Chapter 6

Water Waves

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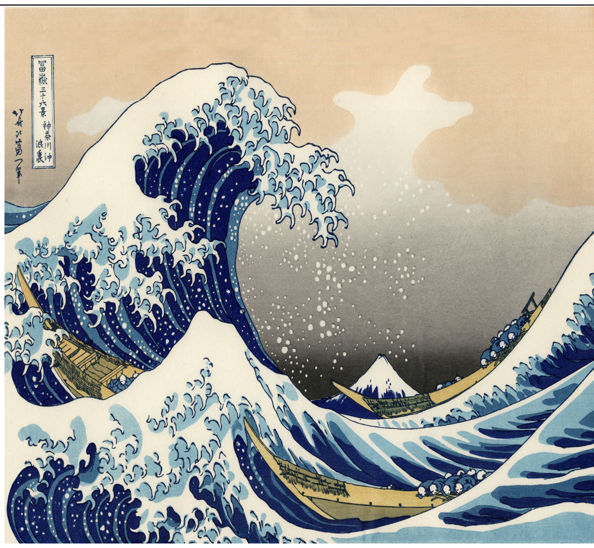


In this final chapter we will look at how waves travel in fluids. One of the most commonly observed waves are surface waves on water. These vary in scale from tsunamis or tidal waves on oceans to small ripples on ponds or bowls of water. There can also be internal waves within fluids, such as atmospheric lee waves.
Real waves are very complex and so in order to analyse them, we will have to make a number of approximations.

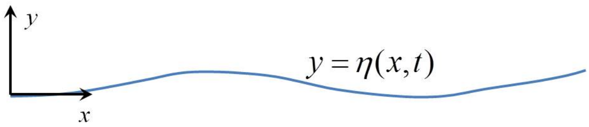
6.1 Surface Gravity Waves on Deep Water

Let us investigate two dimensional water waves where the water surface is given by:

$$y = \eta(x,t)$$

and the fluid velocity for $y < \eta(x,t)$ is:

$$\mathbf{u} = (u(x,y,t), v(x,y,t), 0).$$



We shall assume that the flow is irrotational, so that $\nabla \times \mathbf{u} = 0$ which is a reasonable assumption in water waves. We define a velocity potential $\phi(x, y, t)$, such that

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}.$$

Mass conservation yields:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{6.1}$$

We shall also assume that the water is very deep so that $u = \frac{\partial \phi}{\partial x} \to 0$ and $v = \frac{\partial \phi}{\partial y} \to 0$ as $y \to -\infty$.

6.1.1 Boundary Conditions at the Free Surface

We also require boundary conditions at the free surface $y = \eta(x, t)$. The first of these comes from the Bernoulli equation (5.1):

$$\frac{\partial \phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} - \mathbf{g} \cdot \mathbf{x} = F(t).$$

At the free surface $y = \eta(x, t)$, the pressure is equal to P_{atm} and:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2\right) + g\eta = F(t) - \frac{P_{\text{atm}}}{\rho}.$$

Since adding an arbitrary function of t to the velocity potential does not change the fluid velocity, we define ϕ so that the right-hand-side is zero:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2\right) + g\eta = 0. \tag{6.2}$$

Lastly, the kinematic boundary condition at $y = \eta(x, t)$ reads:

$$\frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v. \tag{6.3}$$

6.1.2 Linearised Boundary Conditions

The boundary conditions (6.2) and (6.3) are rather nasty nonlinear differential equations involving ϕ and η so we need to simplify the problem further. Let us assume that the wave amplitude η is small compared to the wavelength. This allows us to:

i. Impose the boundary conditions at $y = 0$.

ii. Neglect quadratic and higher order terms in both η and ϕ.

