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MATH3620 Fluid Dynamics 2

Example sheet 6

1. Consider the small amplitude, long wave motion on a body of water of depth H, where the free surface is given by $y = \eta(x,t) = a\cos(kx - \omega t)$ and there is rigid boundary at $y = -H$.

   State the boundary conditions on the velocity potential, $\phi(x,y,t)$ and hence show that the velocity potential is of the form

   $$\phi(x,y,t) = f(y)\sin(kx - \omega t)$$

   where the function f satisfies,

   $$\frac{d^2 f}{dy^2} - k^2 f = 0.$$

   Hence show that the velocity potential is given by,

   $$\phi = \frac{ga}{\omega}\frac{\cosh(k(y+H))}{\cosh kH}\sin(kx - \omega t),$$

   and obtain the dispersion relation.

   Find the fluid velocity at the point (x_0, y_0) and show that the fluid particles move in ellipses with semi-axes,

   $$\frac{a\cosh(k(y_0+H))}{\sinh kH}, \qquad \frac{a\sinh(k(y_0+H))}{\sinh kH}.$$

2. *Internal Waves* A fluid of density ρ_1 lies above a fluid of density ρ_2 (with $\rho_2 > \rho_1$ where the interface between the two fluids is at $y = \eta(x,t)$. Both fluid layers are effectively infinitely thick.

   The flows in both fluids are irrotational and given by velocity potentials ϕ_1 and ϕ_2. On the assumption that surface tension is neglible and η is sufficiently small that quadratic terms may be neglected show on $y = 0$ that η, ϕ_1 and ϕ_2 satisfy

   $$\rho_1\frac{\partial\phi_1}{\partial t} - \rho_2\frac{\partial\phi_2}{\partial t} = (\rho_2 - \rho_1)g\eta.$$
   $$\frac{\partial\phi_1}{\partial y} = \frac{\partial\phi_2}{\partial y} = \frac{\partial\eta}{\partial t}.$$

   What are the boundary conditions at $y \to \pm\infty$?

   Find solutions for ϕ_1 and ϕ_2 for the case where $\eta(x,t) = a\cos(kx - \omega t)$ and show that the dispersion relation is given by

   $$\omega^2 = gk\frac{(\rho_2 - \rho_1)}{(\rho_2 + \rho_1)}.$$

Please send any comments or corrections to Dr. C. Beaume.

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