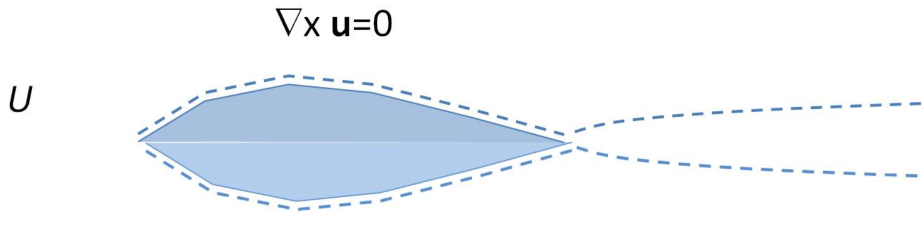
Chapter 5

Irrotational flow — Complex potential and aerofoils

Contents

In the previous chapter, we discussed high Reynolds numbers flow past a streamlined object. In these flows, vorticity is only generated at the object surface and is therefore confined to a thin boundary layer around the surface of the object and to a narrow wake behind it.



Consequently outside these boundary layers the flow has no vorticity and is described as being irrotational.

5.1 Velocity potential

If $\nabla \times \mathbf{u} = 0$, we can write:

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \qquad w = \frac{\partial \phi}{\partial z}.$$

Note that we can add any arbitrary function of t to ϕ.

5.1.1 Incompressible flow

If the fluid is also incompressible, then:

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0,$$

so that ϕ satisfies the Laplace equation.

For example,

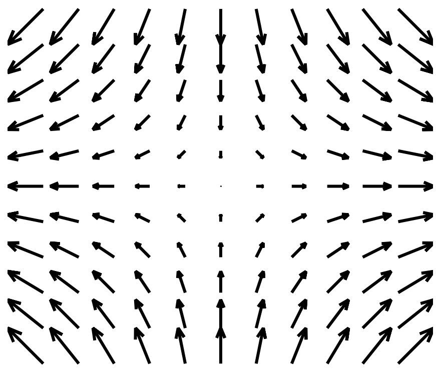
$$\phi = \frac{1}{2} E(x^2 - y^2),$$

satisfies

$$\nabla^2 \phi = 0.$$

and corresponds to the stagnation point flow:

$$\mathbf{u} = \nabla \phi = \begin{pmatrix} Ex \\ -Ey \\ 0 \end{pmatrix} = Ex\hat{\boldsymbol{\imath}} - Ey\hat{\boldsymbol{\jmath}}.$$



In cases where the domain is not infinite, we need boundary conditions. Since we are not including viscous boundary layers, we cannot fully impose the condition $\mathbf{u} = \mathbf{U}$ on a rigid boundary. However, since the volume of the boundary layers is small, we can assume mass is conserved:

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n},$$

where $\mathbf{n}$ is the normal to the surface. The boundary condition on the velocity potential is:

$$\mathbf{n} \cdot \nabla \phi = \frac{\partial \phi}{\partial \mathbf{n}} = \mathbf{U} \cdot \mathbf{n}.$$

This is a *Neumann boundary condition*: it involves the first spatial derivative only. The solution of the Laplace equation is unique up to an arbitrary additive constant. There is a unique solution to the fluid velocity satisfying these boundary conditions.

5.1.2 The Bernoulli equation

Since we neglect the effects of viscosity, the Navier–Stokes equation reduces to the Euler equation (2.5):

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \rho \mathbf{g} - \nabla P.$$

Using the vector identity:

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2} \nabla (\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{u},$$

we can rewrite this as:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\nabla \left(\frac{P}{\rho} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - \mathbf{g} \cdot \mathbf{x} \right).$$

For a potential flow ($\boldsymbol{\omega} = 0$) and since $\mathbf{u} = \nabla \phi$, we obtain:

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - \mathbf{g} \cdot \mathbf{x} \right) = 0,$$

yielding:

$$\frac{\partial \phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} - \mathbf{g}\cdot\mathbf{x} = F(t), \tag{5.1}$$

or, using the dynamic pressure $p = P - \rho\mathbf{g}\cdot\mathbf{x}$:

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} = F(t).$$

As a result, for a steady irrotational flow, $p + \frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}$ is a constant. The pressure is lower in the regions of higher flow speed.

