}}{\sum_{\text{particles in } \delta V} 1}$$

δV: blob of fluid centered on P.

$$\rho(P,t) = \frac{\sum_{\text{particles in } \delta V} m_{\text{particle}}}{\delta V}$$

watch Index Notation

Conservation of mass



V is fixed ⟵ volume of fluid

$\vec{n}$ unit vector : outward pointing normal

$$M_V = \int_V \rho \, dV$$ total mass of fluid in V

The rate of change of mass in time:
$$\frac{dM_V}{dt} = \frac{d}{dt}\left(\int_V \rho \, dV\right)$$
$$= \int_V \frac{\partial \rho}{\partial t} \, dV$$ because V is fixed

Changes of mass of V are due to fluxes:

$$\frac{dM_V}{dt} = -\int_S \rho \, \vec{u} \cdot \vec{n} \, dS$$

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t} \, dV = -\int_S \rho \, \vec{u} \cdot \vec{n} \, dS$$

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t}\, dV = -\int_V \vec{\nabla}\cdot(\rho \vec{u})\, dV$$ (divergence theorem)

$$\Rightarrow \int_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho \vec{u})\right] dV = 0$$

Because we did not impose any condition on the size of V, we can choose V to be infinitesimally small. In this case:

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho \vec{u}) = 0}$$ everywhere

continuity equation

mass conservation

Time derivative: $\frac{\partial f}{\partial t}$: rate of change in time of f at $\vec{x}$ ← Euler

$$\frac{Df}{Dt}(\vec{x}(t), t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

$$= \frac{\partial f}{\partial t} + (\vec{u}\cdot\vec{\nabla})\, f$$ rate of change in time of f for a particle passing through $\vec{x}$ at t

Lagrange

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0 \implies \frac{\partial \rho}{\partial t} + \rho\vec{\nabla}\cdot\vec{u} + \underbrace{(\vec{u}\cdot\vec{\nabla})\rho}$$

$$\implies \frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{u} = 0$$

Incompressibility condition: An incompressible fluid has $\frac{D\rho}{Dt} = 0$

Mass conservation $\implies \rho\vec{\nabla}\cdot\vec{u} = 0$

$\implies \boxed{\vec{\nabla}\cdot\vec{u} = 0}$ incompressibility $\begin{cases}\text{condition}\\ \text{constraint}\end{cases}$

This incompressibility condition impose a condition on the velocity.

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\psi}) = 0$ always.

So we can write $\vec{u} = \vec{\nabla}\times\vec{\psi}$ $\qquad$ $\vec{\psi}$ is the streamfunction

For example: a 2D flow: $\vec{u} = u(x,y)\,\vec{e_x} + v(x,y)\,\vec{e_y}$

$\implies \vec{\psi} = \psi(x,y)\,\vec{e_z}$

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Streamlines: lines where the streamfunction is constant

$d\psi = 0$ on the streamline $\Rightarrow \dfrac{\partial \psi}{\partial x} dx + \dfrac{\partial \psi}{\partial y} dy = 0$

$\Rightarrow -v\,dx + u\,dy = 0$

$\Rightarrow \vec{u} \times \vec{dl} = 0$ where $\vec{dl} = (dx, dy)$ element of displacement along the streamline

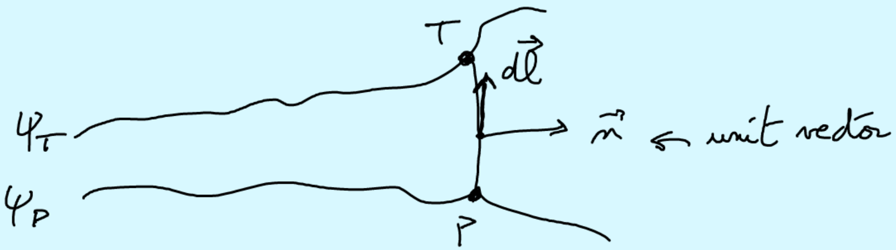
$\Rightarrow \vec{u} \,//\,$ streamlines

$$\vec{u} \cdot \vec{\nabla}\psi = u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y}$$

$$= \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y}$$

$$= 0 \quad \text{(same conclusion)}$$

Flux **between streamlines**:



$$Q = \int_P^T \vec{u} \cdot \vec{n}\, ds$$

$\vec{dl} = dx\, \vec{e_x} + dy\, \vec{e_y}$

$\vec{n}\, ds = dy\, \vec{e_x} - dx\, \vec{e_y}$

$$= ds\left(\frac{\partial y}{\partial s}\vec{e_x} - \frac{\partial x}{\partial s}\vec{e_y}\right)$$

$$\Rightarrow \quad Q = \int_P^T \left(u \frac{\partial y}{\partial s} - v \frac{\partial x}{\partial s} \right) ds$$

$$= \int_P^T \left(\frac{\partial \psi}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial \psi}{\partial x} \frac{\partial x}{\partial s} \right) ds$$

$$= \int_P^T \frac{d\psi}{ds}\, ds$$

$$= \int_P^T d\psi$$

$$= \psi_T - \psi_P$$

$$\|\vec{u}\| = \left\| \frac{\partial \psi}{\partial y} \vec{e_x} - \frac{\partial \psi}{\partial x} \vec{e_y} \right\|$$

$$= \left\| \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right) \right\|$$

$$= \| \vec{\nabla} \psi \|$$

<u>Strain rate and vorticity tensors:</u>

velocity gradient tensor : $\overline{\overline{K}}$

$$K_{ij} = \frac{\partial u_i}{\partial x_j}$$

$$= E_{ij} + \Omega_{ij}$$

E_{ij}: symmetric ($E_{ij} = E_{ji}$), strain-rate tensor

Ω_{ij}: anti-symmetric ($\Omega_{ij} = -\Omega_{ji}$), vorticity tensor

$$E_{ij} = \frac{1}{2}\left(K_{ij} + K_{ji}\right) \qquad \Omega_{ij} = \frac{1}{2}\left(K_{ij} - K_{ji}\right)$$

2D flow incompressible: $\partial_x u + \partial_y v = 0$

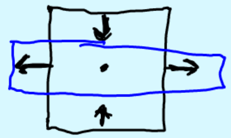
$$\Rightarrow \partial_x u = -\partial_y v$$

$$\bar{\bar{K}} = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & -\partial_x u \end{pmatrix}$$

3 unknowns

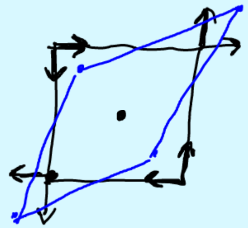
$$\hookrightarrow \quad \bar{\bar{E}} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \qquad \bar{\bar{\Omega}} = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$$

- $\bar{\bar{E}}_a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \partial_x u = 1 \quad \partial_y v = -1$



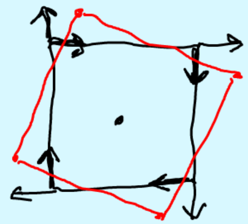
DEFORMATION

- $\bar{\bar{E}}_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \partial_y u = 1 \quad \partial_x v = 1$



DEFORMATION

- $\bar{\bar{\Omega}}_c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad \partial_y u = 1 \quad \partial_x v = -1$



SOLID BODY ROTATION

$\bar{\bar{E}}$: deformation → strain rate tensor

$\bar{\bar{\Omega}}$: solid body rotation → vorticity tensor

Lecture 4

$\vec{\omega} = \vec{\nabla} \times \vec{u}$ vorticity

$$\omega_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} u_k$$

$$\Rightarrow \underline{\varepsilon_{ilm}\, \omega_i} = \varepsilon_{ilm}\, \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

$$= \left(\delta_{lj}\, \delta_{mk} - \delta_{lk}\, \delta_{mj} \right) \frac{\partial u_k}{\partial x_j}$$

$$= \frac{\partial u_m}{\partial x_l} - \frac{\partial u_l}{\partial x_m}$$

$$= 2\, \Omega_{ml}$$

$$\Omega_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

$$\Rightarrow \Omega_{ml} = \frac{1}{2}\, \varepsilon_{ilm}\, \omega_i$$

$$= \frac{1}{2}\, \varepsilon_{klm}\, \omega_k$$

$$\Rightarrow \Omega_{ij} = \frac{1}{2}\, \varepsilon_{kji}\, \omega_k$$

$$= -\frac{1}{2}\, \varepsilon_{ijk}\, \omega_k$$

$$\overline{\overline{\Omega}} = \begin{pmatrix} \pm\omega\,? & \pm\omega\,? \\ \cdot \quad \cdot & \cdot \quad \cdot \end{pmatrix}$$

Chapter 2

Equation of motion

V fixed

$\vec{n}$: unit outward pointing normal

momentum density $\vec{q} = \rho \vec{u}$

total momentum in V : $\int_V \vec{q} \, dV$

rate of change of the total momentum in V with time: $\frac{d}{dt} \int_V \vec{q} \, dV = \int_V \frac{\partial \vec{q}}{\partial t} \, dV$

$= \int_V \underline{\frac{\partial}{\partial t}} (\rho \vec{u}) \, dV$

Newton's second law:

$$\underbrace{\begin{pmatrix} \text{rate of change of } \vec{q} \\ \text{in } V \end{pmatrix}}_{\vec{\mathrm{I}}} = \underbrace{\begin{pmatrix} \text{flux of } \vec{q} \\ \text{through } S \end{pmatrix}}_{\vec{\mathrm{II}}} + \underbrace{\begin{pmatrix} \text{net force} \\ \text{acting on } V \end{pmatrix}}_{\vec{\mathrm{III}}}$$

- $I_i = \int_V \frac{\partial}{\partial t}(\rho u_i)\, dV$

- $\vec{II} = -\int_S \vec{q}\ \vec{u}\cdot\vec{n} \quad dS$

$$II_i = -\int_S q_i\ u_j \boxed{n_j \quad dS}$$

$$= -\int_V \frac{\partial}{\partial x_j}(q_i u_j)\, dV \quad \text{(divergence theorem)}$$

$$= -\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)\, dV$$

- $\vec{III} = \int_V \vec{F}_{body}\, dV + \int_S \vec{F}_{local}\, dS$

force density

$$\vec{F}_{body} = \rho \vec{g}\ (+\ \vec{other}) \qquad F_{body_i} = \rho\, g_i$$

total stress tensor

$$F_{local_i} = n_j\ \tau_{ji}$$

$$\int_S F_{local_i}\, dS = \int_S n_j\ \tau_{ji}\, dS$$

$$= \int_S \tau_{ji}\, n_j \, dS$$

$$= \int_V \frac{\partial}{\partial x_j} (\tau_{ji})\, dV \qquad (\text{divergence } \cancel{\text{theorem}})$$

$$\Rightarrow \int_V \frac{\partial}{\partial t}(\rho u_i)\, dV = -\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)\, dV + \int_V (\rho g_i) dV + \int_V \frac{\partial}{\partial x_j}(\tau_{ji})\, dV$$

$$\Rightarrow \underline{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j)} = \rho g_i + \frac{\partial}{\partial x_j}(\tau_{ji})$$

pointwise (we can take V as small as a "blob" of fluid)

Continuity equation: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

$$\Rightarrow \rho \frac{\partial u_i}{\partial t} + \underbrace{u_i \frac{\partial \rho}{\partial t} + u_i \frac{\partial}{\partial x_j}(\rho u_j)} + \rho u_j \frac{\partial}{\partial x_j}(u_i) = \rho g_i + \frac{\partial}{\partial x_j}(\tau_{ji})$$

$$\Rightarrow \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial}{\partial x_j} u_i = \rho g_i + \frac{\partial}{\partial x_j} \tau_{ji}$$

$$\Rightarrow \boxed{\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = \rho\vec{g} + \vec{\nabla}\cdot\bar{\bar{\tau}}}$$ momentum equation

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Lecture 5

2.2 Constitutive equations

2.2.1 Ideal fluid

$\bar{\bar{\tau}} = -P\bar{\bar{I}}$ ← fluid in which the only internal stress is due to pressure.

$$\Rightarrow \tau_{ji} = -P\,\delta_{ij}$$

$$\Rightarrow \frac{\partial}{\partial x_j}\tau_{ji} = -\frac{\partial P}{\partial x_j}\delta_{ij}$$

$$= -\frac{\partial P}{\partial x_i}$$

$$\Rightarrow \vec{\nabla}\cdot\bar{\bar{\tau}} = -\vec{\nabla}P$$

$$\Rightarrow \boxed{\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = \rho\vec{g} - \vec{\nabla}P}$$ Euler equation

2.2.2 Newtonian fluids

Fluid for which the strain rate and the applied stress are linearly related ($\rightarrow$ rheology).
Isotropic fluid (all directions respond similarly).

$$\overline{\overline{\tau}} = -P\,\overline{\overline{I}} + \overline{\overline{\sigma}}$$

viscous stress tensor

$$= -P\,\overline{\overline{I}} + 2\mu\,\overline{\overline{E}}$$

strain rate tensor

dynamic viscosity

$$\frac{\partial}{\partial x_j}\tau_{ji} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[2\mu\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$$

$$= -\frac{\partial P}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

$$= -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2} + \mu\cancel{\frac{\partial^2 u_j}{\partial x_i \partial x_j}}$$

0 because $\frac{\partial u_j}{\partial x_j} = 0$ (incompressibility condition)

$$\Rightarrow \vec{\nabla}\cdot\overline{\overline{\tau}} = -\vec{\nabla}P + \mu\vec{\nabla}^2\vec{u}$$

$$\Rightarrow \boxed{\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = \underline{\underline{\rho\vec{g}}} - \vec{\nabla}P + \mu\vec{\nabla}^2\vec{u}}$$

Navier-Stokes equation

2.2.3 Different kinds of pressure

No flow, no forces: hydrostatic case: $\vec{0} = \rho \vec{g} - \vec{\nabla} P_H$ ← hydrostatic pressure

$$\Longrightarrow \quad P_H = \underline{\rho \vec{g}} \cdot \vec{x} + P_0 \quad \leftarrow \text{reference pressure}$$

pressure

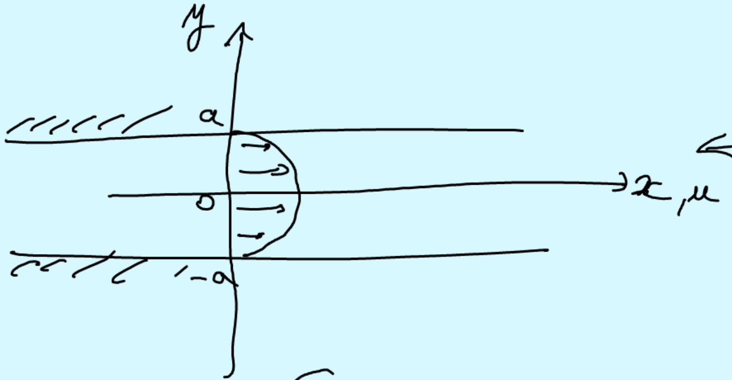
total → $P = P_H + p$ ← dynamic pressure

hydrostatic

$$\hookrightarrow \quad \boxed{\rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} \right] = - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u}}$$