Under this approximation, equations (6.2) and (6.3) reduce to,

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \qquad \text{at } y = 0, \tag{6.4}$$

and

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial y} = 0 \qquad \text{at } y = 0. \tag{6.5}$$

6.1.3 Harmonic Waves

Let us now look for solutions of equation (6.1) that satisfy boundary conditions (6.4),(6.5) and $\partial\phi/\partial y = 0$ as $y \to -\infty$. We seek separable solutions of the form $\phi = X(x)Y(y)T(t)$. Substituting into equation (6.1), we have:

$$X''YT = -XY''T,$$

so that:

$$\frac{Y''}{Y} = -\frac{X''}{X} = k^2,$$

with k a constant. We require $Y' \to 0$ as $y \to \infty$, which is satisfied by:

$$Y = \exp(ky),$$

and $k > 0$. Additionally, we get:

$$X = A\exp(ikx) + B\exp(-ikx).$$

We can eliminate η in equations (6.4) and (6.5) by taking the time derivative of equation (6.4) and substituting into equation (6.5) to get:

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0.$$

Using the above forms for ϕ and Y, we get:

$$\frac{d^2T}{dt^2} = -gkT.$$

The solution for T reads:

$$T = C\exp(i\omega t) + D\exp(-i\omega t),$$

where the frequency ω satisfies $\omega^2 = gk$. The solution for the velocity potential takes the form:

$$\phi(x, y, t) = \exp(ky)\left[A\exp(ikx) + B\exp(-ikx)\right]\left[C\exp(-i\omega t) + D\exp(i\omega t)\right],$$

with $\eta(x, t)$ given by

$$\eta(x, t) = \frac{i\omega}{g}\left[A\exp(ikx) + B\exp(-ikx)\right]\left[C\exp(-i\omega t) - D\exp(i\omega t)\right],$$

which we can rewrite in the form:

$$\eta(x, t) = a\cos(kx - \omega t + \alpha) + b\cos(kx + \omega t - \beta). \tag{6.6}$$

for some constants a, b, α and β. The solutions are in the form of plane waves moving to the left and right, where the angular frequency ω and wavelength k are related by:

$$\omega^2 = gk.$$

This last equation is called the *dispersion relation*.

6.1.4 Fluid Motion

In order to examine the motion of the fluid, let us consider the case of a wave moving to the right and set $\alpha = 0$ so that $\eta(x,t) = a\cos(kx - \omega t)$. To find the corresponding velocity potential, we use equation (6.4):

$$\frac{\partial \phi}{\partial t} = -g\eta = -ga\cos(kx - \omega t) \qquad \text{at } y = 0.$$

The velocity potential is:

$$\phi = \frac{ag}{\omega}\exp(ky)\sin(kx - \omega t). \tag{6.7}$$

Therefore, the fluid velocity is:

$$u = \frac{\partial \phi}{\partial x} = a\omega\exp(ky)\cos(kx - \omega t), \tag{6.8}$$
$$v = \frac{\partial \phi}{\partial y} = a\omega\exp(ky)\sin(kx - \omega t). \tag{6.9}$$

Since the velocity is small and periodic in time we can assume that the fluid particles do not move far from their original positions and hence the position $(x(t), y(t))$ of a particle initially located at (x_0, y_0) satisfies:

$$\frac{dx}{dt} = a\omega\exp(ky_0)\cos(kx_0 - \omega t),$$
$$\frac{dy}{dt} = a\omega\exp(ky_0)\sin(kx_0 - \omega t).$$

and hence:

$$x - x_0 = -a\exp(ky_0)\left(\sin(kx_0 - \omega t) - \sin(kx_0)\right),$$
$$y - y_0 = a\exp(ky_0)\left(\cos(kx_0 - \omega t) - \cos(kx_0)\right).$$

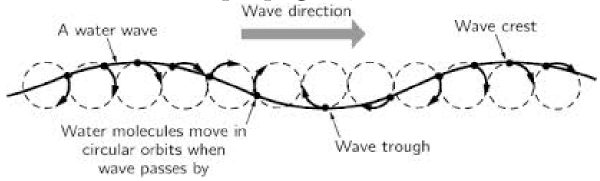
The particle path thus satisfies:

$$(x - x_c)^2 + (y - y_c)^2 = a^2\exp(2ky_0)$$

where

$$x_c = x_0 + a\exp(ky_0)\sin(kx_0),$$
$$y_c = y_0 - a\exp(ky_0)\cos(kx_0).$$

The particle paths are circles of radius $a\exp(ky_0)$. Thus, although the wave propagates in the x-direction, the fluid itself does not propagate, but moves in small circles.