5.2 Planar potential flows

If the flow is confined to the (x, y) plane:

$$\mathbf{u} = (u(x,y), v(x,y), 0)\,,$$

and we can express the fluid velocity for an incompressible flow in terms of the streamfunction ψ:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}.$$

If, additionally, the flow is irrotational, we have:

$$\begin{aligned}\omega = 0 &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ &= -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2},\end{aligned}$$

and therefore the streamfunction ψ satisfies the Laplace equation.
As a result, for planar incompressible irrotational flows, the velocity potential and the streamfunction are related by:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \tag{5.2}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{5.3}$$

The right-hand pair of equalities in the above equations are known as the *Cauchy–Riemann equations* and have a connection with complex variable theory as we shall see in the next section.
One property of the Cauchy–Riemann equations is that this relation between ϕ and ψ is sufficient to show that both functions satisfy the Laplace equation. Summing the x derivative of equation (5.2) with the y derivative of equation (5.3), we get:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0.$$

We can show in a similar way that:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

5.3 The complex derivative

Let f be a complex function of z where $z = x + iy$. The derivative of f with respect to z is defined as:

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}.$$

Note that this definition requires that the limit is the same for all infinitesimal increments δz which can be in any direction in the complex plane. As a consequence, differentiability of a complex function imposes restrictions on its real and imaginary parts.
If $f(z) = g(x, y) + ih(x, y)$ is complex differentiable, where g and h are both real functions then, taking $\delta z = \delta x$, we obtain:

$$\frac{df}{dz} = \frac{\partial g}{\partial x} + i\frac{\partial h}{\partial x}. \tag{5.4}$$

If, instead, we take the increment in the imaginary direction $\delta z = i\delta y$, then:

$$\frac{df}{dz} = \frac{1}{i}\frac{\partial g}{\partial y} + \frac{\partial h}{\partial y} = \frac{\partial h}{\partial y} - i\frac{\partial g}{\partial y}. \tag{5.5}$$

Comparing the real and imaginary terms in equations (5.4) and (5.5), we obtain the Cauchy–Riemann equations:

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}, \tag{5.6}$$

$$\frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}, \tag{5.7}$$

from which we deduce that both g and h satisfy the two-dimensional Laplace equation.
Hence the real and imaginary parts of any complex differentiable function $f(z)$ are solutions of the Laplace equation.

5.4 The complex potential

This powerful result allows us to use complex differentiable functions to derive two-dimensional potential flows. We define the complex potential as:

$$w(z) = \phi(x, y) + i\psi(x, y), \tag{5.8}$$

since the Cauchy–Riemann equations guarantee that ϕ and ψ will have the appropriate properties.
The fluid velocity is obtained from equation (5.4):

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv, \tag{5.9}$$

or alternatively:

$$u + iv = \overline{\frac{dw}{dz}},$$

where the overbar indicates complex conjugation.
Furthermore:

$$|\mathbf{u}| = \sqrt{u^2 + v^2} = \left(\frac{dw}{dz}\overline{\frac{dw}{dz}}\right)^{1/2} = \left|\frac{dw}{dz}\right|, \tag{5.10}$$

and the velocity direction, α, can be expressed as:

$$\alpha = \arg\left(\overline{\frac{dw}{dz}}\right) = -\arg\left(\frac{dw}{dz}\right). \tag{5.11}$$

5.4.1 Examples of complex potentials

i. **Uniform flow:** If $u = U_0 \cos\alpha$, $v = U_0 \sin\alpha$, then:

$$\frac{dw}{dz} = U_0 \cos\alpha - iU_0 \sin\alpha = U_0 e^{-i\alpha},$$

and so the corresponding complex potential is:

$$w = U_0 e^{-i\alpha} z.$$

ii. **Saddle point flow:** Consider:

$$w = Az^2.$$

If A is real, then:

$$w = A(x+iy)^2 = A(x^2 - y^2) + 2iAxy = \phi(x,y) + i\psi(x,y).$$

Differentiating, we have:

$$u - iv = 2Az = 2A(x+iy).$$

iii. **Source/Sink:** The logarithm of a complex number $z = re^{i\theta}$ is given by:

$$\log(z) = \log(re^{i\theta}) = \log(r) + i\theta. \tag{5.12}$$

For a real number A:

$$w = A\log(z) = A\log(r) + iA\theta.$$

Thus, $\phi = A\log(r)$ where $r = \sqrt{x^2 + y^2}$. This flow is most easily recognised in polar coordinates,

$$u_r = \frac{\partial\phi}{\partial r} = \frac{A}{r}.$$

It represents a 2D source (sink) flow out of (into) the origin for $A > 0$ ($A < 0$).