Plane Poiseuille flow

(*)



$\vec{\nabla} p = -G \vec{e_x} \qquad G > 0$

(*) Equations:

$$\begin{cases} \cancel{\rho \dfrac{\partial u}{\partial t}} + \rho u \dfrac{\partial u}{\partial x} + \cancel{\rho v \dfrac{\partial u}{\partial y}} = - \dfrac{\partial p}{\partial x} + \mu \dfrac{\partial^2 u}{\partial x^2} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ \cancel{\rho \dfrac{\partial v}{\partial t}} + \cancel{\rho u \dfrac{\partial v}{\partial x}} + \cancel{\rho v \dfrac{\partial v}{\partial y}} = - \dfrac{\partial p}{\partial y} + \cancel{\mu \dfrac{\partial^2 v}{\partial x^2}} + \cancel{\mu \dfrac{\partial^2 v}{\partial y^2}} \end{cases}$$

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} = 0$$

* Boundary conditions :

$y = a$: $\cancel{v = 0}$ (impenetrability), $u = 0$ (no-slip)

$y = -a$: $\cancel{v = 0}$, $u = 0$

* Hypotheses : • steady flow : $\frac{\partial}{\partial t} \sim 0$

• one dimensional : $v = 0$

↳ Solution :

$$\begin{cases} \rho \cancel{u \frac{\partial u}{\partial x}} = -\frac{\partial p}{\partial x} + \mu \cancel{\frac{\partial^2 u}{\partial x^2}} + \mu \frac{\partial^2 u}{\partial y^2} \\ 0 = -\frac{\partial p}{\partial y} \\ \frac{\partial u}{\partial x} = 0 \longrightarrow u = u(y) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial p}{\partial x} = \mu \frac{\partial u}{\partial y^2} \\ \frac{\partial p}{\partial y} = 0 \longrightarrow p = p(x) \end{cases}$$

$$\underbrace{\frac{\partial p}{\partial x}}_{f(x)} = \mu \underbrace{\frac{\partial u}{\partial y^2}}_{f(y)} = k_1 = -G$$

Example Sheet 1

Problem 1

$$\vec{v} = \begin{pmatrix} 4x + 2z \\ -4y \\ -2x \end{pmatrix}$$

(a)
$$\begin{cases} \dfrac{dx}{dt} = 4x + 2z \\ \dfrac{dy}{dt} = -4y \\ \dfrac{dz}{dt} = -2x \end{cases} \Rightarrow \begin{cases} \dfrac{d^2x}{dt^2} = 4\dfrac{dx}{dt} + 2\dfrac{dz}{dt} = 4\dfrac{dx}{dt} - 4x \\ \dfrac{dz}{dt} = -2x \\ y = k_1 e^{-4t} \end{cases}$$

$$\Delta = 4^2 - 4(-4)^2 = 0$$

$$x_\pm = \frac{4 \pm \sqrt{0}}{2} = 2$$

$$\Rightarrow \begin{cases} x = (k_2 + k_3 t)e^{2t} \\ y = k_1 e^{-4t} \\ z = \dfrac{1}{2}\dfrac{dx}{dt} - 2x = \dfrac{1}{2}k_3 e^{2t} + (k_2 + k_3 t)e^{2t} - 2(k_2 + k_3 t)e^{2t} \end{cases}$$

$$= \underline{\left(-k_2 + \frac{1}{2}k_3 - k_3 t\right)} e^{2t}$$

at $t = 0$,
$x = x_0 \Rightarrow k_2 = x_0$

$y = y_0 \Rightarrow k_1 = y_0$

$z = z_0 \Rightarrow \underline{-k_2 + \frac{1}{2}k_3 = z_0}$

$$\Rightarrow \quad k_3 = 2(z_0 + x_0)$$

$$\Rightarrow \begin{cases} x = [x_0 + 2(z_0+x_0)t]\, e^{2t} \\ y = y_0 e^{-4t} \\ z = [z_0 - 2(z_0+x_0)t]\, e^{2t} \end{cases}$$

$$xyz = [x_0 + 2(x_0+z_0)t]\, y_0\, [z_0 - 2(x_0+z_0)t]$$

$$\frac{xyz}{x_0 y_0 z_0} = \frac{[x_0 + 2(x_0+z_0)t]\,[z_0 - 2(x_0+z_0)t]}{x_0 z_0}$$

$$\rightarrow \quad x + z = [x_0 + z_0]\, e^{2t}$$

$$y = y_0 e^{-4t}$$

$$x+z = [x_0 + 2(z_0+x_0)t]e^{2t} + [z_0 - 2(z_0+x_0)t]e^{2t}$$
$$= [x_0 + 2(z_0+x_0)t + z_0 - 2(z_0+x_0)t]\, e^{2t}$$
$$= [x_0 + z_0]\, e^{2t}$$

$$\Rightarrow \boxed{(x+z)^2 y = (x_0+z_0)^2 y_0}$$

$$\Rightarrow \quad y = \frac{K}{(x+z)^2}, \quad K = (x_0+z_0)^2 y_0$$

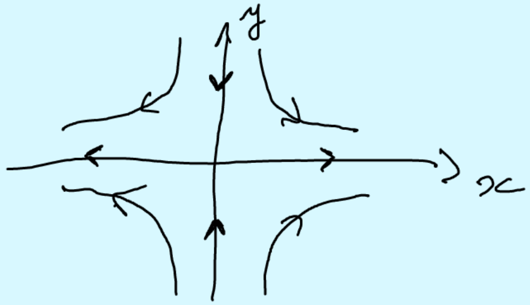
(c) $\vec{\nabla}\vec{u} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} = \begin{pmatrix} 4 & 0 & 2 \\ 0 & -4 & 0 \\ -2 & 0 & 0 \end{pmatrix}$

$\bar{\bar{E}} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\qquad \bar{\bar{\Omega}} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$

$\bar{\bar{E}} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ only $\Rightarrow$ $\frac{\partial u}{\partial x} = 4 \qquad u = 4x + u_0$

$\frac{\partial v}{\partial y} = -4 \qquad v = -4y + v_0$

To isolate the effect of $\bar{\bar{E}}$, we set $u_0 = v_0 = 0$

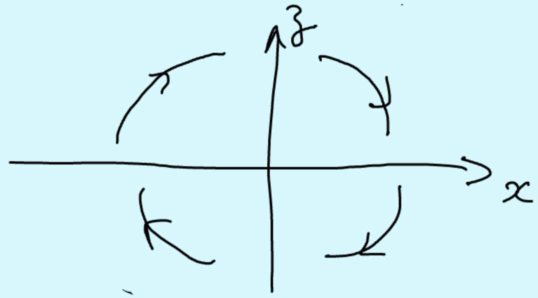


$$\Omega_{xz} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \text{only} \Rightarrow \frac{\partial u}{\partial z} = 2 \qquad u = 2z + u_0$$

$$\frac{\partial w}{\partial z} = -2 \qquad w = -2x + w_0$$

" - - - - - " $\Rightarrow$ $u = 2z$

$w = -2x$



Problem 3

$$\vec{\nabla}^2 \vec{u} = \vec{\nabla}\,(\vec{\nabla}\cdot\vec{u}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{u})$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{u})\Big|_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\vec{\nabla}\times\vec{u}\right)\Big|_k$$

$$= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left[\varepsilon_{klm}\frac{\partial}{\partial x_l}u_m\right]$$

$$= \varepsilon_{ijk}\,\varepsilon_{klm}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_l}u_m\right)$$

$$= \varepsilon_{kij}\, \varepsilon_{klm} \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_l} u_m \right)$$

$$= \left(\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl} \right) \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_l} u_m \right)$$

$$= \delta_{il}\, \delta_{jm} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} u_m - \delta_{im}\, \delta_{jl} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} u_m$$

$$= \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} u_j - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} u_i$$

$$= \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_j} \right) - \frac{\partial^2}{\partial x_j^2} u_i$$

$$\hookrightarrow \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \vec{\nabla}^2 \vec{u}$$

$\Rightarrow \quad \vec{\nabla}^2 \vec{u} = \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{u})$, which is what we were trying to prove.

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} = -G$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -\frac{G}{\mu}$$

$$\Rightarrow \quad u = -\frac{G}{2\mu} y^2 + k_2 y + k_3$$

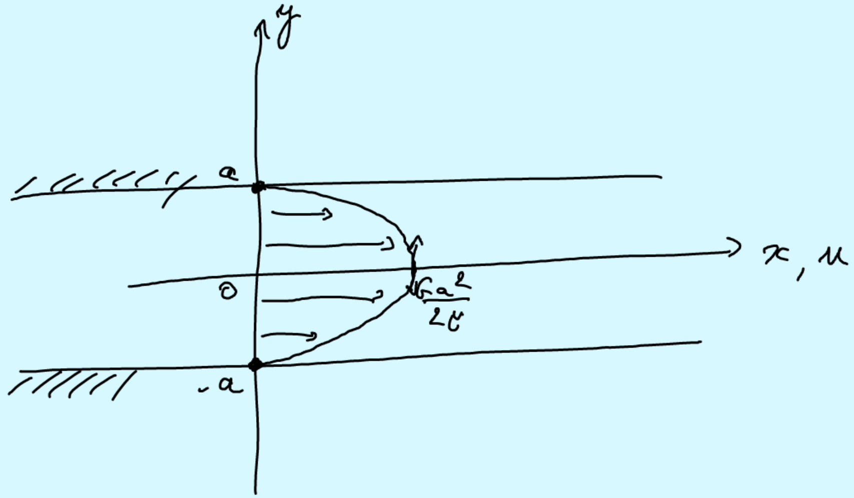
BC: $y = a : \quad u = 0 \quad \Rightarrow \quad 0 = -\frac{Ga^2}{2\mu} + k_2 a + k_3$

$y = -a : \quad u = 0 \quad \Rightarrow \quad 0 = -\frac{Ga^2}{2\mu} - k_2 a + k_3$

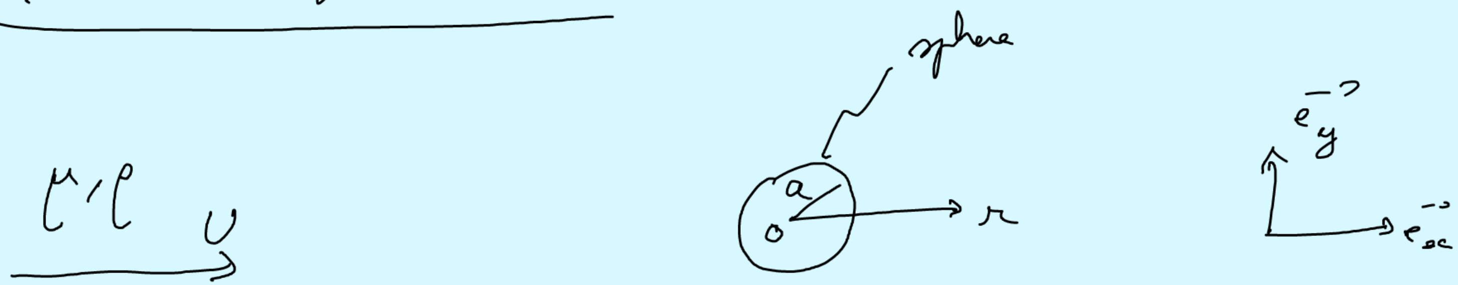
$$\Rightarrow \quad k_2 = 0 \;, \quad k_3 = \frac{Ga^2}{2\mu}$$

$$\Rightarrow \quad \boxed{u = \frac{G}{2\mu}\left[a^2 - y^2\right]}$$

(f) Sketch:



2.6 The Reynolds number



$$(*)\quad \begin{cases} \rho\left[\dfrac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u} \\ \vec{\nabla}\cdot\vec{u} = 0 \end{cases}$$

BC: when $r \to \infty$ $\quad \vec{u} = U \vec{e_x}$

$r = a \quad \vec{u} = \vec{0}$

Non-dimensionalization: $\vec{x} = a\,\vec{x}^*$

$[x] = L$

$[a] = L$

$[x^*] = 1 \rightarrow$ x^* is dimensionless

$\vec{u} = U\,\vec{u}^*$

$[u] = L.T^{-1}$

$[U] = L.T^{-1}$

$[u^*] = 1$

$t = \frac{a}{U}\, t^*$

$p = P\,p^*$

not the total pressure
(the non-dimensionalizing pressure scale)

Into the equations: $\vec{\nabla}\cdot\vec{u} = 0 \Rightarrow \vec{\nabla}\cdot\left(U\vec{u}^*\right) = 0$

$\Rightarrow U\,\vec{\nabla}\cdot(\vec{u}^*) = 0$

$\Rightarrow \frac{U}{a}\,\vec{\nabla}^*\cdot(u^*) = 0$

$\Rightarrow \boxed{\vec{\nabla}^*\cdot\vec{u}^* = 0}$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial(a x^*)} = \frac{1}{a}\frac{\partial}{\partial x^*}$$

$$\rho\left[\frac{U^2}{a}\frac{\partial \vec{u}^*}{\partial t^*}+\frac{U^2}{a}\left(\vec{u}^*\cdot\vec{\nabla}^*\right)\vec{u}^*\right]=-\frac{P}{a}\vec{\nabla}^* p^*+\frac{\mu U}{a^2}\vec{\nabla}^{*2}\vec{u}^*$$

$$\Rightarrow\quad \frac{\rho U^2}{a}\left[\frac{\partial \vec{u}^*}{\partial t^*}+\left(\vec{u}^*\cdot\vec{\nabla}^*\right)\vec{u}^*\right]=-\frac{P}{a}\vec{\nabla}^* p^*+\frac{\mu U}{a^2}\vec{\nabla}^{*2}\vec{u}^*$$

$$\Rightarrow\quad \underbrace{\frac{\partial \vec{u}^*}{\partial t^*}}_{[\;]=1}+\underbrace{\left(\vec{u}^*\cdot\vec{\nabla}^*\right)\vec{u}^*}_{[\;]=1}=\underbrace{-\frac{P}{\rho U^2}\vec{\nabla}^* p^*}_{[\;]=1}+\underbrace{\frac{\mu}{\rho U a}\vec{\nabla}^{*2}\vec{u}^*}_{[\;]=1}$$

$$\longrightarrow\quad \left[\vec{\nabla}^{*2}\vec{u}^*\right]=1$$

$$\Rightarrow\quad \left[\frac{\mu}{\rho U a}\right]=1$$

This is a nondimensional number

$Re=\frac{\rho U a}{\mu}$ is the Reynolds number

$$Re=\frac{\text{inertia}}{\text{viscous effects}}$$

$Re \gg 1 \longrightarrow$ the flow is dominated by inertia

$Re \ll 1 \longrightarrow$ the flow is dominated by viscous effects

$$\hookrightarrow \quad \frac{\partial \vec{u}^*}{\partial t^*} + \left(\vec{u}^* \cdot \vec{\nabla}^*\right)\vec{u}^* = -\frac{P}{\rho U^2}\vec{\nabla}^* p^* + \frac{1}{Re}\vec{\nabla}^{*2}\vec{u}^*$$

