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MATH3620 Fluid Dynamics 2

Example sheet 4

1. A vortex sheet is created in the (x, y) plane along the line $y = 0$ such that at time $t = 0$:

$$u = \begin{cases} U & y > 0 \\ -U & y < 0 \end{cases}.$$

   (a) On the assumption that the velocity is one-dimensional and that u is independent of x and that there are no pressure gradients, show that $u(y, t)$ satisfies the diffusion equation:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{1}$$

   and give a definition for the diffusivity ν.

   (b) Show that equation (1) admits similarity solutions of the form $u(y, t) = f(\eta)$, where $\eta = yt^a$ for some constant a which you should determine, with f satisfying:

$$\frac{d^2 f}{d\eta^2} + \frac{\eta}{2\nu}\frac{df}{d\eta} = 0.$$

   State the boundary conditions that apply on f for $\eta \to \pm\infty$.

   (c) Show that for $t > 0$ the velocity is given by:

$$u = U\text{erf}\left(\frac{y}{2\sqrt{\nu t}}\right).$$

2. A viscous fluid is contained in the gap between two stationary walls at $y = 0$ and $y = h$ and is initially at rest. At $t = 0$, a pressure gradient $\frac{dp}{dx} = -G$ is suddenly applied.

   (a) On the assumption that the velocity is one-dimensional and that u is independent of x, show that, for $t > 0$, $u(y, t)$ satisfies the equation:

$$\frac{\partial u}{\partial t} = \frac{G}{\rho} + \nu \frac{\partial^2 u}{\partial y^2}. \tag{2}$$

   (b) Show that:

$$u_{SS}(y) = \frac{Gy(h - y)}{2\rho\nu},$$

   is a solution of this equation.

   (c) By positing:

$$u(y, t) = u_{SS}(y) + v(y, t),$$

   show that $v(y, t)$ satisfies the diffusion equation:

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial y^2},$$

   and state the boundary conditions on v.

(d) Using separation of variables to obtain $v(y,t)$, show that the general solution of equation (2) satisfying the boundary conditions $u = 0$ on $y = 0$ and $y = h$ is given by:

$$u(y,t) = \frac{Gy(h-y)}{2\rho\nu} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{\nu n^2 \pi^2}{h^2} t\right) \sin\left(\frac{n\pi y}{h}\right).$$

(e) Obtain the solution for $u(y,t)$ satisfying the initial condition $u = 0$ at $t = 0$.

3. Consider the motion of a viscous fluid in the region $y > 0$ caused by oscillating a wall at $y = 0$ so that:

$$\mathbf{u}(0,t) = (U \exp(i\Omega t), 0, 0).$$

(a) On the assumption that the fluid velocity in the region $y > 0$ is of the form:

$$\mathbf{u} = (u(y,t), 0, 0),$$

and that the pressure gradient is zero, show that the fluid velocity satisfies:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}.$$

(b) Seek a solution of the form $u(y,t) = f(y)\exp(i\Omega t)$ and show that such a solution satisfying the boundary condition at $y = 0$ and decaying to zero as $y \to \infty$ is given by:

$$u(y,t) = U \exp(-ky) \exp(i\Omega t - iky),$$

where $k = \sqrt{\Omega/2\nu}$ (Hint: recall that $i = (1+i)^2/2$).

Please send any comments or corrections to Dr. C. Beaume.

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