6.2 Wave Speed

Calculating how fast the waves travel turns out to be more complicated than it first appears.

6.2.1 Phase Velocity

We can determine the speed at which the wave crests move. For a wave of the form:

$$\eta(x,t) = a\cos(kx - \omega t) = a\cos\left[k\left(x - \frac{\omega}{k}t\right)\right],$$

the angle within the cosine function remains constant along the trajectory:

$$x = x_0 + \frac{\omega}{k}t.$$

We define the *phase speed* or *phase velocity* as:

$$c_p = \frac{\omega}{k}. \tag{6.10}$$

for the case of gravity waves on deep water, we have:

$$c_p = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}, \tag{6.11}$$

where $\lambda = 2\pi/k$ is the wavelength. In particular the crests of long waves travel faster than shorter waves. For example a wave of length 10cm has a wave speed of 0.4 m·s^{-1} whereas a wave of 1 km in length travels at 40 m·s^{-1}. Consequently, unlike for acoustic waves, waves of different lengths travel at different speeds and will disperse. These waves are called *dispersive waves.*

6.2.2 Energy Flow

Let us now determine the energy associated with the wave motion. The potential energy at position x is purely gravitational:

$$V(x) = \int_0^{\eta} \rho g y dy = \frac{\rho g \eta^2}{2}. \tag{6.12}$$

For the case $\eta = a\cos(kx - \omega t)$, we have:

$$V(x,t) = \frac{\rho g a^2}{2}\cos^2(kx - \omega t),$$

which, averaged over a cycle yields:

$$\overline{V} = \frac{\rho g a^2}{4}.$$

To find the kinetic energy per unit length, we use the solution for the fluid velocity (equations (6.8) and (6.9)):

$$\overline{T}(x,t) = \int_{-\infty}^{0}\left(\frac{\rho}{2}\right)(u^2 + v^2)dy = \frac{\rho a^2 g^2 k^2}{2\omega^2}\int_{-\infty}^{0}\exp(2ky)dy = \frac{\rho a^2 g^2 k}{4\omega^2} = \frac{\rho a^2 g}{4},$$

using the dispersion relation $gk = \omega^2$. Hence adding this to the potential energy, the average energy per unit length associated with the wave is:

$$\overline{E} = \overline{V} + \overline{T} = \frac{\rho g a^2}{2}.$$

This energy is transported by the fluid pressure. The rate of work of the fluid pressure at x is given by

$$\frac{dW}{dt} = \int_{-\infty}^{0}(P - P_H)udy$$

where P_H is the hydrostatic pressure. Using the Bernoulli equation, we find $P - P_H = -\rho\frac{\partial\phi}{\partial t}$, so:

$$\frac{dW}{dt} = -\rho\int_{-\infty}^{0}\frac{\partial\phi}{\partial t}udy = \frac{a^2g^2k}{\omega}\cos^2(kx-\omega t)\int_{-\infty}^{0}\exp(2ky)dy = \frac{a^2g\omega}{2k}\cos^2(kx-\omega t).$$

Averaged over a cycle, we obtain:

$$\overline{\frac{dW}{dt}} = \frac{a^2g\omega}{4k}.$$

The speed of energy transport is:

$$\frac{1}{\overline{\overline{E}}}\overline{\frac{dW}{dt}} = \frac{\omega}{2k},$$

which is equal to half the phase speed. This unexpected result demonstrates that the wave energy is not transported at the same rate.