2 cases for P: * $P = \rho U^2$:

$$\frac{\partial \vec{u}^*}{\partial t^*} + \left(\vec{u}^* \cdot \vec{\nabla}^*\right)\vec{u}^* = -\vec{\nabla}^* p^* + \frac{1}{Re}\vec{\nabla}^{*2}\vec{u}^* \quad , \quad \vec{\nabla}^* \cdot \vec{u}^* = 0$$

BC: $r^* \longrightarrow \infty \quad , \quad \vec{u}^* = 1\,\vec{e}_x$

$r = a r^*$

$r^* = 1 \quad , \quad \vec{u}^* = \vec{0}$

only 1 parameter

SIMILARITY THEORY

* $P = \frac{\mu U}{a}$

$$Re\left[\frac{\partial \vec{u}^*}{\partial t^*} + \left(\vec{u}^* \cdot \vec{\nabla}^*\right)\vec{u}^*\right] = -\vec{\nabla}^* p^* + \vec{\nabla}^{*2}\vec{u}^* \quad , \quad \vec{\nabla}^* \cdot \vec{u}^* = 0$$

BC: $r^* \longrightarrow \infty \quad \vec{u}^* = 1\,\vec{e}_x$

$r^* = 1 \quad \vec{u}^* = \vec{0}$

Chapter 3: Low Re flows

$$Re = \frac{\rho U a}{\mu} \ll 1$$

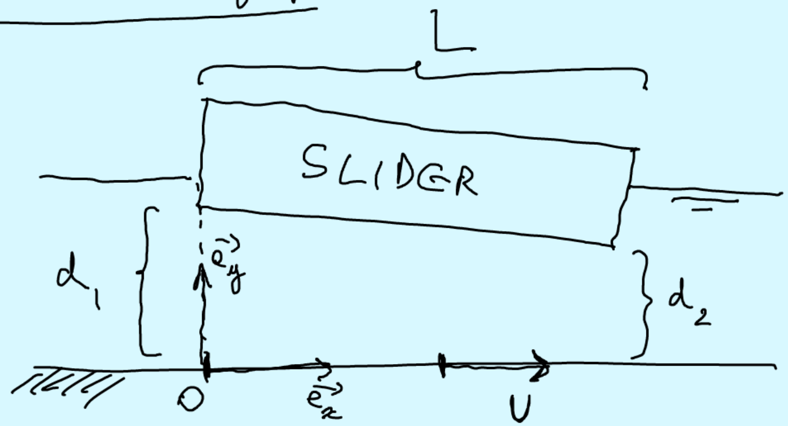
inertia $\ll$ viscous effects, the flow is dominated by viscous effects.

Stokes flow

$$\begin{cases} \vec{\nabla} p = \mu \vec{\nabla}^2 \vec{u} & \text{Stokes equation} \\ \vec{\nabla} \cdot \vec{u} = 0 \end{cases}$$

3.3.1 Slider bearing flow

⊛



height of fluid between the slider and the wall

$$h(x) = d_1 + \frac{d_2 - d_1}{L} x$$

$$\bar{h} = \frac{d_1 + d_2}{2}$$

$$h' = \frac{dh}{dx}$$

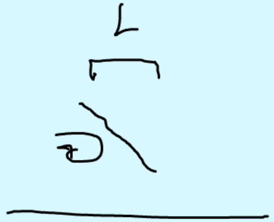
$$|h'| \ll 1$$

⊛ Equations:

$$\begin{cases} \dfrac{\partial p}{\partial x} = \mu \dfrac{\partial^2 u}{\partial x^2} + \mu \dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial p}{\partial y} = \mu \dfrac{\partial^2 v}{\partial x^2} + \mu \dfrac{\partial^2 v}{\partial y} \end{cases}$$

Lecture 9

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$



* BC. $y=0$: $u=U$, $v=0$
  $y=h(x)$: $u=0$, $v=0$

* Hypothesis : $|h'| \ll 1$ $\longleftarrow$

* Nondimensionalization :

$$x=\frac{\bar{h}}{h'}x^*$$
$$y=\bar{h}\,y^*$$
$$u=U\,u^*$$
$$v=V\,v^*=U h' v^*$$ V to be determined
$$p=P\,p^*$$ P to be determined

Look at h': $h'=\dfrac{dh}{dx}$

$$\frac{1}{h'}=\frac{dx}{dh}$$

$$\frac{\bar{h}}{h'}=\bar{h}\,\frac{dx}{dh}$$

* Determination of V and P :

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \Rightarrow \frac{U h'}{\bar{h}}\frac{\partial u^*}{\partial x^*}+\frac{V}{\bar{h}}\frac{\partial v^*}{\partial y^*}=0$$

$$\Rightarrow U h'\frac{\partial u^*}{\partial x^*}+V\frac{\partial v^*}{\partial y^*}=0$$

$\Rightarrow$ $V=Uh'$ to allow the flow to be more complex than 1D.

$$
\begin{cases}
\dfrac{P h'}{\ell} \dfrac{\partial p^*}{\partial x^*} = \mu \left(\dfrac{U h'^2}{\ell^2} \dfrac{\partial^2 u^*}{\partial x^{*2}} + \dfrac{U}{\ell^2} \dfrac{\partial^2 u^*}{\partial y^{*2}} \right) \\[2ex]
\dfrac{P}{\ell} \dfrac{\partial p^*}{\partial y^*} = \mu \left(\dfrac{U h'^3}{\ell^2} \dfrac{\partial^2 v^*}{\partial x^{*2}} + \dfrac{U h'}{\ell^2} \dfrac{\partial^2 v^*}{\partial y^{*2}} \right)
\end{cases}
$$

$$
\Rightarrow
\begin{cases}
P \dfrac{\partial p^*}{\partial x^*} = \mu \left(\cancel{\dfrac{U h'}{\ell} \dfrac{\partial^2 u^*}{\partial x^{*2}}} + \dfrac{U}{h' \ell} \dfrac{\partial^2 u^*}{\partial y^{*2}} \right) \\[2ex]
P \dfrac{\partial p^*}{\partial y^*} = \mu \left(\cancel{\dfrac{U h'^3}{\ell} \dfrac{\partial^2 v^*}{\partial x^{*2}}} + \dfrac{U h'}{\ell} \dfrac{\partial^2 v^*}{\partial y^{*2}} \right)
\end{cases}
$$

$$
\Rightarrow
\begin{cases}
P \dfrac{\partial p^*}{\partial x^*} \simeq \dfrac{\mu U}{\ell h'} \dfrac{\partial^2 u^*}{\partial y^{*2}} & \longrightarrow P = \dfrac{\mu U}{\ell h'} \\[2ex]
P \dfrac{\partial p^*}{\partial y^*} \simeq \dfrac{\mu U h'}{\ell} \dfrac{\partial^2 v^*}{\partial y^{*2}} & \longrightarrow P = \dfrac{\mu U h'}{\ell}
\end{cases}
$$

we use this scale because it is the largest, and therefore it includes the smallest contributions to the pressure

① Recap:

$$x = \frac{\bar{h}}{h'} x^*$$

$$y = \bar{h}\, y^*$$

$$u = U u^*$$

$$v = U h' v^*$$

$$p = \frac{\mu U}{\bar{h} h'} p^*$$

scales

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{\partial p^*}{\partial x^*} = \frac{\partial^2 u^*}{\partial y^{*2}}$$

$$\frac{\partial p^*}{\partial y^*} = 0$$

equations

$y^* = 0$, $u^* = 1$, $v^* = 0$

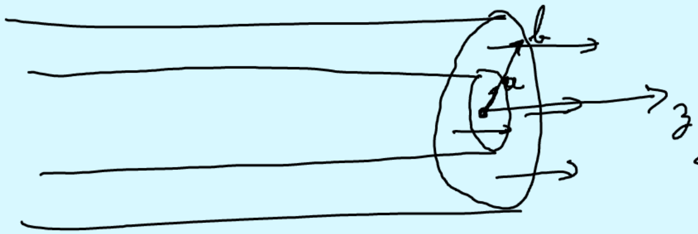
$y^* = h^*$, $u^* = 0$, $v^* = 0$

$h = \bar{h}\, h^*$

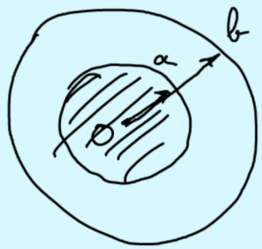
BC

EXAMPLE SHEET 2

(3)



$$\frac{\partial p}{\partial z}\vec{e}_z = -G\vec{e}_z$$



$$\vec{u} = u\,\vec{e}_r + v\,\vec{e}_\theta + w\,\vec{e}_z$$

$$\Rightarrow \vec{u} = w(r)\,\vec{e}_z$$

Navier - Stokes in cylindrical coordinates :

$$\left\{\begin{array}{l} 0 = -\dfrac{\partial p}{\partial r} \\[2ex] 0 = -\dfrac{1}{r}\dfrac{\partial p}{\partial \theta} \\[2ex] 0 = \underbrace{-\dfrac{\partial p}{\partial z}}_{f(z)} + \underbrace{\mu\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial w}{\partial r}\right)\right]}_{f(r)} \end{array}\right.$$

(first two equations) $\Rightarrow p = p(z)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = -\frac{G}{\mu} \Rightarrow r\frac{\partial w}{\partial r} = -\frac{Gr^2}{2\mu} + k_1$$

$$\Rightarrow \frac{\partial w}{\partial r} = -\frac{Gr}{2\mu} + \frac{k_1}{r}$$

$$\Rightarrow w = -\frac{Gr^2}{4\mu} + k_1 \ln r + k_2$$

$$\bar{\bar{\tau}} = \begin{pmatrix} \times & o & o \\ o & \times & o \\ o & o & \times \end{pmatrix}$$

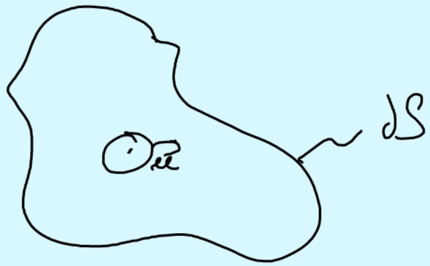
$\times$ normal stresses

o tangential stresses $\rightarrow$ shear

Only component of the velocity gradient which is non-zero is $\frac{\partial w}{\partial r}$

$$\tau_{rz} = \mu \frac{\partial w}{\partial r}$$

4 (a) $$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{G}{\mu} \qquad (1)$$



(b) For $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $w \equiv 0$

So, the velocity satisfies a no-slip boundary condition on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Into (1): $$w = A\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$$\Rightarrow \frac{2A}{a^2} + \frac{2A}{b^2} = -\frac{G}{\mu}$$

$$\Rightarrow A\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = -\frac{G}{2\mu}$$

$$\Rightarrow A = -\frac{G}{2\mu}\,\frac{a^2 b^2}{a^2 + b^2}$$

So, the velocity is, in fact: $$w = -\frac{G}{2\mu}\,\frac{a^2 b^2}{a^2 + b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

If $a=b$: $w = -\frac{Ga^2}{4\mu}\left(\frac{x^2}{a^2}+\frac{y^2}{a^2}-1\right)$

$$\Rightarrow w = -\frac{G}{4\mu}\left(x^2+y^2-a^2\right)$$

In polar coordinates $(r^2 = x^2+y^2)$: $w = \frac{G}{4\mu}\left(a^2-r^2\right)$

(c) $w = B\, y\left[(y-\sqrt{3}\,a)^2 - 3x^2\right]$

3 sides: $y=0$ (i)
$y=\sqrt{3}\,(a-x)$ (ii)
$y=\sqrt{3}\,(a+x)$ (iii)

(i): $w = B\; 0\;[\;\dots\;] = 0$

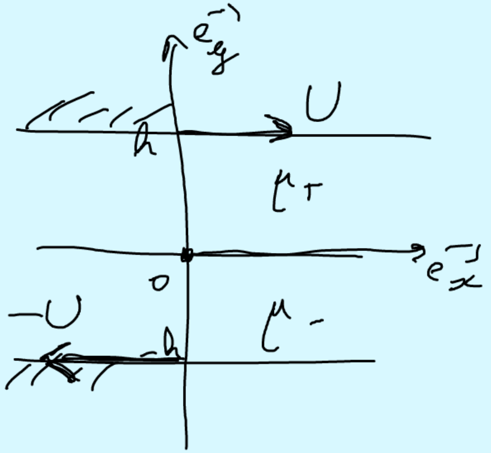
(ii): $w = B\,y\left[(\underbrace{\sqrt{3}a-\sqrt{3}x-\sqrt{3}a})^2-3x^2\right]$
$= B\,y\left[3x^2-3x^2\right]$
$= 0$

(iii) : $w = B y \left[(\sqrt{3} a + \sqrt{3} x - \sqrt{3} a)^2 - 3x^2 \right]$

$$= B y \left[3x^2 - 3x^2 \right]$$
$$= 0$$

↳ $\underline{w = B y \left[(y - \sqrt{3} a)^2 - 3x^2 \right]}$ satisfies no-slip BC on the equilateral triangle with edges $y = 0$, $y = \sqrt{3}(a - x)$, $y = \sqrt{3}(a + x)$

②



(a) $u_+(y=0) = u_-(y=0)$

$$\mu_+ \frac{\partial u_+}{\partial y}(y=0) = \mu_- \frac{\partial u_-}{\partial y}(y=0)$$