The strength of the source is defined from the flow rate through a circle centered around the origin:

$$q = \int_0^{2\pi} u_r a d\theta = 2\pi A,$$

and the complex potential for a source of strength q at the origin is given by:

$$w(z) = \frac{q}{2\pi}\log(z).$$

iv. **Vortex:** If A is imaginary, $iA = k$, then $\phi = k\theta$ which represents the vortex:

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{k}{r}.$$

The circulation about the origin is:

$$\Gamma = \int_0^{2\pi} v_\theta a d\theta = 2\pi k,$$

so that the complex potential for a vortex of circulation Γ at the origin is given by:

$$w(z) = -\frac{i\Gamma}{2\pi}\log(z).$$

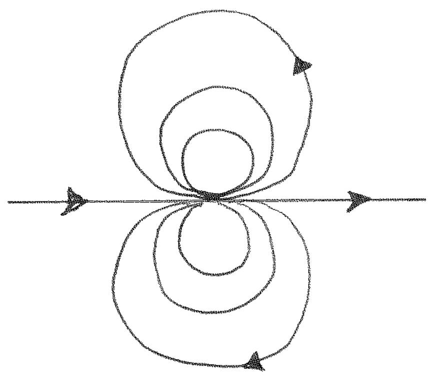
More generally, the complex potential for a vortex at $z = z_0$ is given by:

$$w(z) = -\frac{i\Gamma}{2\pi}\log(z - z_0).$$

v. **Dipole:** Consider the complex potential

$$w = -\frac{m}{z}.$$

We find the velocity by differentiating:



$$\begin{aligned} u - iv = \frac{dw}{dz} = \frac{m}{z^2} = \frac{m\bar{z}^2}{|z^4|} \\ = \frac{m}{(x^2+y^2)^2}(x-iy)^2 \\ = \frac{m}{(x^2+y^2)^2}\left(x^2 - y^2 - i2xy\right), \end{aligned}$$

so that:

$$\begin{aligned} u &= \frac{m(x^2-y^2)}{(x^2+y^2)^2}, \\ v &= \frac{2mxy}{(x^2+y^2)^2}, \end{aligned}$$

which is a dipole flow.

We can also obtain other flows through combinations of these complex potentials.

5.5 Imposing boundary conditions

As we noted above, we can only impose the continuity of the normal component of velocity for potential flows. In order to describe a fixed boundary, it is sufficient to impose that the streamfunction is constant along the boundary. We thus need to ensure that the imaginary part of the complex potential is constant on the boundary.

5.5.1 Method of images

For simple boundaries, this can be achieved using the method images. For example, the complex potential due to a line vortex at $(0, d)$ is given by:

$$w(z) = -\frac{i\Gamma}{2\pi}\log(z - id).$$

Let us try to find the flow in $y > 0$ when there is a wall at $y = 0$. We need to impose that the real axis $y = 0$ is a streamline. We can do this by placing an "image" vortex at the point $y = -id$ of opposite strength. This will cancel out the y component of the velocity on $y = 0$. The complex potential for this image system is given by:

$$w(z) = -\frac{i\Gamma}{2\pi}\log(z - id) + \frac{i\Gamma}{2\pi}\log(z + id) = -\frac{i\Gamma}{2\pi}\log\left(\frac{z - id}{z + id}\right).$$

As a consequence, the streamlines are given by:

$$\left|\frac{z - id}{z + id}\right| = \text{cst}.$$

In particular, on the real axis $z \pm id = x \pm id$,

$$|z - id| = |z + id| = \sqrt{d^2 + x^2},$$

and so $\psi = \Gamma/2\pi \log 1 = 0$.
We can achieve a generalisation of this principle for any complex potential $f(z)$ by considering the function:

$$w(z) = f(z) + \overline{f(\overline{z})},$$

for which w is a function of z (and not another combination of x and y).
Remark: Taking the complex conjugate of a function is done by replacing each term by its complex conjugate (in other words we replace i by $-i$). For example, if:

$$f(z) = i \log(z + b),$$

then

$$\overline{f(z)} = -i \log(\overline{z} + \overline{b}).$$

In this case, $\overline{f(z)}$ is a function of $\overline{z}$ and not z. However $\overline{f(\overline{z})}$ is a function of z:

$$\overline{f(\overline{z})} = -i \log(z + \overline{b}).$$

Returning to the general case, on the real axis, $z = x$,

$$w(x) = f(x) + \overline{f(x)},$$

which is real (from the definition of the complex conjugate) and so $\psi = 0$. In the case of the line vortex above:

$$f(z) = -\frac{i\Gamma}{2\pi} \log(z - id),$$

and so:

$$\overline{f(\overline{z})} = \frac{i\Gamma}{2\pi} \log(\overline{\overline{z} - id}) = \frac{i\Gamma}{2\pi} \log(z + id).$$

