Dr. C. Beaume

MATH3620 Fluid Dynamics 2

Example sheet 1

1. Consider the velocity field
$$\mathbf{u} = (4x + 2z, -4y, -2x)\,.$$
   (a) Find the equation of the particle path that passes through the general point (x_0, y_0, z_0) at $t = 0$.

   (b) Calculate $\nabla \cdot \mathbf{u}$, $\nabla \times \mathbf{u}$ and $(\mathbf{u} \cdot \nabla)\mathbf{u}$.

   (c) Find the velocity gradient tensor $\nabla\mathbf{u}$ and determine the strain-rate $\mathbf{E}$ and vorticity tensor $\mathbf{\Omega}$. Verify that $\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k$ where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. Describe the flows associated with $\mathbf{E}$ and $\mathbf{\Omega}$.

2. Show that the velocity field $\mathbf{u} = (\alpha x, -\alpha y, 0)$, where α is a constant, satisfies the Navier–Stokes equation:
$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u},$$
   and determine the associated pressure p.

3. Use index notation to show that
$$(\mathbf{u} \cdot \nabla)\,\mathbf{u} = \frac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u} \times (\nabla \times \mathbf{u}),$$
$$\nabla^2\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}).$$
   Hence show for an incompressible fluid that the Navier–Stokes equation:
$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \rho\mathbf{g} + \mu \nabla^2 \mathbf{u},$$
   can be re-written as:
$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\nabla\left(\frac{P}{\rho} + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{g} \cdot \mathbf{x}\right) - \nu \nabla \times \boldsymbol{\omega},$$
   where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity and $\nu = \mu/\rho$ is the kinematic viscosity.

Please send any comments or corrections to Dr. C. Beaume.

`c.m.l.beaume@leeds.ac.uk`