(b) $\vec{u} = u(y)\, \vec{e_x}$

↳ NS = ... $\Rightarrow$ $0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u_-}{\partial y^2} = 0 \qquad \frac{\partial^2 u_+}{\partial y^2} = 0$$

$$\Rightarrow u_- = k_1 y + k_2 \qquad \Rightarrow u_+ = k_3 y + k_4$$

BC: $y = -h \quad u_- = -U$ $\qquad$ BC: $y = h \quad u_+ = U$

$$\Rightarrow u_- = k_1 y - U + k_1 h \qquad \Rightarrow u_+ = k_3 y + U - k_3 h$$

$$\Rightarrow u_- = k_1 (y + h) - U \qquad \Rightarrow u_+ = k_3 (y - h) + U$$

$$u_+(y=0) = u_-(y=0) = u_0$$

$$k_1 h - U = u_0 \qquad -k_3 h + U = u_0$$

$$\Rightarrow k_1 = \frac{u_0 + U}{h} \qquad \Rightarrow k_3 = \frac{U - u_0}{h}$$

$$\Rightarrow u_- = \frac{u_0 + U}{h}(y + h) - U \qquad u_+ = \frac{U - u_0}{h}(y - h) + U$$

$$\mu_- \frac{\partial u_-}{\partial y}\bigg|_0 = \mu_+ \frac{\partial u_+}{\partial y}\bigg|_0 \Rightarrow \mu_- \frac{u_0 + U}{h} = \mu_+ \frac{U - u_0}{h}$$

$$\Rightarrow \quad \mu_-(\mu_0+U) = \mu_+(U-\mu_0)$$

$$\Rightarrow \quad \mu_0(\mu_- + \mu_+) = U(\mu_+ - \mu_-)$$

$$\Rightarrow \quad \mu_0 = \frac{U(\mu_+ - \mu_-)}{\mu_+ + \mu_-}$$

Lecture 11

$$x = \frac{\bar{h}}{h'} x^*$$

$$y = \bar{h}\, y^*$$

$$u = U u^*$$

$$v = V \bar{h}\, v^*$$

$$p = \frac{\mu U}{\bar{h}\, h'} p^*$$

$$\frac{\partial p^*}{\partial x^*} = \frac{\partial^2 u^*}{\partial y^{*2}}$$

$$\frac{\partial p^*}{\partial y^*} = 0$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$y^* = 0$

$u^* = 1$

$v^* = 0$

$y^* = h^*$

$u^* = 0$

$v^* = 0$

Solution: $p^* = p^*(x^*)$

$$u^* = \frac{1}{2} \frac{\partial p^*}{\partial x^*} y^{*2} + k_1 y^* + k_2$$

BC: $y^* = 0,\ u^* = 1 \Rightarrow k_2 = 1$

$$y^* = h^*,\ u^* = 0 \Rightarrow \frac{1}{2} \frac{\partial p^*}{\partial x^*} h^{*2} + k_1 h^* + 1 = 0$$

$$\Rightarrow k_1 = -\frac{1}{h^*} - \frac{1}{2} \frac{\partial p^*}{\partial x^*} h^*$$

$$\hookrightarrow u^* = \frac{1}{2}\frac{\partial p^*}{\partial x^*}\left[y^{*2} - h^* y^*\right] + 1 - \frac{y^*}{h^*}$$

$$\frac{\partial v^*}{\partial y^*} = -\frac{\partial u^*}{\partial x^*} \Longrightarrow v^*(y^*) = v^*(0) - \int_0^{y^*} \frac{\partial u^*}{\partial x^*}\, dy^*$$

$$h^* = h^*(x^*)$$

$$\vdots$$

$$\Longrightarrow v^*(y^*) = \frac{1}{2}\frac{\partial^2 p^*}{\partial x^{*2}}\left(\frac{h^* y^{*2}}{2} - \frac{y^{*3}}{3}\right) + \frac{1}{4}\frac{\partial p^*}{\partial x^*}\frac{\partial h^*}{\partial x^*} y^{*2} - \frac{y^{*2}}{2h^{*2}}\frac{\partial h^*}{\partial x^*}$$

BC: $y^* = h^* \Longrightarrow v^* = 0$

$$\Rightarrow \frac{1}{12}\frac{\partial^2 p^*}{\partial x^{*2}} h^{*3} + \frac{1}{4}\frac{\partial p^*}{\partial x^*}\frac{\partial h^*}{\partial x^*} h^{*2} - \frac{1}{2}\frac{\partial h^*}{\partial x^*} = 0$$

$$\Rightarrow \frac{\partial^2 p^*}{\partial x^{*2}} h^{*3} + 3\frac{\partial p^*}{\partial x^*}\frac{\partial h^*}{\partial x^*} h^{*2} = 6\frac{\partial h^*}{\partial x^*}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x^*}\left(\frac{\partial p^*}{\partial x^*} h^{*3}\right) = 6\frac{\partial h^*}{\partial x^*}}$$

<u>Reynolds equation</u>

$$\hookrightarrow \frac{\partial p^*}{\partial x^*} = \frac{6}{h^{*2}} + \frac{k_3}{h^{*3}}$$

Note: $h = d_1 + h'x \Rightarrow h^* = \frac{d_1}{\bar{h}} + x^*$

$$\Rightarrow dh^* = dx^*$$

$$\int_0^{L^*} \frac{\partial p^*}{\partial x^*} dx^* = \int_0^{L^*} \left(\frac{6}{h^{*2}} + \frac{k_3}{h^{*3}}\right) dx^*$$

$$\Longrightarrow p^*(L^*) - p^*(0) \simeq \int_{d_1^* = \frac{d_1}{h}}^{d_2^* = \frac{d_2}{h}} \left(\frac{6}{h^{*2}} + \frac{k_3}{h^{*3}} \right) dh^*$$

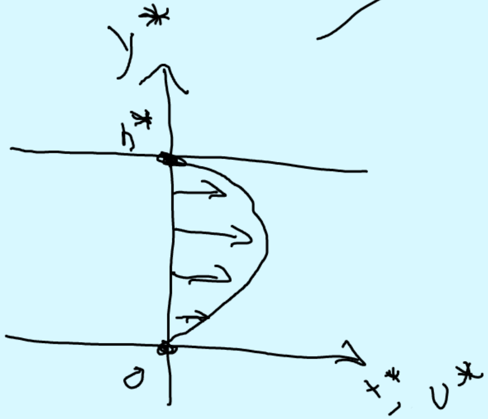
$$\Longrightarrow 0 = -\frac{6}{h^*}\Bigg|_{d_1^*}^{d_2^*} - \frac{k_3}{2h^{*2}}\Bigg|_{d_1^*}^{d_2^*}$$

$$\vdots$$

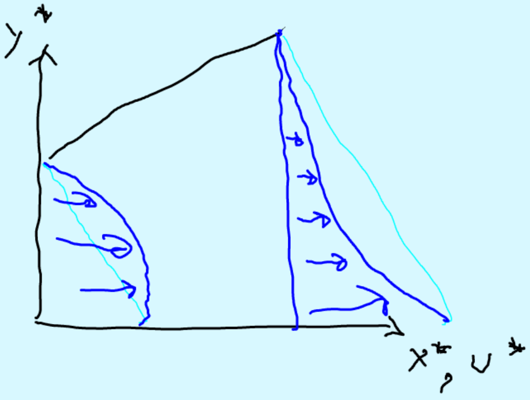
$$\Longrightarrow k_3 = -\frac{12\, d_1^* d_2^*}{(d_1^* + d_2^*)}$$

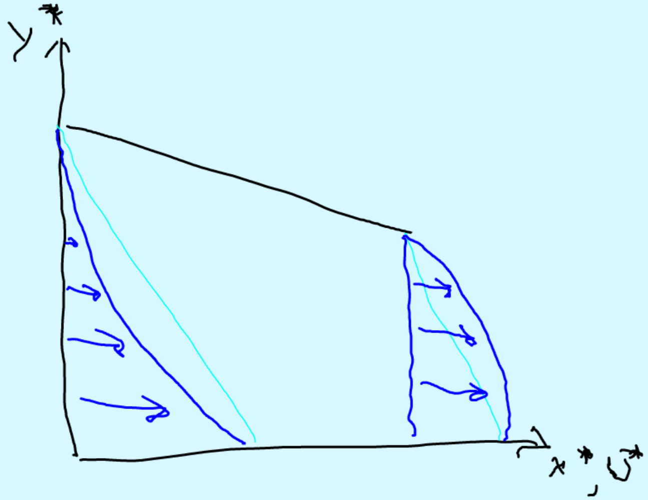
$$\hookrightarrow \frac{dp^*}{dx^*} = \frac{6}{h^{*2}} - \frac{12\, d_1^* d_2^*}{h^{*3}(d_1^* + d_2^*)}$$

$$\hookrightarrow u^* = 3\left[\frac{1}{h^{*2}} - \frac{2\, d_1^* d_2^*}{h^{*3}(d_1^* + d_2^*)} \right](y^{*2} - h^* y^*) + 1 - \frac{y^*}{h^*}$$



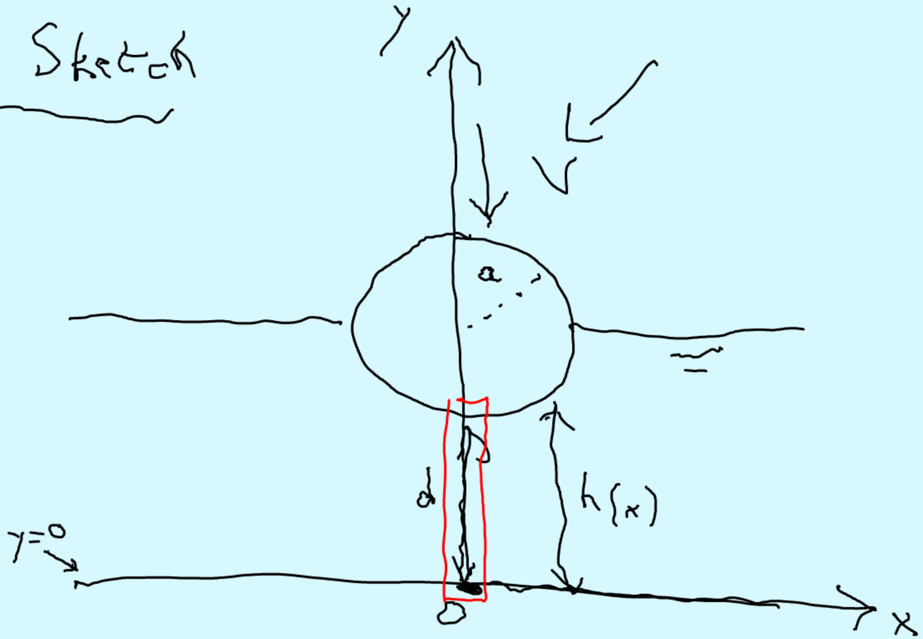
Note: $d_1^* + d_2^* = 2 \Rightarrow u^* = \frac{3}{h^{*3}}\left[h^* - d_1^* d_2^*\right]\left(y^{*2} - h^* y^*\right) + 1 - \frac{y^*}{h^*}$



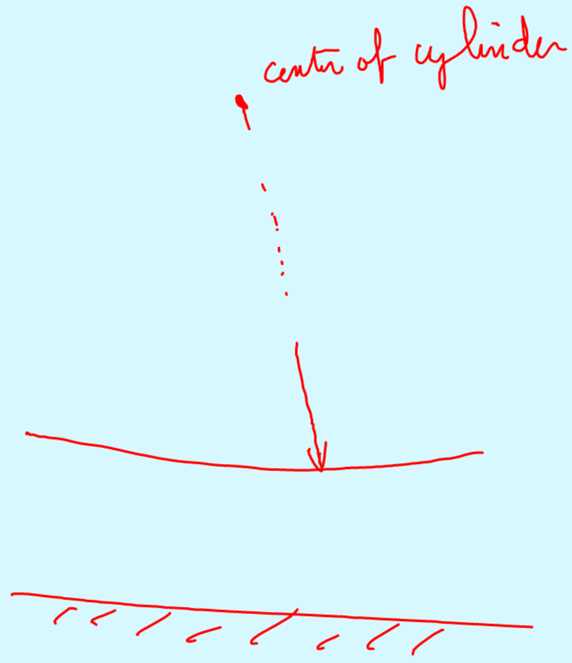


Fluid expulsion between a cylinder and a wall

Sketch



$d \ll a$



$$h(x) = d + a - \sqrt{a^2 - x^2}$$

$$= d + a - a\sqrt{1 - \frac{x^2}{a^2}}$$

$x \ll a \Rightarrow$

$$h(x) \simeq d + a - a\left(1 - \frac{x^2}{2a^2}\right)$$

$$\simeq d\left(1 + \frac{x^2}{2ad}\right)$$

* Equations:

$$
\begin{cases}
\dfrac{\partial p}{\partial x} = \mu \dfrac{\partial^2 u}{\partial x^2} + \mu \dfrac{\partial^2 u}{\partial y^2} \\[2ex]
\dfrac{\partial p}{\partial y} = \mu \dfrac{\partial^2 v}{\partial x^2} + \mu \dfrac{\partial^2 v}{\partial y^2} \\[2ex]
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0
\end{cases}
$$

* BC:

$y = 0: \; u = v = 0$

$y = h: \; u = 0, \; v = -V$

* Non-dimensionalization:

$x = ?\, x^*$

$y = d\, y^*$

$u = ?\, u^*$

$v = V v^*$

$p = ?\, p^*$

$$\frac{\partial p}{\partial y} = \mu \frac{\partial^2 v}{\partial y^2} + \cdots$$

$$x = \sqrt{ad}\, x^* = d\sqrt{\frac{a}{d}}\, x^* = d\varepsilon^{-1} x^* \qquad \varepsilon << 1$$

$$u = V\varepsilon^{-1} u^*$$

$$p = \frac{\mu V}{d} \varepsilon^{-2} p^*$$

* Recap:

$$\frac{\partial p^*}{\partial x^*} = \frac{\partial^2 u^*}{\partial y^{*2}} \qquad y^* = 0, \quad u^* = v^* = 0$$

$$\frac{\partial p^*}{\partial y^*} = 0 \qquad y^* = h^*, \quad u^* = 0, \quad v^* = -1$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \qquad \text{Note: } h^* = 1 + \frac{1}{2} x^{*2}$$

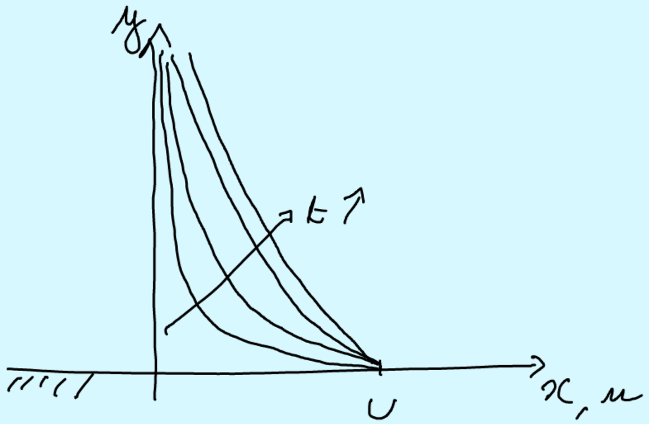
* Solution:

$$\frac{\partial}{\partial x^*}\left(\frac{\partial p^*}{\partial x^*} h^{*3} \right) = -12 \qquad \text{Reynolds equation}$$

Chapter 4: Vorticity and boundary layers

Lecture 13

4.1.1 Stokes first problem



$$(*)\quad \begin{cases} \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \cancel{v\dfrac{\partial u}{\partial y}}\right) = -\dfrac{\partial p}{\partial x} + \mu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ \rho\left(\cancel{\dfrac{\partial v}{\partial t}} + u\cancel{\dfrac{\partial v}{\partial x}} + v\cancel{\dfrac{\partial v}{\partial y}}\right) = -\dfrac{\partial p}{\partial y} + \mu\left(\cancel{\dfrac{\partial^2 v}{\partial x^2}} + \cancel{\dfrac{\partial^2 v}{\partial y^2}}\right) \\ \dfrac{\partial u}{\partial x} + \cancel{\dfrac{\partial v}{\partial y}} = 0 \end{cases}$$

$(*)$ $y = 0$: $v = 0$ $\quad u = U$

$y \to \infty$: $u \to 0$ $\quad v \to 0$

$t = 0$: $u = 0$ $\quad v = 0$

$(*)$
- $\vec{u} = u(x, y, t)\,\vec{e_x}$
- $\vec{\nabla} p = \vec{0}$

Solution :

$$\begin{cases} \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x}\right) = -\dfrac{\partial p}{\partial x} + \mu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ \dfrac{\partial p}{\partial y} = 0 \quad \rightarrow \quad p(x) \\ \dfrac{\partial u}{\partial x} = 0 \quad \rightarrow \quad u(y, t) \end{cases}$$

$$\Rightarrow \quad \rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \quad \rho \frac{\partial u}{\partial t} - \mu \frac{\partial^2 u}{\partial y^2} = -\frac{\partial p}{\partial x} = -G = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}} \qquad \nu = \frac{\mu}{\rho} \quad \text{kinematic viscosity}$$

$y = 0 \qquad u = U \qquad \cancel{v = 0}$

$y \to \infty \qquad u \to 0 \qquad \cancel{v \to 0}$

$t = 0 \qquad u = \cancel{v} = 0$

(i) Separation of variables : $u(y,t) = Y(y)\, T(t)$ . . .