5.5.2 Milne-Thompson circle theorem

There exists a similar construction for the case when the boundary is the circle $|z| = a$:

$$w(z) = f(z) + \overline{f\left(\frac{a^2}{\overline{z}}\right)}.$$

On the circle $z = ae^{i\theta}$, we have:

$$w(ae^{i\theta}) = f(ae^{i\theta}) + \overline{f(ae^{i\theta})},$$

and so $\psi = 0$.
Flow past a cylinder. We can use the Milne-Thompson circle theorem to find the complex potential for a flow past a cylinder. We start with the uniform flow $f(z) = U_0 z e^{-i\alpha}$. Applying the circle theorem:

$$w(z) = U_0 z e^{-i\alpha} + \frac{\overline{U_0 a^2 e^{-i\alpha}}}{\overline{z}} = U_0 z e^{-i\alpha} + U_0 e^{i\alpha} \frac{a^2}{z}.$$

We can now add a line vortex at $z = 0$ so that the circle $|z| = a$ remains a streamline:

$$w(z) = U_0 z e^{-i\alpha} + U_0 e^{i\alpha} \frac{a^2}{z} - \frac{i\Gamma}{2\pi} \log z. \tag{5.13}$$

5.5.3 Conformal mapping

Another way to find flow solutions for particular boundary conditions is to use a transformation that maps the boundary onto that of a different problem for which we know the solution.

Flow round a corner. Let us start with the simple complex potential:

$$w(z) = U_0 z,$$

which represents the uniform flow in the positive x direction in the upper half-plane ($y > 0$). This flow admits a straight boundary at $y = 0$. Let us map this onto the quadrant $x > 0, y > 0$ by taking:

$$Z = z^{1/2}.$$

This maps the point $z = re^{i\theta}$ onto $Z = r^{1/2}e^{i\theta/2}$, mapping the negative x-axis $\theta = \pi$ onto the positive y-axis $\theta = \pi/2$. The corresponding complex potential is given by:

$$W(Z) = U_0 Z^2.$$

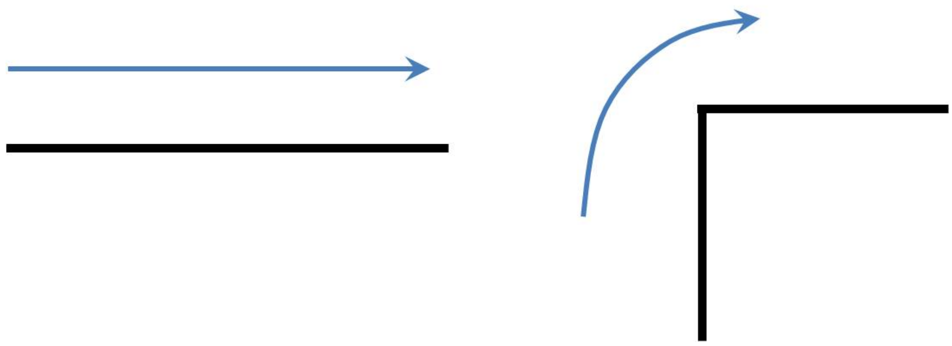
This is the stagnation point flow which describes the flow round a right-angled corner.

We can generalise this to a flow around a corner of angle α by taking $Z = z^{\alpha/\pi}$ so that $z = Z^{\pi/\alpha}$ and:

$$W(Z) = U_0 Z^{\pi/\alpha}.$$

For example, $\alpha = 3\pi/2$ models the flow around the outside of a right-angled corner:

$$W(Z) = U_0 Z^{2/3}.$$



5.5.4 The Joukowski transformation

Another useful transformation is the Joukowski transformation:

$$Z = z + \frac{c^2}{z}. \tag{5.14}$$

Let us consider the effect of this transformation on the circle $|z| = a$:

$$Z(ae^{i\theta}) = a\left(e^{i\theta} + \frac{c^2}{a^2}e^{-i\theta}\right) = a\left[\left(1+\frac{c^2}{a^2}\right)\cos\theta + i\left(1-\frac{c^2}{a^2}\right)\sin\theta\right].$$

Hence, for $0 \le c \le a$, it maps the $|z| = a$ circle onto the ellipse with semi-axes $(a + c^2/a)$ and $(a - c^2/a)$. Note that if we take $c = a$, the ellipse collapses onto a flat plate along the X axis running from $X = -2a$ to $X = 2a$.
The inverse mapping is given by:

$$z = \frac{Z}{2} + \left(\frac{Z^2}{4} - c^2\right)^{1/2},$$

where we have taken the positive square root so that z maps onto Z when $c = 0$.
Using the solution for the flow past a cylinder with circulation Γ in equation (5.13), we can obtain the flow past an ellipse at an angle α to the major axis and with circulation Γ:

$$W(Z) = U_0\left[\frac{Z}{2} + \left(\frac{Z^2}{4} - c^2\right)^{1/2}\right]e^{-i\alpha} + \frac{U_0 a^2 e^{i\alpha}}{\frac{Z}{2} + \left(\frac{Z^2}{4} - c^2\right)^{1/2}}$$
$$- \frac{i\Gamma}{2\pi}\log\left[\frac{Z}{2} + \left(\frac{Z^2}{4} - c^2\right)^{1/2}\right].$$

