Formulae Sheet for Fluid Dynamics

Cartesian coordinates

pressure, p, velocity, $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$,

$$\nabla p = \frac{\partial p}{\partial x}\mathbf{e}_x + \frac{\partial p}{\partial y}\mathbf{e}_y + \frac{\partial p}{\partial z}\mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

$$\nabla \times \mathbf{u} = \begin{pmatrix} \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z} \\ \\ \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x} \\ \\ \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \end{pmatrix} \qquad \mathbf{u} \cdot \nabla \mathbf{u} = \begin{pmatrix} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} \\ \\ u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} \\ \\ u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} \end{pmatrix}$$

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \qquad \boldsymbol{\nabla^2}\mathbf{u} = \begin{pmatrix} \nabla^2 u \\ \\ \nabla^2 v \\ \\ \nabla^2 w \end{pmatrix}$$

Cylindrical Polar Coordinates

velocity, $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$.

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z},$$

$$\nabla \times \mathbf{u} = \begin{pmatrix} \dfrac{1}{r}\dfrac{\partial w}{\partial \theta} - \dfrac{\partial v}{\partial z} \\ \\ \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial r} \\ \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(rv) - \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{pmatrix} \qquad \mathbf{u} \cdot \nabla \mathbf{u} = \begin{pmatrix} u\dfrac{\partial u}{\partial r} + \dfrac{v}{r}\dfrac{\partial u}{\partial \theta} + w\dfrac{\partial u}{\partial z} - \dfrac{v^2}{r} \\ \\ u\dfrac{\partial v}{\partial r} + \dfrac{v}{r}\dfrac{\partial v}{\partial \theta} + w\dfrac{\partial v}{\partial z} + \dfrac{uv}{r} \\ \\ u\dfrac{\partial w}{\partial r} + \dfrac{v}{r}\dfrac{\partial w}{\partial \theta} + w\dfrac{\partial w}{\partial z} \end{pmatrix}$$

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} \qquad \nabla^{\mathbf{2}}\mathbf{u} = \begin{pmatrix} \nabla^2 u - \dfrac{2}{r^2}\dfrac{\partial v}{\partial \theta} - \dfrac{u}{r^2} \\ \nabla^2 v + \dfrac{2}{r^2}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r^2} \\ \nabla^2 w \end{pmatrix}$$

Spherical Polar Coordinates

$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\phi$

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\mathbf{e}_\phi,$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi},$$

$$\nabla \times \mathbf{u} = \begin{pmatrix} \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}\left(w\sin\theta\right) - \dfrac{1}{r}\dfrac{\partial v}{\partial \phi} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial u}{\partial \phi} - \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(rw\right) \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(rv\right) - \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{pmatrix}$$

$$\mathbf{u} \cdot \nabla\mathbf{u} = \begin{pmatrix} u\dfrac{\partial u}{\partial r} + \dfrac{v}{r}\dfrac{\partial u}{\partial \theta} + \dfrac{w}{r\sin\theta}\dfrac{\partial u}{\partial \phi} - \dfrac{v^2 + w^2}{r} \\ u\dfrac{\partial v}{\partial r} + \dfrac{v}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{w}{r\sin\theta}\dfrac{\partial v}{\partial \phi} + \dfrac{uv}{r} - \dfrac{w^2\cot\theta}{r} \\ u\dfrac{\partial w}{\partial r} + \dfrac{v}{r}\dfrac{\partial w}{\partial \theta} + \dfrac{w}{r\sin\theta}\dfrac{\partial w}{\partial \phi} + \dfrac{uw}{r} + \dfrac{vw\cot\theta}{r} \end{pmatrix}$$

$$\nabla^2 p = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 p}{\partial\phi^2}$$

$$\boldsymbol{\nabla^2}\mathbf{u} = \begin{pmatrix} \nabla^2 u - \dfrac{2u}{r^2} - \dfrac{2}{r^2}\dfrac{\partial v}{\partial\theta} - \dfrac{2v\cot\theta}{r^2} - \dfrac{2}{r^2\sin\theta}\dfrac{\partial w}{\partial\phi} \\[2ex] \nabla^2 v + \dfrac{2}{r^2}\dfrac{\partial u}{\partial\theta} - \dfrac{v}{r^2\sin^2\theta} - \dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial w}{\partial\phi} \\[2ex] \nabla^2 w - \dfrac{w}{r^2\sin^2\theta} + \dfrac{2}{r^2\sin\theta}\dfrac{\partial u}{\partial\phi} + \dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial v}{\partial\phi} \end{pmatrix}$$