(ii) Similarity variable method : $\begin{cases} \eta = y\, t^a \\ f(\eta) = u(y,t) \end{cases}$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial t} = a\, t^{a-1} y \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial y} = t^a \frac{\partial}{\partial \eta} \quad \Rightarrow \quad \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}\left(t^a \frac{\partial}{\partial \eta}\right)$$

$$= \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial y} \left(t^{a} \frac{\partial}{\partial \eta} \right)$$

$$= t^{2a} \frac{\partial^2}{\partial \eta^2}$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \Rightarrow a t^{a-1} y f' = \nu t^{2a} f''$$

$$\Rightarrow f'' = \frac{a t^{-a-1} y}{\nu} f'$$

$$y t^{-a-1} = \eta \quad \Rightarrow y t^{-a-1} = y t^{a}$$

$$\Rightarrow t^{-2a-1} = 1$$

$$\Rightarrow -2a-1=0$$

$$\Rightarrow \boxed{a = -\frac{1}{2}}$$

$$\Rightarrow f'' = -\frac{\eta}{2\nu} f' \quad , \quad \underline{\eta = y t^{-1/2} \qquad \left(\eta = \frac{y}{\sqrt{t}} \right)}$$

$$+ \text{BC}$$

$$F = f' \qquad F' = -\frac{\eta}{2\nu} F$$

$$\Rightarrow \frac{F'}{F} = -\frac{\eta}{2\nu}$$

$$\Rightarrow f' = k_1 \exp\left(-\frac{\eta^2}{4\nu} \right)$$

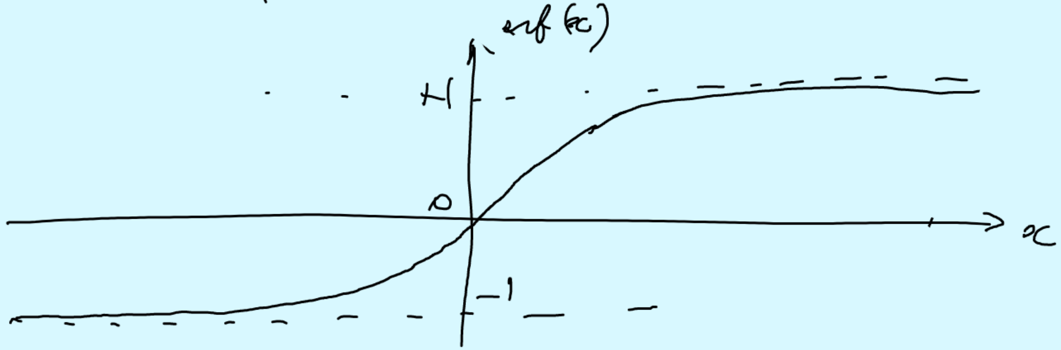
$$\Rightarrow f = k_1 \int_0^{\eta} \exp\left(-\frac{\xi^2}{4\nu} \right) d\xi + k_2$$

Error function : $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^{x} \exp(-y^2)\, dy$

$y = \dfrac{\xi}{2\sqrt{\nu}}$ $\qquad \mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^{2\sqrt{\nu}\, x} \exp\left(-\dfrac{\xi^2}{4\nu}\right) \dfrac{d\xi}{2\sqrt{\nu}}$

$2\sqrt{\nu}\, x = \eta$ $\qquad \mathrm{erf}\left(\dfrac{\eta}{2\sqrt{\nu}}\right) = \dfrac{1}{\sqrt{\pi\nu}} \displaystyle\int_0^{\eta} \exp\left(-\dfrac{\xi^2}{4\nu}\right) d\xi$

$\hookrightarrow \quad f = k_1 \sqrt{\pi\nu}\; \mathrm{erf}\left(\dfrac{\eta}{2\sqrt{\nu}}\right) + k_2$



$u = k_3\, \mathrm{erf}\left(\dfrac{y}{2\sqrt{\nu t}}\right) + k_2$

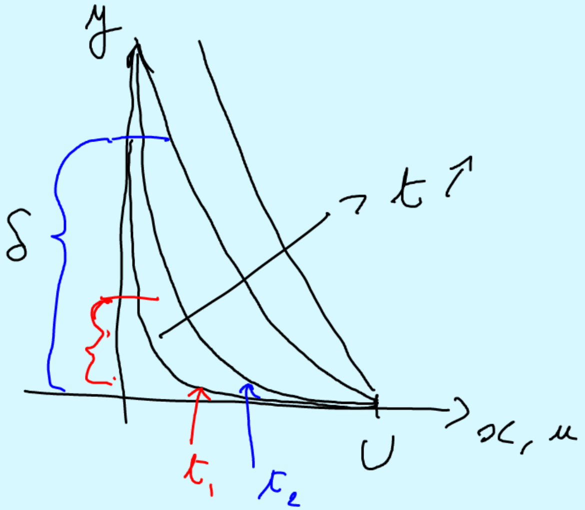
BC: $\quad y = 0 : \quad u = U \quad \Longrightarrow \quad k_2 = U$

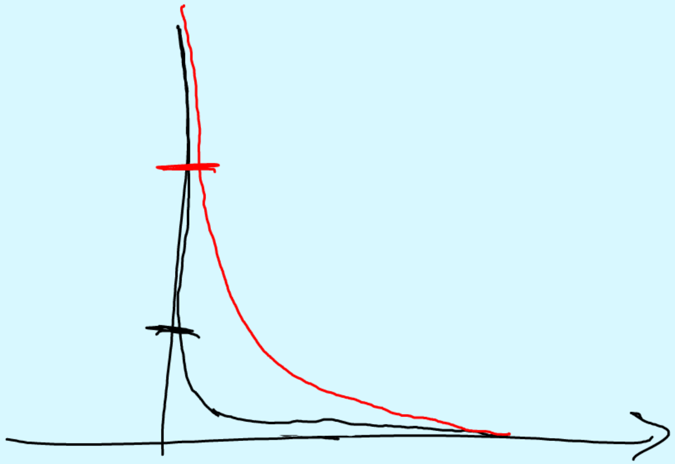
$\qquad y \to \infty : \quad u \to 0 \quad \Longrightarrow \quad k_3 + k_2 = 0$

$\qquad\qquad \Longrightarrow k_3 = -U$

$$\boxed{u = U\left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right]}$$

IC: $t=0$ $\quad u=0$ which is compatible with the above solution.



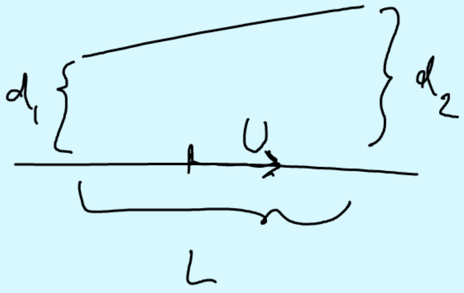


$\delta \sim \sqrt{\nu t}$ is the distance over which the effect of the moving wall is felt.
↑ thickness of the boundary layer

$\operatorname{erf}(2) \simeq 0.99 \Rightarrow \dfrac{\delta_{99}}{2\sqrt{\nu t}} = 2$ is the thickness of the boundary layer given by the 99% criterion

$$\Rightarrow \delta_{99} = 4\sqrt{\nu t}$$

Problem 3



$u \sim U$

$v \sim U h'$

$x \sim \bar{h}/h'$

$y \sim \bar{h}$

$\rho\, u \dfrac{\partial u}{\partial x}$ $\quad \rho \dfrac{U^2 h'}{\bar{h}}$

$\rho\, v \dfrac{\partial u}{\partial y}$ $\quad \rho \dfrac{U^2 h'}{\bar{h}}$

$\rho\, u \dfrac{\partial v}{\partial x}$ $\quad \to \rho \dfrac{U^2 h'^2}{\bar{h}}$

$\rho\, v \dfrac{\partial v}{\partial y}$ $\quad \rho \dfrac{U^2 h'^2}{\bar{h}}$

$$\rho(\vec{u}\cdot\vec{\nabla})\vec{u} \sim \rho \frac{U^2 h'}{\bar{h}}$$

$\mu \vec{\nabla}^2 \vec{u}$

$\mu \dfrac{\partial^2 u}{\partial x^2}$ $\quad \mu \dfrac{U h'^2}{\bar{h}^2}$

$\mu \dfrac{\partial^2 u}{\partial y^2}$ $\quad \mu \dfrac{U}{\bar{h}^2}$

$\mu \dfrac{\partial^2 v}{\partial x^2}$ $\quad \mu \dfrac{U h'^3}{\bar{h}^2}$

$\mu \dfrac{\partial^2 v}{\partial y^2}$ $\quad \mu \dfrac{U h'}{\bar{h}^2}$

Inertia is negligible provided: $\rho \frac{U^2 |h'|}{h} \ll \frac{\mu U}{h^2}$

$$\Rightarrow \quad U \ll \frac{\mu}{\rho h |h'|}$$

$$\Rightarrow \quad U \ll \frac{\mu \, 2 \, L}{\rho (d_1 + d_2) |d_2 - d_1|}$$

$$\Rightarrow \quad U \ll \frac{2 \mu L}{\rho |d_2^2 - d_1^2|}$$

$$\Rightarrow \quad \frac{\rho U (d_1 + d_2)}{2 \mu} \ll \frac{2 \mu L}{\rho |d_2^2 - d_1^2|} \, \frac{\rho (d_1 + d_2)}{2 \mu}$$

$$\Rightarrow \quad Re \ll \frac{L (d_1 + d_2)}{|d_2^2 - d_1^2|}$$

$$\Rightarrow \quad Re \ll \frac{L}{|d_2 - d_1|}$$

↳ If $|d_2 - d_1| \ll 1$, then we can neglect inertial for large Re.

Vorticity equation

Navier - Stokes : $\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u}$

Incompressibility condition : $\vec{\nabla}\cdot\vec{u} = 0$

Note: $\vec{\nabla} \times \vec{\nabla} F = \vec{0}$ always (take cartesian example, or index notation, or vector calculus)

Remember from ES1: $\vec{u}\times(\vec{\nabla}\times\vec{u}) = \frac{1}{2}\vec{\nabla}(\vec{u}^2) - (\vec{u}\cdot\vec{\nabla})\vec{u}$

$$\hookrightarrow \rho\left[\frac{\partial \vec{u}}{\partial t} + \frac{1}{2}\vec{\nabla}(\vec{u}^2) - \vec{u}\times\underbrace{(\vec{\nabla}\times\vec{u})}_{\vec{\omega}}\right] = -\vec{\nabla} p + \mu\vec{\nabla}^2\vec{u}$$

$$\Rightarrow \rho\left[\frac{\partial \vec{u}}{\partial t} - \vec{u}\times\vec{\omega}\right] = -\vec{\nabla}\left[p + \frac{1}{2}\vec{u}^2\right] + \mu\vec{\nabla}^2\vec{u}$$

We take the curl: $\rho\frac{\partial\vec{\omega}}{\partial t} - \rho\vec{\nabla}\times(\vec{u}\times\vec{\omega}) = \vec{0} + \mu\vec{\nabla}^2\vec{\omega}$

$$\vec{\nabla}\times(\vec{u}\times\vec{\omega})\Big|_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{u}\times\vec{\omega})_k$$

$$= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\varepsilon_{klm}u_l\omega_m\right)$$

$$= \varepsilon_{kij}\,\varepsilon_{klm}\frac{\partial}{\partial x_j}(u_l\omega_m)$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\frac{\partial}{\partial x_j}(u_l\omega_m)$$

$$= \frac{\partial}{\partial x_j}\left(u_i\,\omega_j\right) - \frac{\partial}{\partial x_j}\left(u_j\,\omega_i\right)$$

$$= \underbrace{u_i \cancel{\frac{\partial \omega_j}{\partial x_j}}}_{=0 \text{ by incompressibility}} + \omega_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial \omega_i}{\partial x_j} - \underbrace{\omega_i \cancel{\frac{\partial u_j}{\partial x_j}}}_{=0 \text{ by incompressibility}}$$

$$= \left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{u}\Big|_i - \left(\vec{u}\cdot\vec{\nabla}\right)\vec{\omega}\Big|_i$$

$$\hookrightarrow \quad \rho\frac{\partial \vec{\omega}}{\partial t} + \rho\left(\vec{u}\cdot\vec{\nabla}\right)\vec{\omega} = \rho\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{u} + \mu\vec{\nabla}^2\vec{\omega}$$

$$\boxed{\rho\Bigg[\underbrace{\underbrace{\frac{\partial \vec{\omega}}{\partial t}}_{\substack{\text{rate of change}\\ \text{of vorticity}\\ \text{with time}}} + \underbrace{\left(\vec{u}\cdot\vec{\nabla}\right)\vec{\omega}}_{\substack{\text{advection}\\ \text{of}\\ \text{vorticity}}}}_{\substack{\text{inertia}\\ \text{rate of change of vorticity}\\ \text{as we follow a particle}}}\Bigg] = \underbrace{\rho\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{u}}_{\substack{\text{vortex}\\ \text{stretching}}} + \underbrace{\mu\vec{\nabla}^2\vec{\omega}}_{\text{viscous effects}}}$$