5.5.5 Flow past a finite flat plate

Rather than working with the rather messy expression above, we can determine the fluid velocity using the conformal mapping and the complex potential in the original z-space. Since $w(z) = W(Z)$:

$$\frac{dw}{dz} = \frac{dW}{dZ}\frac{dZ}{dz}.$$

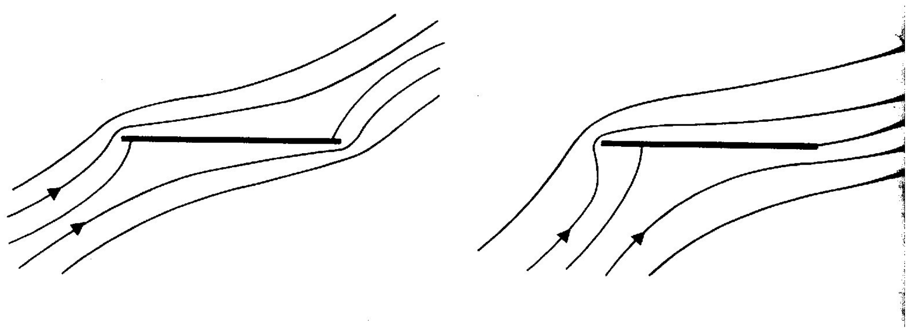
The velocity components u and v are given by:

$$U - iV = \frac{dW}{dZ} = \frac{dw}{dz}\left(\frac{dZ}{dz}\right)^{-1}, \tag{5.15}$$

and so, for the flow past a flat plate, we have:

$$U - iV = \frac{U_0 z^2 e^{-i\alpha} - U_0 e^{i\alpha}a^2 - \frac{i\Gamma z}{2\pi}}{z^2 - a^2}. \tag{5.16}$$

Note that the denominator vanishes at $z = \pm a$, indicating that the fluid velocity is infinite at the edges of the plate unless the numerator also vanishes there.



(a) $\Gamma=0$ (b) $\Gamma=-4\pi Ua\sin\alpha$

The position of the points of flow separation is controlled by the circulation Γ. Consequently, we can choose Γ so that the separation occurs at the trailing edge. This is referred to as the *Kutta condition*: we want to cancel the numerator in equation (5.16) at $z = a$ (trailing edge):

$$U_0 e^{-i\alpha} - U_0 e^{i\alpha} - \frac{i\Gamma}{2\pi a} = 0,$$

yielding: $\Gamma = -4\pi U_0 a \sin\alpha$.

5.5.6 The Joukowski aerofoil

While the flat-plate provides a rough approximation to a thin aerofoil (such as in a paper aeroplane), real aerofoils have a rounded leading and sharp trailing edge. We can generate a symmetric aerofoil of this kind by moving the circle centre along the x-axis to $z = -b$ before applying the Joukowski transformation. The result is an aerofoil whose shape is given by:

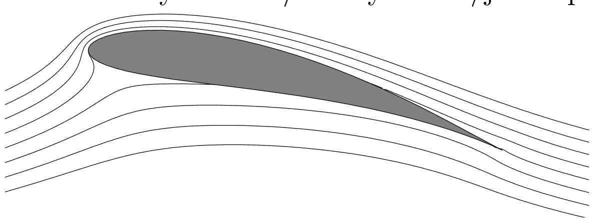
$$Z = -b + (a+b)e^{i\theta} + \frac{a^2}{-b + (a+b)e^{i\theta}}.$$

The corresponding fluid velocity is given by:

$$U - iV = \frac{\left(U_0 e^{-i\alpha} - U_0 e^{i\alpha}\frac{a+b^2}{(z+b)^2} - \frac{i\Gamma}{2\pi(z+b)}\right)}{\left(1 - \frac{a^2}{z^2}\right)},$$

and the Kutta condition gives $\Gamma = -4\pi U_0(a+b)\sin\alpha$.

Better still is a cambered aerofoil, which is formed by moving the centre of the circle to $z = be^{i\beta}$ (see http://s6.aeromech.usyd.edu.au/aerodynamics/jflow2.php).