6.2.3 The Group Velocity

To see how the energy can move at a different velocity to the wave crests let us consider the evolution of a wave packet containing waves of frequency close to k_0. At $t = 0$, we can define η from its Fourier transform:

$$\eta(x,0) = f(x) = \int_{-\infty}^{\infty}F(k)\exp(ikx)dk \tag{6.13}$$

but where $F(k)$ is only non-zero for values of k close to k_0. At time t, the surface is given by:

$$\eta(x,t) = \int_{-\infty}^{\infty}F(k)\exp(ikx - i\omega(k)t)dk$$

where $\omega(k)$ is the corresponding frequency. Now since $F(k)$ is non-zero only for $k \approx k_0$, we can approximate $\omega(k)$ by its Taylor series:

$$\omega(k) \approx \omega(k_0) + \frac{d\omega}{dk}(k-k_0),$$

and hence:

$$\eta(x,t) \approx \int_{-\infty}^{\infty}F(k)\exp\left[ikx - i\omega(k_0)t - i\frac{d\omega}{dk}(k-k_0)t\right]dk,$$

giving:

$$\eta(x,t) \approx \exp\left[i(k_0x - \omega(k_0))t\right]\int_{-\infty}^{\infty}F(k)\exp\left[i(k-k_0)\left(x - \frac{d\omega}{dk}t\right)\right]dk,$$

which indicates a wave with wavenumber k_0 moving at wavespeed $\omega(k_0)$ within an envelope moving at speed:

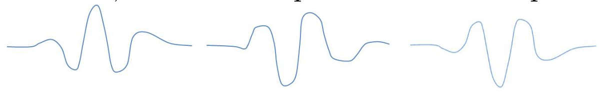
$$c_g = \frac{d\omega}{dk}, \tag{6.14}$$

where c_g is called the *group velocity*.
For gravity waves, $\omega = \sqrt{gk}$, and therefore:

$$c_g = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{c_p}{2}.$$

Note that for non-dispersive waves, ω/k is constant and $c_g = c_p$.

Although the crests of gravity waves travel at c_p, the disturbance itself travels at half this speed. You can see this if you drop a stone in a pond. The disturbance travels radially outwards at the group velocity but individual crests appear at the back of this disturbance, travel to the front and then disappear again. Consequently while the phase speed for a 1km wave is 40 m·s^{-1} the disturbance only travels at 20m·s^{-1}. However ocean waves can still travel quickly. Tsunamis or tidal waves associated with earthquakes can have wavelengths in excess of 100 km and so in the deep ocean can travel at speeds of around 200 m·s^{-1}. Note that the average ocean is around 4.3 km, so the assumption of infinite depth is no longer valid.



6.3 Gravity Waves on Shallower Water

Let us now consider the case where there is an ocean bed at $y = -H$. The boundary conditions at the free surface, $y = 0$, remain given by equations (6.4) and (6.5):

$$\frac{\partial \phi}{\partial t} + g\eta = 0,$$
$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial y} = 0.$$

The only difference is that the boundary condition $v = 0$ must be imposed at $y = -H$ rather than at $y \to \infty$. This problem is set as an exercise on the example sheet. The solution for ϕ corresponding to the surface wave disturbance:

$$\eta = a\cos(kx - \omega t), \tag{6.15}$$

is:

$$\phi = \frac{ga}{\omega}\frac{\cosh(k(y+H))}{\cosh kH}\sin(kx - \omega t), \tag{6.16}$$

with the dispersion relation given by

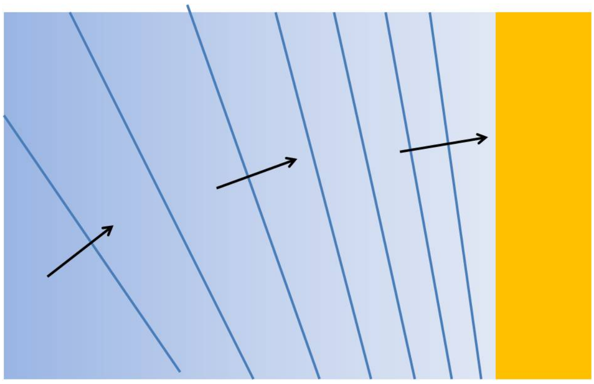
$$\omega^2 = gk\tanh kH. \tag{6.17}$$

Hence, the phase speed is:

$$c_p = \left[\frac{g\tanh kH}{k}\right]^{1/2}. \tag{6.18}$$

In the limit $kH \to \infty$, $\tanh kH \to 1$ and we recover the deep water case where $c_p = \sqrt{g/k}$. However, in the opposite limit when $kH \ll 1$, $\tanh kH \simeq kH$ yielding $c_p \to \sqrt{gH}$. Thus, in shallow water the phase speed is independent of k. The waves are non-dispersive and the group velocity is equal to the phase velocity.

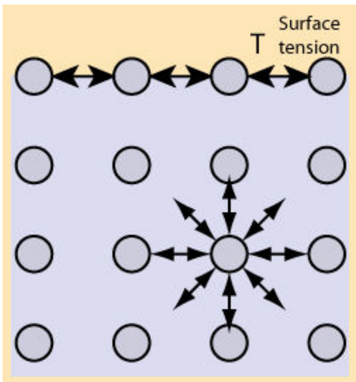
One consequence of this result is that waves slow down as the sea gets shallower. This causes the waves to become shorter and steeper. In the deep ocean a Tsunami wave is only around a metre in height with a wavelengths of around 100km and is hard to detect. However, as it reaches the coast, its wavelength reduces and the height increases around 30 fold. This is also the reason why waves always appear to come in parallel to the shore-line.



6.4 Capillary Waves

So far we have considered waves in which the force trying to restore equilibrium is due to gravity. While this is the dominant force for waves on open water, for short waves there is an alternative restoring force due to surface tension.

6.4.1 Surface Tension



In a liquid, there are short-range attractive forces between molecules. For molecules away from the boundaries there are equal numbers of molecules in all directions and so these forces cancel each other out. However, for molecules at the surface there is an imbalance in the force due to the greater affinity that the molecules have for themselves as opposed to molecules of the neighbouring fluid which results in a tension in the surface that resists expansion of the surface area.

The surface tension between air and water is 0.072 $\mathrm{N{\cdot}m^{-1}}$. This is sufficient to prevent a paper-clip from sinking and allows small insects called pond skaters to “skate” over the surface of ponds.

If the surface is flat, then the tension forces on opposite sides cancel, however, if the surface is curved, then surface tension produces a net surface force in the normal direction. As a consequence the surface force is not continuous (as assumed in equation (2.12)), but has a jump given by:

$$\mathbf{n}\cdot\boldsymbol{\tau} = -P\mathbf{n} + \mathbf{n}\cdot\boldsymbol{\sigma} = -P_{\text{atm}}\mathbf{n} - \gamma(\nabla\cdot\mathbf{n})\mathbf{n}, \tag{6.19}$$

where γ is the surface tension and $\nabla\cdot\mathbf{n}$ is the surface curvature.

For an inviscid fluid, $\boldsymbol{\sigma}$ is assumed to be negligible and we get:

$$P = P_{\text{atm}} + \gamma(\nabla\cdot\mathbf{n}).$$

If the surface is given by $f(x,y) = y - \eta(x,t) = 0$:

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{1 + (\frac{\partial \eta}{\partial x})^2}} \left(-\frac{\partial \eta}{\partial x}, 1 \right),$$

and hence the curvature is:

$$\nabla \cdot \mathbf{n} = \frac{\partial}{\partial x} \left(-\frac{\frac{\partial \eta}{\partial x}}{\sqrt{1 + (\frac{\partial \eta}{\partial x})^2}} \right).$$