VORTICITY EQUATION

viscous effects

$$\frac{\partial \omega}{\partial t} = \nu\frac{\partial^2 \omega}{\partial x^2} \xrightarrow{\text{Fourier transform}} \frac{\partial \hat{\omega}}{\partial t} = -\nu k^2 \hat{\omega}$$

$$\hookrightarrow \hat{\omega} = \Omega\, e^{-\nu k^2 t}$$

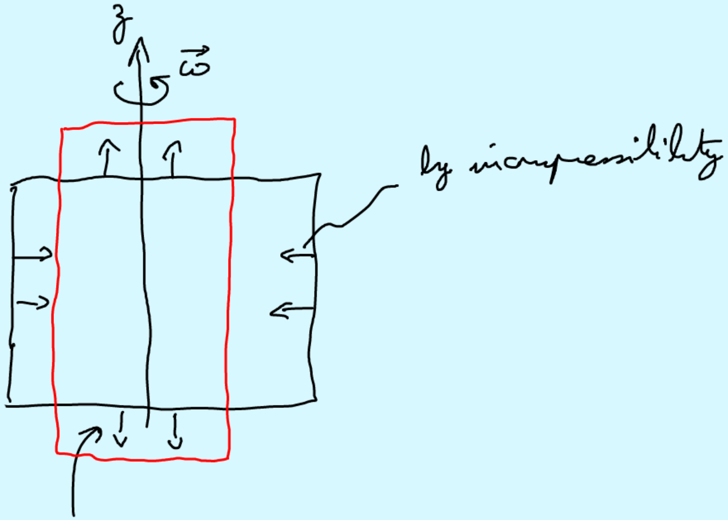
energy dissipation.

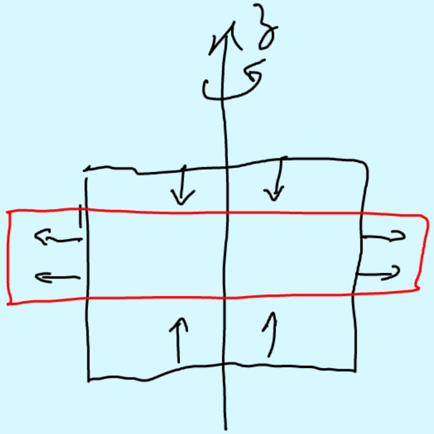
$$\frac{d\vec{\omega}}{dt} = \left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{u}$$

Let's orient the axes such that $\vec{\omega} = \omega\,\vec{e}_z$

$$\Rightarrow \frac{\partial \omega}{\partial t} = \omega \frac{\partial}{\partial z} w$$

$$\Rightarrow \underbrace{\frac{1}{\omega}\frac{\partial \omega}{\partial t}}_{\text{relative rate of change of } \omega \text{ with time}} = \frac{\partial w}{\partial z}$$





$$\frac{\partial w}{\partial z} > 0$$

$$\Rightarrow \frac{1}{\omega}\frac{\partial \omega}{\partial t} > 0$$

$$\Rightarrow |\omega| \nearrow$$

↳ blob of fluid gets thinner (and longer) and spins faster

$$\frac{\partial w}{\partial z} < 0$$

$$\Rightarrow \frac{1}{\omega}\frac{\partial \omega}{\partial t} < 0$$

$$\Rightarrow |\omega| \searrow$$

↳ blob of fluid becomes wider (and shorter) and spins more slowly.

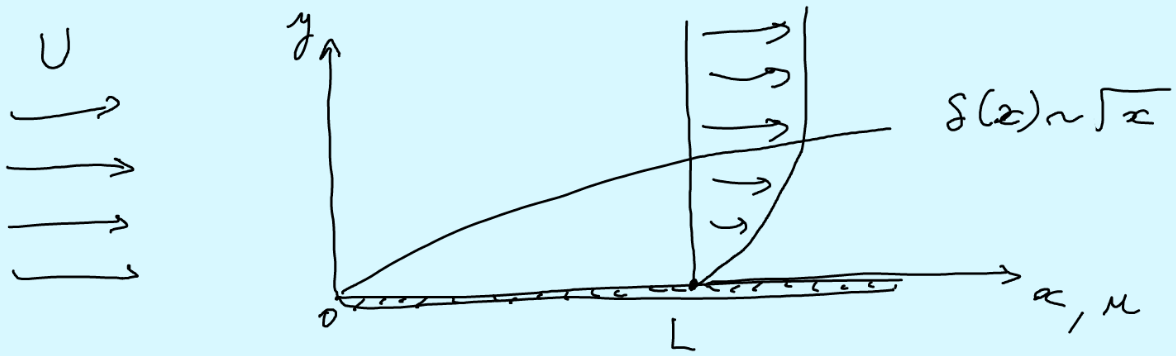
Remark: 2D flow $\rightarrow$ $\vec{u} = u(x,y)\,\vec{e_x} + v(x,y)\,\vec{e_y}$

$$\Rightarrow \vec{\omega} = \omega(x,y)\,\vec{e_z}$$

$$\rho\left[\frac{\partial \vec{\omega}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{\omega}\right] = \rho(\vec{\omega}\cdot\vec{\nabla})\vec{u} + \mu\vec{\nabla}^2\vec{\omega}$$

$$\Rightarrow \rho\left[\frac{\partial \omega}{\partial t} + (\vec{u}\cdot\vec{\nabla})\,\omega\right] = \mu\vec{\nabla}^2\omega$$

Blasius boundary layer



(*) Hypotheses:
- 2D
- steady
- Large $Re = \varepsilon^{-2}$; $\rho \frac{UL}{\mu} \gg 1$; $L \gg \frac{\mu}{U\rho}$
- thin BL: $\frac{\delta}{L} = \varepsilon \ll 1$

(*) Equations:

$$\begin{cases} \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} \\ \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial x^2} + \mu \frac{\partial^2 v}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{cases}$$

3 state variables (u, v, p)
2D problem (x, y)

(*) BC: $y = 0$: $u = v = 0$

$y \to \infty$: $u \to U$, $v \to 0$

(*) Nondimensionalization:

$$x = L x^*$$
$$y = \delta y^* = \varepsilon L y^*$$
$$u = U u^*$$
$$v = U\varepsilon v^* \longleftarrow \text{incompressibility condition}$$
$$p = \rho U^2 p^*$$

$$\hookrightarrow \begin{cases} \rho\dfrac{U^2}{L}\, u^* \dfrac{\partial u^*}{\partial x^*} + \rho\dfrac{U^2}{L}\, v^*\dfrac{\partial u^*}{\partial y^*} = -\rho\dfrac{U^2}{L}\dfrac{\partial p^*}{\partial x^*} + \dfrac{\mu U}{L^2}\dfrac{\partial^2 u^*}{\partial x^{*2}} + \dfrac{\mu U}{\varepsilon^2 L^2}\dfrac{\partial^2 u^*}{\partial y^{*2}} \\[2ex] \rho\dfrac{\varepsilon U^2}{L}\, u^*\dfrac{\partial v^*}{\partial x^*} + \rho\dfrac{\varepsilon U^2}{L}\, v^* \dfrac{\partial v^*}{\partial y^*} = -\rho\dfrac{U^2}{\varepsilon L}\dfrac{\partial p^*}{\partial y^*} + \dfrac{\mu\varepsilon U}{L^2}\dfrac{\partial^2 v^*}{\partial x^{*2}} + \dfrac{\mu U}{\varepsilon L^2}\dfrac{\partial^2 v^*}{\partial y^{*2}} \\[2ex] \dfrac{U}{L}\dfrac{\partial u^*}{\partial x^*} + \dfrac{\varepsilon U}{\varepsilon L}\dfrac{\partial v^*}{\partial y^*} = 0\end{cases}$$

$$\Rightarrow \begin{cases} \overbrace{u^*\dfrac{\partial u^*}{\partial x^*} + v^*\dfrac{\partial u^*}{\partial y^*}}^{\text{inertia}} = -\dfrac{\partial p^*}{\partial x^*} + \cancel{\dfrac{1}{Re_L}\dfrac{\partial^2 u^*}{\partial x^{*2}}} + \overbrace{\dfrac{1}{\varepsilon^2 Re_L}\dfrac{\partial^2 u^*}{\partial y^{*2}}}^{\text{diffusion}} \\[2ex] u^*\cancel{\dfrac{\partial v^*}{\partial x^*}} + v^*\cancel{\dfrac{\partial v^*}{\partial y^*}} = -\dfrac{1}{\varepsilon^2}\dfrac{\partial p^*}{\partial y^*} + \cancel{\dfrac{1}{Re_L}\dfrac{\partial^2 v^*}{\partial x^{*2}}} + \cancel{\dfrac{1}{\varepsilon^2 Re_L}\dfrac{\partial^2 v^*}{\partial y^{*2}}} \\[2ex] \dfrac{\partial u^*}{\partial x^*} + \dfrac{\partial v^*}{\partial y^*} = 0 \end{cases}$$

$$Re_L = \rho\frac{UL}{\mu}$$

$\hookrightarrow$ to keep inertia and diffusion in balance, we require $\varepsilon^2 Re_L = 1$

$$\hookrightarrow Re = \varepsilon^{-2}$$

$$\Rightarrow \rho\frac{UL}{\mu} = \frac{L^2}{\delta^2}$$

$$\Rightarrow \delta = \sqrt{\frac{L\mu}{\rho U}}$$

$$\Rightarrow \begin{cases} \rho\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) = -\dfrac{\partial p}{\partial x} + \mu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial p}{\partial y} = 0 \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

Since $\dfrac{\partial p}{\partial y} = 0$, we can get the pressure from a streamline located infinitely far away from the BL.

$$\hookrightarrow \quad p + \frac{1}{2}\rho U_\infty^2 = cst$$

$$\Rightarrow p = cst$$

$$\Rightarrow \frac{\partial p}{\partial x} = 0$$

$$\hookrightarrow \begin{cases} \rho\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) = \mu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

2 state variables (u, v)

2D (x, y)

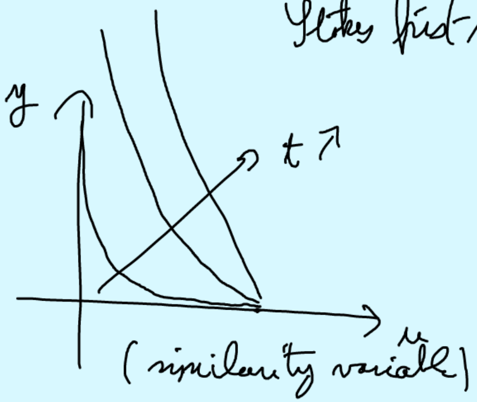
Lecture 17

Streamfunction : $u = \frac{\partial \psi}{\partial y}$ $\quad v = -\frac{\partial \psi}{\partial x}$ $\quad \nu = \frac{\mu}{\rho}$

$$\Rightarrow \boxed{\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}}$$

1 state variable (ψ)
2D (x, y)

Stokes first-problem



Blasius made the assumption that the boundary layer is self-similar:

- $\eta = y\left(\frac{U}{\nu x}\right)^{1/2}$

$$= \frac{y}{\delta(x)} \qquad \delta(x) = \left(\frac{\nu x}{U}\right)^{1/2}$$

- $u = U f'$ $\qquad f' = \frac{df}{d\eta}$ ← similarity function

↳ $u = \frac{\partial \psi}{\partial y}$ $\Rightarrow U f' = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y}$

$$\Rightarrow \frac{\partial \psi}{\partial \eta} = U f' \delta$$

$$\Rightarrow \underline{\psi = U \delta f}$$

$\delta(x)$

$$\eta = \frac{y}{\delta}$$

$$\frac{\partial \eta}{\partial x} = -\frac{y\delta'}{\delta^2}$$

- $$\frac{\partial \psi}{\partial x} = U\delta' f + U\delta \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= U\delta' f - \frac{U y \delta'}{\delta} f' = -v \;!$$

- $$\frac{\partial \psi}{\partial y} = U\delta \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= U f' = u \;!$$

- $$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left(U f'\right)$$

$$= U \frac{\partial f'}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= \frac{U}{\delta} f''$$

- $$\frac{\partial^3 \psi}{\partial y^3} = \frac{U}{\delta^2} f'''$$

- $$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(U f'\right)$$

$$= U \frac{\partial f'}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= -\frac{U\delta' y}{\delta^2} f''$$

In the ψ equation :

$$U f' \left(- \frac{U \delta' y}{\delta^2} f'' \right) - \left[U \delta' f - \frac{U y \delta'}{\delta} f' \right] \left(\frac{U f''}{\delta} \right) = \nu \frac{U f'''}{\delta^2}$$

$$\Rightarrow - \frac{U^2 \delta' y}{\delta^2} f' f'' - \frac{U^2 \delta'}{\delta} f f'' + \frac{U^2 y \delta'}{\delta^2} f' f'' = \nu \frac{U f'''}{\delta^2}$$

$$\Rightarrow \quad - \frac{U^2 \delta'}{\delta} f f'' = \nu \frac{U f'''}{\delta^2}$$

$$\Rightarrow \quad f''' + \frac{U \delta \delta'}{\nu} f f'' = 0$$

$$\delta = \left(\frac{\nu x}{U} \right)^{1/2} \quad \Rightarrow \quad \delta' = \frac{1}{2} \left(\frac{\nu}{x U} \right)^{1/2}$$

$$\Rightarrow \quad \delta \delta' = \frac{1}{2} \frac{\nu}{U}$$

$$\Rightarrow \quad \boxed{f''' + \frac{1}{2} f f'' = 0}$$

1 state variable (f)
1D (η)

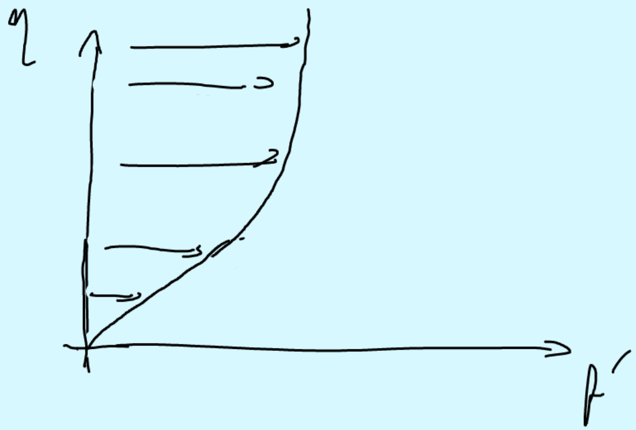
BC: $y = 0$ $\quad u = 0$, $v = 0$ $\quad\Big|\quad$ $\eta = 0$ $\quad f' = 0$, $f = 0$