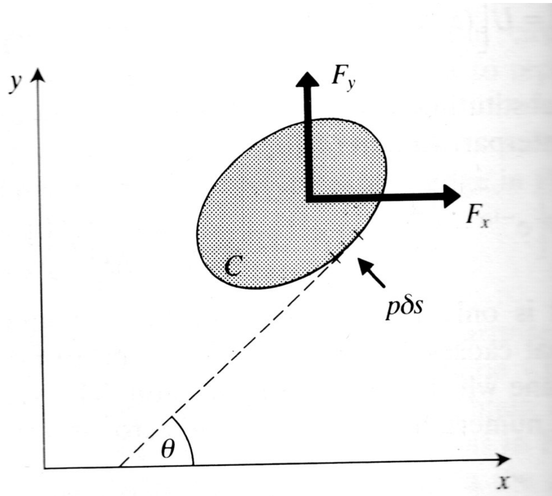


5.6 Forces on streamlined bodies

Our remaining task is to calculate the force exerted by the flow on the aerofoil. Since we ignore viscous effects, this force comes from the fluid pressure and is given by:

$$\mathbf{F} = -\int_S p\mathbf{n}ds, \tag{5.17}$$

where $\mathbf{n}$ is the normal to the surface.



Consider an element of the surface δS that is at an angle θ to the x-axis. The outward normal to this surface element is $(\sin\theta, -\cos\theta)$. The contribution from this element to the force is:

$$\delta \mathbf{F} = -p(\sin\theta, -\cos\theta)\delta s.$$

For a steady flow, the pressure is given by the Bernoulli equation (5.1):

$$p = E - \frac{\rho}{2}(u^2 + v^2),$$

for some constant E. The contribution from E integrates to zero around a closed loop, so:

$$\mathbf{F} = \frac{\rho}{2}\int_S (u^2 + v^2)(\sin\theta, -\cos\theta)ds.$$

Let us write the force in complex form:

$$F_x - iF_y = \frac{\rho}{2}\oint_C \left|\frac{dw}{dz}\right|^2 (\sin\theta + i\cos\theta)ds,$$

where C is the contour in the complex plane corresponding to the surface S of the body, and we have used equation (5.10) to write the speed as $|dw/dz|$. Furthermore:

$$(\sin\theta + i\cos\theta)ds = i(\cos\theta - i\sin\theta)ds = i\overline{dz},$$

so:

$$F_x - iF_y = \frac{i\rho}{2}\oint_C \frac{dw}{dz}\overline{\left(\frac{dw}{dz}\right)}\overline{dz}.$$

However,

$$\overline{\left(\frac{dw}{dz}\right)}\overline{dz} = \overline{dw},$$

and since C is a streamline, dw is real:

$$\overline{dw} = dw = \frac{dw}{dz}dz,$$

yielding:

$$F_x - iF_y = \frac{i\rho}{2}\oint_C \left(\frac{dw}{dz}\right)^2 dz, \tag{5.18}$$

which is known as the *Blasius theorem*.

5.6.1 Forces on cylinder

Applying the Blasius theorem to the flow past a cylinder, we have:

$$F_x - iF_y = \frac{i\rho}{2}\oint_C \left(U_0 e^{-i\alpha} - U_0 e^{i\alpha}\frac{a^2}{z^2} - \frac{i\Gamma}{2\pi z}\right)^2 dz,$$

where C is the circle $|z| = a$. To evaluate this integral, let us consider:

$$\oint_{|z|=a} z^{-n} dz,$$

where n is an integer. Substituting $z = ae^{i\theta}$, we have:

$$\begin{aligned}\oint_{|z|=a} z^{-n} dz &= \int_0^{2\pi} a^{1-n} i e^{i(1-n)\theta} d\theta \\ &= \begin{cases} \frac{a^{1-n}}{1-n}\left[e^{i(1-n)\theta}\right]_0^{2\pi} = 0 & n \neq 1 \\ i\left[\theta\right]_0^{2\pi} = 2\pi i & n = 1 \end{cases}.\end{aligned}$$

This is a special case of the Cauchy residue theorem and it can be shown more generally that the result holds for any closed curve around the origin. The only non-zero contribution to the integral thus comes from the coefficient of z^{-1}:

$$F_x - iF_y = \frac{i\rho}{2} 2\pi i\left(-\frac{iU_0 e^{-i\alpha}\Gamma}{\pi}\right) = i\rho U_0 \Gamma e^{-i\alpha} = \rho U_0 \Gamma(\sin\alpha + i\cos\alpha).$$