For linear waves, $|\frac{\partial \eta}{\partial x}| \ll 1$, and:

$$\nabla \cdot \mathbf{n} \simeq -\frac{\partial^2 \eta}{\partial x^2}.$$

Thus, the pressure at the surface $y = \eta(x,t)$ is:

$$P = P_{\text{atm}} - \gamma \frac{\partial^2 \eta}{\partial x^2}.$$

6.4.2 Short waves on deep water

Applying the Bernoulli equation at the surface (and neglecting the term $(u^2 + v^2)$), we get:

$$\frac{\partial \phi}{\partial t} + \frac{P_{\text{atm}}}{\rho} - \frac{\gamma}{\rho} \frac{\partial^2 \eta}{\partial x^2} + g\eta = E(t),$$

and so by choosing the origin of ϕ suitably the Bernoulli equation at $y = 0$ becomes:

$$\frac{\partial \phi}{\partial t} - \frac{\gamma}{\rho} \frac{\partial^2 \eta}{\partial x^2} + g\eta = 0. \tag{6.20}$$

The kinematic boundary condition remains given by equation (6.5):

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial y} = 0$$

and the velocity potential ϕ satisfies:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

with $\frac{\partial \phi}{\partial y} \to 0$ as $y \to -\infty$.
We again look for a solution of the form $\eta(x,t) = a\cos(kx - \omega t)$, so that the boundary conditions at $y = 0$ become:

$$\frac{\partial \phi}{\partial t} = -a\cos(kx - \omega t) \left(\frac{\gamma k^2}{\rho} + g \right),$$
$$\frac{\partial \phi}{\partial y} = a\omega \sin(kx - \omega t).$$

A suitable solution of Laplace's equation is:

$$\phi = B\exp(ky)\sin(kx - \omega t),$$

where

$$-B\omega\cos(kx-\omega t) = -a\cos(kx-\omega t)\left(\frac{\gamma k^2}{\rho}+g\right),$$
$$Bk\sin(kx-\omega t) = a\omega\sin(kx-\omega t).$$

The dispersion relation becomes:

$$\omega^2 = gk + \frac{\gamma k^3}{\rho}. \tag{6.21}$$

The phase speed is given by:

$$c_p = \frac{\omega}{k} = \left(\frac{g}{k}+\frac{\gamma k}{\rho}\right)^{1/2},$$

and the group velocity by:

$$c_g = \frac{g+3\gamma k^2/\rho}{2\left(gh+\gamma k^3/\rho\right)^{1/2}}.$$

For small k, the phase speed decreases with k for $k < k_c$, where

$$k_c = \sqrt{\frac{\rho g}{\gamma}}.$$

This limit corresponds to gravity waves. However, above this critical wave number, the phase speed increases with k and surface tension becomes the dominant mechanism. Waves in this regime are referred to as *capillary waves*. The wavelength corresponding to k_c is:

$$\lambda_c = 2\pi\sqrt{\frac{\gamma}{\rho g}}$$

is around 1.7 cm in water.
Thus water waves of lengths shorter than around 1 cm are driven by surface tension rather than gravity, and in the limit $k \gg k_c$,

$$c_p = \sqrt{\frac{\gamma k}{\rho}}, \qquad c_g = \frac{3}{2}\sqrt{\frac{\gamma k}{\rho}},$$

indicating that the group velocity is faster than the phase velocity. Hence crests appear at the front of the disturbance and travel backwards relative to the wave packet.

6.5 Nonlinear waves

All the cases in this section have been based on the assumption that the amplitude is small compared with wavelength so that $|\frac{\partial\eta}{\partial x}| \ll 1$. However, this is not always the case in practice. In particular waves approaching a beach become nonlinear and overturn causing them to break. Recently it has also been recognised that nonlinear waves can occur in the deep ocean where they appear as dangerous high amplitude steep waves, known as rogue waves. There are also special nonlinear waves called solitary waves that can travel long distances without changing shape, as famously discovered by John Scott Russell on the Glasgow-Edinburgh Canal.