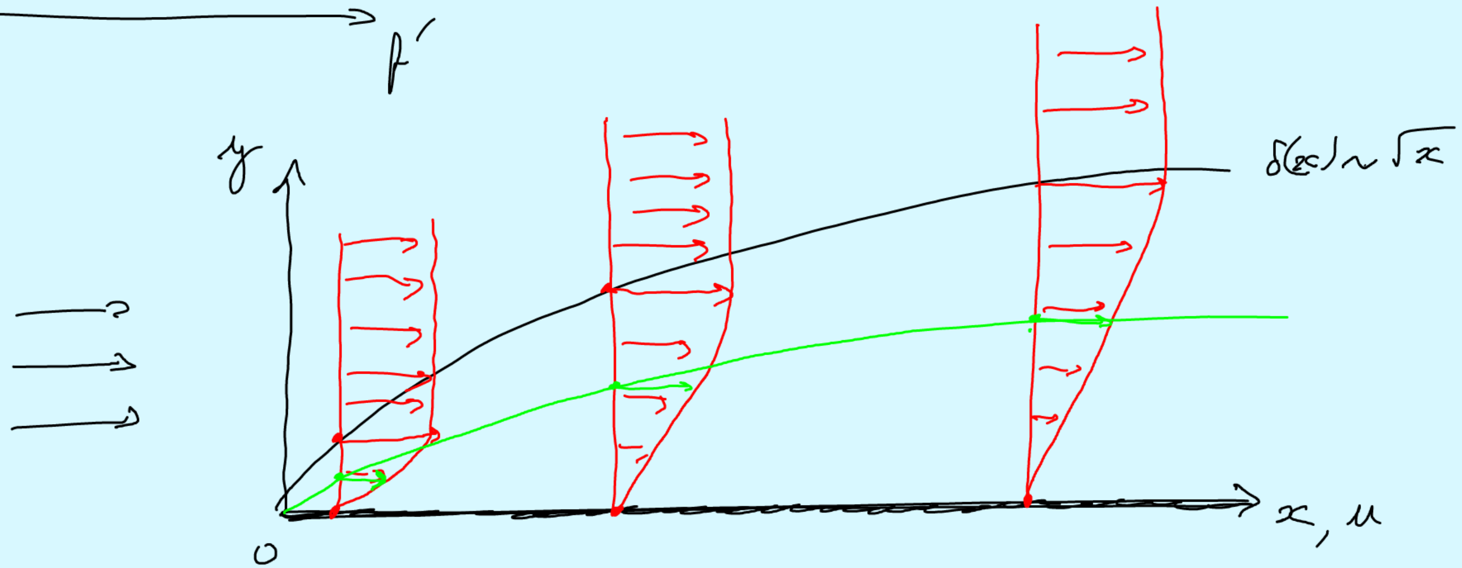
$\eta \to \infty$ $\quad u \to U$, $(v \to 0)$ $\quad\Big|\quad$ $\eta \to \infty$ $\quad f' = 1$

Boundary layer thickness: δ : $\dfrac{u}{U} \lesssim 0.99$

$\Rightarrow f' \lesssim 0.99$

$\Rightarrow \eta \lesssim 5$ (numerically)





$$\eta \leq 5 \Rightarrow \delta\left(\frac{U}{\nu x}\right)^{1/2} \leq 5$$

$$\Rightarrow \delta \leq 5\left(\frac{\nu x}{U}\right)^{1/2}$$

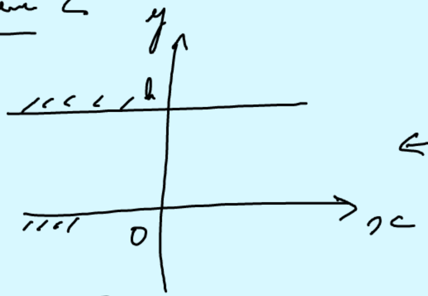
$$\hookrightarrow \delta_{99} = 5\left(\frac{\nu x}{U}\right)^{1/2}$$

$$\Rightarrow \frac{\delta_{99}}{x} = 5\left(\frac{\nu}{Ux}\right)^{1/2}$$

$$\Rightarrow \frac{\delta_{99}}{x} = 5 \, Re_x^{-1/2}$$ $\leftarrow$ boundary layer aspect ratio.

Example Sheet 4

Problem 2.



$\longleftarrow \quad \vec{\nabla} p = -G\,\vec{e}_x$

u is not a function of x

(a) 2D :
$$\begin{cases} \rho\,\partial_t u + \rho\,\cancel{u\,\partial_x u} + \rho\,\cancel{v\,\partial_y u} = \overset{G}{-\partial_x p} + \mu\,\cancel{\partial_x^2 u} + \mu\,\partial_y^2 u \\ \cancel{\rho\,\partial_t v} + \rho\,\cancel{u\,\partial_x v} + \rho\,\cancel{v\,\partial_y v} = -\partial_y p + \mu\,\cancel{\partial_x^2 v} + \cancel{\mu\,\partial_y^2 v} \\ \cancel{\partial_x u} + \cancel{\partial_y v} = 0 \end{cases}$$

$$\Rightarrow \quad \rho\,\partial_t u = G + \mu\,\partial_y^2 u$$

$$\Rightarrow \quad \boxed{\partial_t u = \frac{G}{\rho} + \nu\,\partial_y^2 u}$$

(b) $u = u_{ss}(y) \Rightarrow \partial_y^2 u_{ss} = -\dfrac{G}{\rho\nu}$

$$\Rightarrow \quad u_{ss} = -\frac{G y^2}{2\rho\nu} + k_1 y + k_2$$

BC : $y = 0 \; : \; u_{ss} = 0 \Rightarrow k_2 = 0$

$y = h \; : \; u_{ss} = 0 \Rightarrow -\dfrac{G h^2}{2\rho\nu} + k_1 h = 0$

$$\Rightarrow \quad k_1 = \frac{G h}{2\rho\nu}$$

$$\hookrightarrow \quad u_{ss} = \frac{G}{2\rho\nu}\left[h y - y^2 \right]$$

$$= \frac{G y}{2\rho\nu}\left(h - y \right)$$

(c) $u = u_{ss} + v$

$$\partial_t u = \frac{G}{\rho} + \nu\, \partial_y^2 u \quad \Rightarrow \quad \partial_t (u_{ss} + v) = \frac{G}{\rho} + \nu\, \partial_y^2 (u_{ss} + v)$$

$$\Rightarrow \quad \partial_t v = \underbrace{\frac{G}{\rho} + \nu\, \partial_y^2 u_{ss}}_{=0} + \nu\, \partial_y^2 v$$

$$\Rightarrow \quad \partial_t v = \nu\, \partial_y^2 v$$

$$\text{BC: } \left.\begin{array}{l} u = 0 \text{ at } y = 0 \\ u = 0 \text{ at } y = h \end{array}\right\} \Rightarrow \left\{\begin{array}{l} u_{ss} + v = 0 \text{ at } y = 0 \\ u_{ss} + v = 0 \text{ at } y = h \end{array}\right\} \Rightarrow \left\{\begin{array}{l} v = 0 \text{ at } y = 0 \\ v = 0 \text{ at } y = h \end{array}\right.$$

(d) $v(y,t) = Y(y)\, T(t)$

$$\partial_t v = \nu\, \partial_y^2 v \quad \Rightarrow \quad Y T' = \nu\, Y'' T$$

$$\Rightarrow \underset{(t)}{\frac{T'}{\nu T}} = \underset{(y)}{\frac{Y''}{Y}} = k_1$$

$$\Rightarrow \begin{cases} T' = \nu k_1 T & \Rightarrow \quad T = k_2 \exp(\nu k_1 t) \\ Y'' = k_1 Y & \Rightarrow \quad Y = k_3 \exp(\sqrt{k_1}\, y) + k_4 \exp(-\sqrt{k_1}\, y) \end{cases}$$

$$\Rightarrow v = \exp(\nu k_1 t)\left[\overset{k_2 k_3}{k_5} \exp(\sqrt{k_1}\, y) + \overset{k_2 k_4}{k_6} \exp(-\sqrt{k_1}\, y) \right]$$

BC: $v = 0$ at $y = 0$ $\longrightarrow$ $\exp(\nu k_1 t)\left[k_5 + k_6 \right] = 0 \Rightarrow k_5 = -k_6$

($v = 0$ at $y = h$)

$$\Rightarrow v = k_5 \exp(\nu k_1 t)\left[\exp(\sqrt{k_1}\, y) - \exp(-\sqrt{k_1}\, y) \right]$$

$v = 0$ at $y = h$ $\longrightarrow$ $\exp(\nu k_1 t)\left[\exp(\sqrt{k_1}\, h) - \exp(-\sqrt{k_1}\, h) \right] = 0$

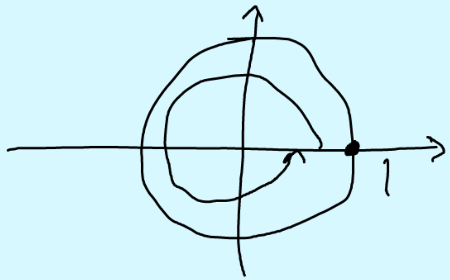
$$\Rightarrow \exp(\sqrt{k_1}\, h) = \exp(-\sqrt{k_1}\, h)$$

$$\Rightarrow \exp(2\sqrt{k_1}\, h) = 1$$

$$\Rightarrow \exp(2\sqrt{k_{1,n}}\, h) = e^{i 2 n \pi}$$

$$\Rightarrow \quad 2\sqrt{k_{1_n}}\, h = 2 i n \pi$$

$$\Rightarrow \quad k_{1_n} = -\frac{n^2\pi^2}{h^2}$$



$$\Rightarrow \quad v_n = k_{5_n} \exp\left(-\nu \frac{n^2\pi^2}{h^2} t\right) \left[\exp\left(\frac{i n \pi}{h} y\right) - \exp\left(-\frac{i n \pi}{h} y\right)\right]$$

$$\Rightarrow \quad v_n = \overset{k_{5_n} 2i}{k_{6_n}} \exp\left(-\frac{\nu n^2 \pi^2}{h^2} t\right) \sin\left(\frac{n\pi}{h} y\right)$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\hookrightarrow \quad u = \frac{G y (h-y)}{2\rho\nu} + \sum_{n=1}^{\infty} \underbrace{k_{6_n}}_{a_n} \exp\left(-\frac{\nu n^2\pi^2}{h^2} t\right) \sin\left(\frac{n\pi}{h} y\right)$$

(e) $u(t=0) = 0 \quad \Rightarrow \quad \dfrac{G y (h-y)}{2\rho\nu} + \displaystyle\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{h} y\right) = 0$

$$\Rightarrow \quad \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{h} y\right) = \frac{G y (y-h)}{2\rho\nu}$$

$$\int_0^h \sin\left(\frac{n\pi}{h} y\right) \sin\left(\frac{m\pi}{h} y\right) dy = \begin{cases} 0 & \text{for } n \neq m \\ \frac{2}{h} & \text{for } n = m \end{cases} \quad \leftarrow \text{check}$$

Chapter 5: Potential flows

Hypotheses: planar
$$\vec{u} = u(x,y)\vec{e_x} + v(x,y)\vec{e_y}$$

incompressible
$$\vec{\nabla}\cdot\vec{u} = 0$$

irrotational
$$\vec{\nabla}\times\vec{u} = \vec{0}$$

Do we need Navier-Stokes?

Route 1: Incompressible $\longrightarrow$ streamfunction ψ

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$
$$\left(\vec{u} = \vec{\nabla}\times\psi\vec{e_z}\right)$$
$$\vec{\nabla}\cdot\vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2\psi}{\partial y\partial x} = 0 \quad \checkmark$$

Irrotational: $\vec{\nabla}\times\vec{u} = 0$

$$\Rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$
$$\Rightarrow -\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} = 0$$
$$\Rightarrow \boxed{\vec{\nabla}^2\psi = 0}$$

Route 2: Irrotational $\longrightarrow$ velocity potential ϕ

$$u = \frac{\partial\phi}{\partial x} \qquad v = \frac{\partial\phi}{\partial y}$$
$$\vec{u} = \vec{\nabla}\phi$$
$$\vec{\nabla}\times\vec{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x} = 0 \quad \checkmark \quad \left(\vec{\nabla}\times\vec{\nabla}\phi = \vec{0}\right)$$

Incompressible: $\vec{\nabla}\cdot\vec{u} = 0$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$
$$\Rightarrow \boxed{\vec{\nabla}^2\phi = 0}$$

Bernoulli equations : Inviscid flow $\rightarrow$ Euler equation

$$\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\,\vec{u}\right] = -\vec{\nabla}P - \rho g\,\vec{e_z}$$

vector calculus (ES 1)

$$\Rightarrow \rho\left[\frac{\partial \vec{u}}{\partial t} - \vec{u}\times\vec{\omega}\right] = -\vec{\nabla}P - \vec{\nabla}\left(\frac{1}{2}\rho\vec{u}^2\right) - \vec{\nabla}(\rho g z)$$

$$\vec{\omega} = \vec{\nabla}\times\vec{u}$$

$$\Rightarrow \rho\left[\frac{\partial \vec{u}}{\partial t} - \vec{u}\times\vec{\omega}\right] = -\vec{\nabla}\left[P + \frac{1}{2}\rho\vec{u}^2 + \rho g z\right]$$

B1 Hypotheses : steady, irrotational

$$\hookrightarrow \vec{0} = -\vec{\nabla}\left[P + \frac{1}{2}\rho\vec{u}^2 + \rho g z\right]$$

$$0 = \frac{\partial f}{\partial x} \Rightarrow f = cst$$

$$0 = \frac{\partial f}{\partial y} \Rightarrow f = cst$$

$$\Rightarrow P + \frac{1}{2}\rho\vec{u}^2 + \rho g z = cst(t)$$

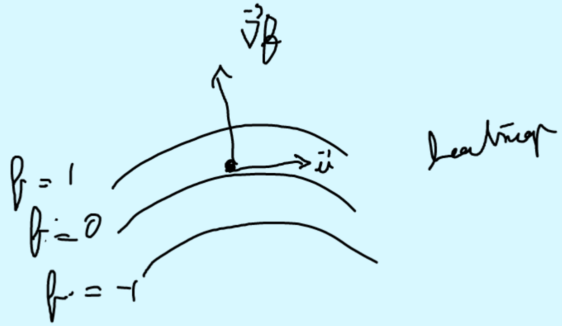
$$\Rightarrow \boxed{P + \frac{1}{2}\rho\vec{u}^2 + \rho g z = cst}$$ (because of steady flow)

B2 Hypotheses : steady

$$\hookrightarrow \rho\,\vec{u}\times\vec{\omega} = \vec{\nabla}\left[P + \frac{1}{2}\rho\vec{u}^2 + \rho g z\right]$$

$$\Rightarrow \quad \rho\, \underbrace{\vec{u}\cdot \underbrace{\left[\vec{u}\times\vec{\omega}\right]}_{\perp\, \vec{u}}}_{=0} = \vec{u}\cdot\vec{\nabla}\left[P+\frac{1}{2}\rho\vec{u}^2+\rho g z\right]$$

$$\Rightarrow \quad \vec{u}\cdot\vec{\nabla}\left[P+\frac{1}{2}\rho\vec{u}^2+\rho g z\right]=0$$



$$\Rightarrow \quad \vec{u}\cdot\vec{\nabla} f = 0$$

$$\Rightarrow \quad \vec{u} \perp \vec{\nabla} f$$

$\Rightarrow$ f varies the fastest in the direction orthogonal to $\vec{u}$

$\Rightarrow$ $\vec{u}$ is in the direction in which f does not vary

$\Rightarrow$ $f = \text{cst}(\psi)$ ← streamfunction value

$$\Rightarrow \quad \boxed{P+\frac{1}{2}\rho\vec{u}^2+\rho g z = \text{cst}(\psi)}$$

$$\boxed{P+\frac{1}{2}\rho\vec{u}^2+\rho g z = \text{cst} \qquad \text{steady, along a streamline}}$$

Complex potential :

$$f : \mathbb{C} \longmapsto \mathbb{C}$$

f is complex differentiable $\Rightarrow$ $\dfrac{df}{dz} = \lim\limits_{|\delta z| \to 0} \dfrac{f(z+\delta z) - f(z)}{\delta z}$ has to give the same result no matter the direction of δz.