Hence: $F_x = \rho U_0 \Gamma \sin\alpha$ and $F_y = -\rho U_0 \Gamma \cos\alpha$. Let us consider $\alpha = 0$. In the case $\Gamma = 0$, there is no force acting on the cylinder. For $\Gamma \neq 0$, the only force component is a lift force directed perpendicular to the direction of flow: $F_y = -\rho U_0 \Gamma$. This is called the *Magnus effect* and is responsible for aerodynamic forces acting on balls in sports (e.g. football, tennis, table tennis).
Video: https://www.youtube.com/watch?v=3ECoR__tJNQ

5.6.2 Forces on an elliptical body

Let us now consider the flow around an ellipse. One way to calculate it to consider the velocity potential $W(Z)$. It is however easier to use the conformal transformation to perform the integral around the cylinder. From equation (5.18):

$$F_x - iF_y = \frac{i\rho}{2}\oint_C \left(\frac{dW}{dZ}\right)^2 dZ,$$

where C is the curve around the surface of the ellipse. We can transform this surface back to a circle of radius a using the Joukowski transformation:

$$Z(z) = z + \frac{c^2}{z}.$$

Noting from equation (5.15) that $dW/dZ = (dw/dz) \cdot (dZ/dz)^{-1}$:

$$\begin{aligned}F_x - iF_y &= \frac{i\rho}{2}\oint_{|z|=a} \left(\frac{dw}{dz}\right)^2 \left(\frac{dZ}{dz}\right)^{-1} dz \\ &= \frac{i\rho}{2}\oint_{|z|=a} \left(U_0 e^{-i\alpha} - U_0 e^{i\alpha}\frac{a^2}{z^2} - \frac{i\Gamma}{2\pi z}\right)^2 \left(1 - \frac{c^2}{z^2}\right)^{-1} dz.\end{aligned}$$

Since $c < a$, we can use the binomial expansion:

$$\left(1 - \frac{c^2}{z^2}\right)^{-1} = 1 + \frac{c^2}{z^2} + \dots,$$

and the property:

$$\oint_{|z|=a} z^{-n} dz = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1 \end{cases},$$

to evaluate the integral by identifying the coefficient of z^{-1}. We obtain:

$$F_x - iF_y = i\rho U_0 \Gamma e^{-i\alpha}.$$

Note that this result is independent of c and is identical to the case of the cylinder. The case of the flat-plate requires a little more care because the integrand is singular at $z^2 = a^2$. We can get around this problem by performing the integral on a larger circle (by exploiting the Cauchy theorem for contour integration, since the integrand does not contain any singularities in the flow domain).

5.6.3 The Kutta–Joukowski lift theorem

The result we have just obtained for an ellipse applies in general to any shaped body. The proof requires using a few results from complex analysis, but it is worth it.
We choose the origin $z = 0$ to lie inside the body and apply the Blasius theorem around the surface contour of the body C_b:

$$F_x - iF_y = \frac{i\rho}{2} \oint_{C_b} \left(\frac{dw}{dz}\right)^2 dz.$$

We now take a second contour C_R, which is a large circle around the origin of radius R where R is much larger than the body we are considering.
A key result from complex analysis is that for a complex differentiable function $f(z)$ that has no singularities inside a closed curve C:

$$\oint_C f(z) dz = 0.$$

In this case, $f(z) = (dw/dz)^2$ has no singularities outside the body. Taking C to be the curve composed of $C_R \cup -C_b$, the Blasius integral can be changed into an integral around C_R:

$$F_x - iF_y = \frac{i\rho}{2} \oint_{C_R} \left(\frac{dw}{dz}\right)^2 dz.$$

Far away from the object, we can expand dw/dz as a power series in z:

$$\frac{dw}{dz} = U - \frac{i\Gamma}{2\pi z} + \dots,$$

where U is the fluid velocity at large distance and Γ is the circulation around C_R. Note that Γ must be equal to the circulation around C_b since the flow is irrotational. Using the result we used for the cylinder, we find:

$$F_x - iF_y = \frac{i\rho}{2} 2\pi i \left(-\frac{iU\Gamma}{\pi}\right) = i\rho U\Gamma. \tag{5.19}$$

This is the *Kutta–Joukowski theorem.* The force is in the direction iU and so is perpendicular to the flow direction. In particular, if U is in the positive x-direction then the force is equal to $F_y = -\rho U\Gamma$. Since this force is perpendicular to the direction of flow it is referred to as a *lift force.*

5.6.4 Lift on a flat plate

Using the Kutta condition, the lift force on the flat plate is given by:

$$F_x - iF_y = i\rho U\Gamma = i\rho U_0 e^{-i\alpha}(-4\pi U_0 a \sin\alpha) = -4\rho U_0^2 \pi a \sin\alpha\,(\sin\alpha + i\cos\alpha)\,.$$

Since the length of the flat plate is $L = 4a$, the lift force per unit length on a wing reads:

$$F_L = F_y \cos\alpha - F_x \sin\alpha = \rho U_0^2 L\pi \sin\alpha. \tag{5.20}$$

Generating lift on a symmetric aerofoil requires a finite angle of attack. Lift then increases with the angle α. However, if this angle becomes too large, the flow separates and generates a catastrophic loss of lift and increase in drag. This leads to stalling.

