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MATH3620 Fluid Dynamics 2

Example sheet 5

1. Using just the Cauchy–Riemann equations:
$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y},$$
$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$
obtain the following results:
   (a) Both ϕ and ψ satisfy the Laplace equation: $\nabla^2\phi = \nabla^2\psi = 0$.
   (b) The curves $\phi =$cst and $\psi =$cst intersect at right angles (Hint: show that $\nabla\psi \cdot \nabla\phi = 0$).

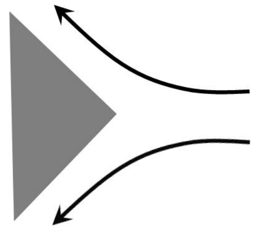
2. Show that the real part of:
$$\int_C \frac{dw}{dz} dz,$$
is equal to the circulation:
$$\Gamma = \int_C \mathbf{u} \cdot \mathbf{dx}.$$

3. Consider the complex potential:
$$w(z) = \frac{U}{a-b}\left[az - b(z^2 - a^2 + b^2)^{1/2}\right],$$
where a and b are real numbers.
   (a) Show that the imaginary part of $w(z)$ is zero on $z = a\cos\theta + ib\sin\theta$ and deduce that the ellipse:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
is a streamline.
   (b) Show that $w(z) \simeq Uz$ as $|z| \to \infty$ and deduce that $w(z)$ is the complex potential for flow past an ellipse with semi-axes a and b.

4. Show that if $f(z)$ is a complex potential then:
$$w(z) = f(z) + \overline{f(-\overline{z})},$$
is a complex potential with the imaginary axis as a streamline. Then, find the image system corresponding to a vortex at (a, b) in the presence of a wall at $x = 0$.

   By using the above result together with the image system when the real axis is a streamline obtain a complex potential in the case both the real and imaginary axes are streamlines. Lastly, obtain the image system corresponding to a vortex at (a, b) in the presence of walls along $x = 0$ and $y = 0$.

5. Show that $Z = z^{3/4}$ maps the complex plane onto the region $-3\pi/4 \leq \arg(Z) \leq 3\pi/4$. By considering $w(z) = -Uz$, find the complex potential for a flow divided by a right-angled wedge like the one in the figure below.



6. *Aerodynamic Torque on an Ellipse*

   In the lectures, it was shown that the aerodynamic torque T on a streamlined body due to the complex potential $W(Z)$ is given by:

$$T = -\frac{\rho}{2}\Re\left\{\oint_C Z\left(\frac{dW}{dZ}\right)^2 dZ\right\},$$

   where C is the contour around the body. Show that if there is a conformal mapping $Z(z)$ such that $W(Z) = w(z)$ then the torque can be evaluated in the z-plane as:

$$T = -\frac{\rho}{2}\Re\left\{\oint_{C_z} Z(z)\left(\frac{dw}{dz}\right)^2\left(\frac{dZ}{dz}\right)^{-1} dz\right\},$$

   where C_z is the contour corresponding to C in the z-plane. Hence, by using the mapping:

$$Z = z + \frac{c^2}{z},$$

   and the complex potential for flow past a cylinder, show that the torque on an ellipse inclined at an angle α is given by:

$$T = -2\rho\pi c^2 U_0^2 \sin 2\alpha.$$

Please send any comments or corrections to Dr. C. Beaume.

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