Let us write $f(z) = g(x,y) + i\, h(x,y)$, $z = x + iy$, $(x,y) \in \mathbb{R}^2$

$$g : \mathbb{R}^2 \longmapsto \mathbb{R}$$
$$h : \mathbb{R}^2 \longmapsto \mathbb{R}$$

$\delta z = \delta x$: $\dfrac{df}{dz} = \dfrac{\partial g}{\partial x} + i \dfrac{\partial h}{\partial x}$

$\delta z = i \delta y$: $\dfrac{df}{dz} = \dfrac{\partial g}{i \partial y} + i \dfrac{\partial h}{i \partial y} = \dfrac{\partial h}{\partial y} - i \dfrac{\partial g}{\partial y}$

f complex differentiable $\Rightarrow$ $\begin{cases} \dfrac{\partial g}{\partial x} = \dfrac{\partial h}{\partial y} & (1) \\ \dfrac{\partial h}{\partial x} = - \dfrac{\partial g}{\partial y} & (2) \end{cases}$

Cauchy – Riemann equations

$$\frac{\partial (1)}{\partial x} + \frac{\partial (2)}{\partial y} \Longrightarrow \frac{\partial^2 g}{\partial x^2} + \cancel{\frac{\partial^2 h}{\partial x \partial y}} = \cancel{\frac{\partial^2 h}{\partial x \partial y}} - \frac{\partial^2 g}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0$$

$$\Rightarrow \quad \vec{\nabla}^2 g = 0$$

$$\frac{\partial(1)}{\partial y} - \frac{\partial(2)}{\partial x} \Rightarrow \cancel{\frac{\partial^2 g}{\partial x \partial y}} - \frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 h}{\partial y^2} + \cancel{\frac{\partial^2 g}{\partial x \partial y}}$$

$$\Rightarrow \quad \vec{\nabla}^2 h = 0$$

We can thus introduce the complex potential w for an incompressible, irrotational, planar flow:

$$\boxed{\underbrace{w}_{\text{complex potential}} = \underbrace{\phi}_{\text{velocity potential}} + i \underbrace{\psi}_{\text{streamfunction}}}$$

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

w is complex differentiable $\Rightarrow \vec{\nabla}^2 \phi = \vec{\nabla}^2 \psi = 0$

$$\left.\begin{aligned} \frac{dw}{dz} &= \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = u - iv \\ &= \frac{\partial \phi}{i \partial y} + i \frac{\partial \psi}{i \partial y} = \frac{\partial \psi}{\partial y} - i \frac{\partial \phi}{\partial y} = u - iv \end{aligned}\right\} \Rightarrow u + iv = \overline{\frac{dw}{dz}}$$

complex conjugate

$$\Rightarrow \begin{cases} u = \boxed{\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}} \\[2ex] v = \boxed{\dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x}} \end{cases}$$

Because the Laplace equations for ϕ and ψ are linear, we can use the principle of superposition : ϕ_1 and ϕ_2 solutions $\Rightarrow$ $\alpha\phi_1 + \beta\phi_2$ is a solution.

Building blocks of potential flows:

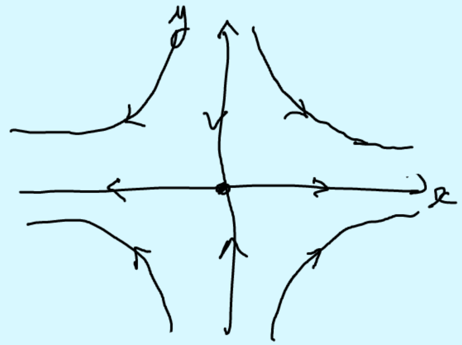
* uniform flow: $\boxed{w = U_0 z e^{-i\alpha}}$

$$\frac{dw}{dz} = U_0 e^{+i\alpha} \rightarrow \text{flow with speed magnitude } U \text{, angle } \alpha \text{ with } \vec{e}_x$$

* saddle point flow: $\boxed{w = A z^2}$

$$\frac{dw}{dz} = 2Az = 2Ax + 2iAy$$

$\hookrightarrow$ $u = 2Ax$
$v = -2Ay$



$A > 0$

* source/sink: $w = A \ln z$

$$= A \ln(re^{i\theta})$$

$$= \underbrace{A \ln r} + \underbrace{iA\theta}$$

$\phi \qquad \psi$

$$u_r = \frac{\partial \phi}{\partial r} = \frac{A}{r} \qquad A>0 \text{ source}, \quad A<0 \text{ sink}$$

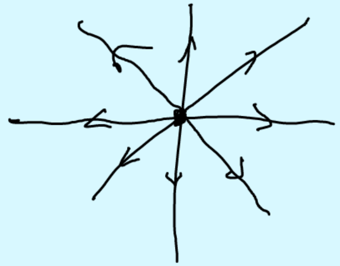
$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

$$q = \int_0^{2\pi} u_r\Big|_a a\, d\theta$$

$$= \int_0^{2\pi} \frac{A}{a} a\, d\theta$$

$$= 2\pi A \implies A = \frac{q}{2\pi}$$

$$\boxed{w = \frac{q}{2\pi}\ln z}$$

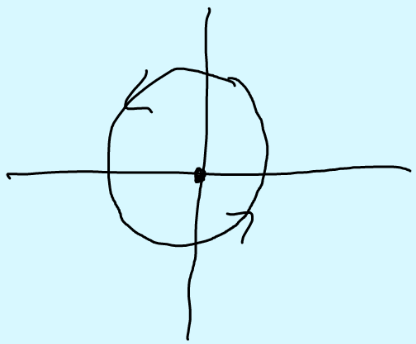


* vortex:

$$\boxed{w = -\frac{i\Gamma}{2\pi}\ln z}$$

$$= -\frac{i\Gamma}{2\pi}\ln r - \frac{i\Gamma}{2\pi} i\theta$$

$$= \underbrace{-\frac{i\Gamma}{2\pi}\ln r}_{\psi} + \underbrace{\frac{\Gamma\theta}{2\pi}}_{\phi}$$



$$u_r = \frac{\partial \phi}{\partial r} = 0$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r}$$

$$\left(\Gamma = \int_0^{2\pi} u_\theta \Big|_a \, a\, d\theta \right)$$

circulation

$$\left.\begin{aligned} &\vec{\nabla}^2 \phi = 0 \\ &\vec{\nabla}^2 \psi = 0 \end{aligned}\right\}$$

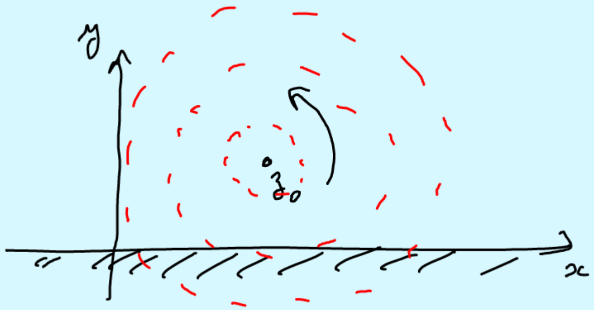
We have the equations (and the building blocks) but how to apply boundary conditions? What are boundary conditions?

Lecture 21

Boundary conditions

(1) Method of images

Imagine a vortex at z_0



$$f(z) = -\frac{i\Gamma}{2\pi} \ln (z - z_0)$$

To create a wall at $y = 0$, we apply: $w(z) = f(z) + \overline{f(\bar{z})}$

check: $z = x \longrightarrow$

$$\begin{aligned} w(x) &= f(x) + \overline{f(\bar{x})} \\ &= f(x) + \overline{f(x)} \\ &\in \mathbb{R} \end{aligned}$$

$$w = \phi + i\psi$$

$\psi(x) = 0$

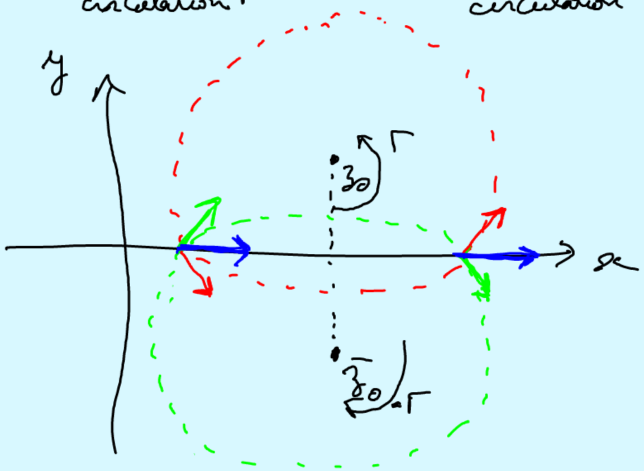
$\Rightarrow y = 0$ is a streamline

Example: $w(z) = -\dfrac{i\Gamma}{2\pi}\ln(z - z_0) \overline{- \dfrac{i\Gamma}{2\pi}\ln(\bar{z} - z_0)}$

$$= -\frac{i\Gamma}{2\pi}\ln(z - z_0) + \frac{i\Gamma}{2\pi}\ln(z - \bar{z}_0)$$

vortex at z_0
circulation Γ

vortex at $\bar{z}_0$
circulation $-\Gamma$



To generate a wall at $x = 0$, we apply: $w(z) = f(z) + \overline{f(-\bar{z})}$

(2) The Milne - Thompson circle theorem

To generate a wall at $|z| = a$, we apply: $w(z) = f(z) + \overline{f\left(\frac{a^2}{\bar{z}}\right)}$

at the wall: $z = a e^{i\theta}$ $\rightarrow$ $w(a e^{i\theta}) = f(a e^{i\theta}) + \overline{f\left(\frac{a^2}{\overline{a e^{i\theta}}}\right)}$

$$= f(a e^{i\theta}) + \overline{f\left(\frac{a^2}{a e^{-i\theta}}\right)}$$

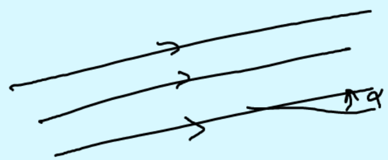
$$= f(a e^{i\theta}) + \overline{f(a e^{i\theta})}$$

$$\in \mathbb{R}$$

↳ Therefore, $|z| = a$ is a streamline (or a wall)

Example: flow past a cylinder

step 1, generate a flow $f(z) = U_0 z e^{-i\alpha}$

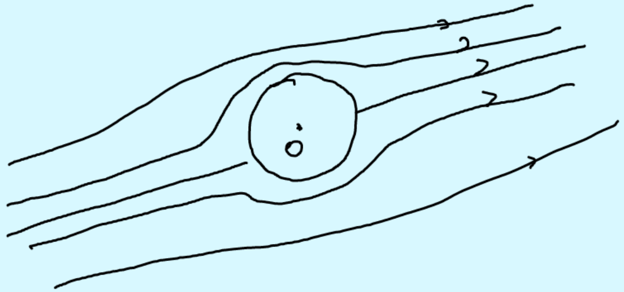


step 2: Milne-Thompson theorem

$$w(z) = U_0 z e^{-i\alpha} + \overline{U_0 \frac{a^2}{\bar{z}} e^{-i\alpha}}$$

$$= U_0 z e^{-i\alpha} + U_0 \frac{a^2}{z} e^{i\alpha}$$

$$= U_0 \left[z e^{-i\alpha} + \frac{a^2}{z} e^{i\alpha} \right]$$



step 3: add circulation: $w(z) = U_0 \left[z e^{-i\alpha} + \frac{a^2}{z} e^{i\alpha} \right] - \frac{i\Gamma}{2\pi} \ln z$

(3) Conformal mapping

First, we generate a uniform flow: $w(z) = U_0 z$

To create the above corner, we apply: $Z = z^{\alpha/\pi} \rightarrow z = Z^{\pi/\alpha}$

Then: $W(Z) = U_0 Z^{\pi/\alpha}$

The $y=0$ axis can be parametrized by: $z = re^{i\theta} \rightarrow \theta = 0$

$\searrow \theta = \pm\pi$

$* \; \theta = 0 : \quad z = r e^{i\theta} \rightarrow Z = r^{\alpha/\pi} e^{i\theta\alpha/\pi}$ ←

$= r^{\alpha/\pi} \rightarrow$ remains $y = 0$, $x > 0$

$* \; \theta = \pi : \quad Z = r^{\alpha/\pi} e^{i\theta\alpha/\pi}$

$= r^{\alpha/\pi} e^{i\alpha} \rightarrow$ rotated to make an angle α with $\theta = 0$.

$* \; \theta = -\pi : \quad Z = r^{\alpha/\pi} e^{-i\alpha} \rightarrow$ rotated to make an angle $-\alpha$ with $\theta = 0$