5.6.5 Torque on a streamlined body

The torque $\mathbf{T}$ about the origin due to a force $\mathbf{F}$ acting at $\mathbf{x}$ is given by $\mathbf{T} = \mathbf{x} \times \mathbf{F}$. Hence, for a planar flow, the torque about the origin due to the force (dF_x, dF_y) acting at (x, y) is:

$$dT = xdF_y - ydF_x.$$

This can be written in complex variable notation:

$$dT = \Re\{iz(dF_x - idF_y)\}.$$

where $\Re\{\dots\}$ denotes the real part. Using the Blasius formula for the force on the surface of a rigid body (equation (5.18)), we get:

$$dF_x - idF_y = \frac{i\rho}{2}\left(\frac{dw}{dz}\right)^2 dz,$$

and the torque on the body is:

$$T = -\frac{\rho}{2}\Re\left\{\oint_C z\left(\frac{dw}{dz}\right)^2 dz\right\}.$$

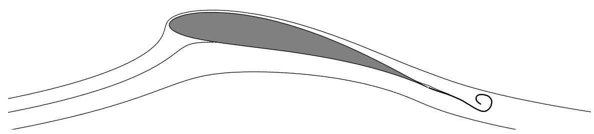
For the flow past a cylinder where the complex potential is given by equation (5.13), we have:

$$\begin{aligned}
T &= -\frac{\rho}{2}\Re\left\{\oint_C z\left(U_0 e^{-i\alpha} - U_0 e^{i\alpha}\frac{a^2}{z^2} - \frac{i\Gamma}{2\pi z}\right)^2 dz\right\} \\
&= -\frac{\rho}{2}\Re\left\{\oint_C z\left(U_0^2 e^{-2i\alpha} - \frac{i\Gamma U_0 e^{-i\alpha}}{\pi z} - \frac{2U_0 a^2}{z^2} - \frac{\Gamma^2}{4\pi^2 z^2} + \dots\right) dz\right\} \\
&= -\frac{\rho}{2}\Re\left\{2\pi i\left(-2U_0 a^2 - \frac{\Gamma^2}{4\pi^2}\right)\right\} \\
&= 0.
\end{aligned}$$

Consequently, there is no aerodynamic torque on a cylinder.

5.7 Origin of circulation around a wing

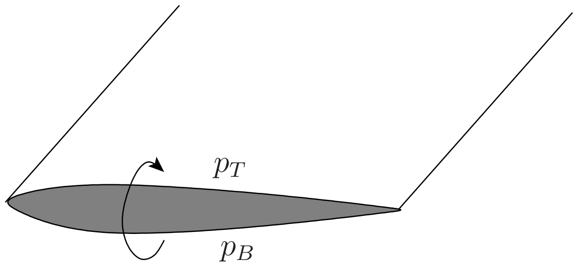
When the plane is stationary on the runway, there is no circulation around the wings. In the absence of viscosity the flow remains vortex free. However, when the plane starts to move, small viscous effects in the boundary layer allow the aerofoil to shed a vortex off the trailing edge.



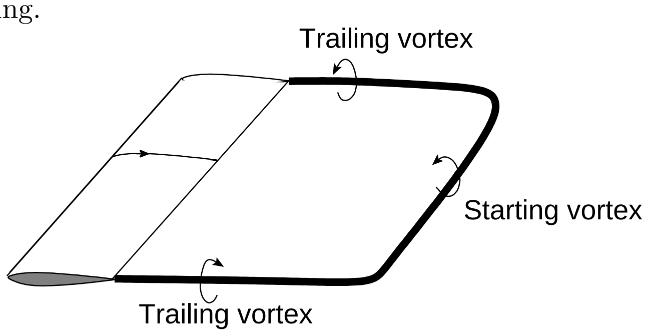
This vortex, called the starting vortex, remains behind on the runway. Its circulation is equal and opposite to the circulation around the wing.

5.8 Three-dimensional aerofoils

No wing is infinitely long. Special care needs to be taken with wingtips.



Since the pressure is different on the upper and lower surfaces of the wing, there is a pressure gradient driving the flow around the edge of the wing. This leads to the creation of a vortex at the edge of the wing.



These trailing vortices are part of a single vortex tube formed by the wings and comprising the trailing vortices and the starting vortex.