Chapter 4

Vorticity diffusion and boundary layers

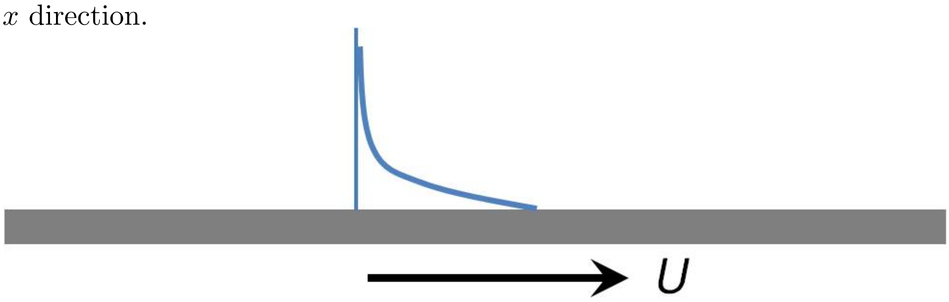
Contents

In this chapter we will turn our attention to high Reynolds number flows. As we noted earlier, it is mathematically dangerous to ignore the viscous terms in the Navier–Stokes equation because, by removing the highest spatial derivative, we are no longer able to satisfy all the boundary conditions at a solid boundary.

4.1 Start-up flows

4.1.1 Flow near an impulsively moved boundary

To illustrate what happens near a boundary, let us consider the flow above a solid wall at $y = 0$. Initially, the fluid is at rest. At time $t = 0$, the boundary starts to move with velocity U in the x direction.



In this simple flow, we can assume that:

$$\mathbf{u} = (u(y,t), 0, 0).$$

Also, since this flow is driven by the motion of the boundary and not an external pressure gradient, we can assume that $\partial p/\partial x = 0$.
The Navier–Stokes equation:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \mathbf{u},$$

reduces to:

$$\rho\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial y^2}, \tag{4.1}$$

and is accompanied with the boundary conditions: $u = U$ on $y = 0$ and $U \to 0$ as $y \to \infty$. We also impose the initial condition: $u = 0$ at $t = 0$.
The velocity $u(x,t)$ thus satisfies the diffusion equation with diffusivity $\nu = \mu/\rho$, where ν is the kinematic viscosity. This problem is equivalent to that of finding the temperature distribution in a semi-infinite bar when one end is suddenly heated to a constant temperature.
We seek a *similarity solution*:

$$u(y,t)=f(\eta),\ \text{where } \eta = yt^a,$$

for some constant a. Using the chain rule:

$$\frac{\partial}{\partial y}=t^a\frac{d}{d\eta},$$
$$\frac{\partial}{\partial t}=ayt^{a-1}\frac{d}{d\eta},$$

so that equation (4.1) becomes:

$$ayt^{a-1}\frac{df}{d\eta}=\nu t^{2a}\frac{d^2 f}{d\eta^2},$$

and therefore:

$$\frac{d^2 f}{d\eta^2}-\frac{ayt^{-a-1}}{\nu}\frac{df}{d\eta}=0.$$

For the similarity solution to exist, this equation must only contain y and t in the combination $\eta = yt^a$ and therefore $-a-1=a$. We get: $a=-\frac{1}{2}$. Solutions thus exists for the similarity variable $\eta = y/\sqrt{t}$ and satisfy:

$$\frac{d^2 f}{d\eta^2}+\frac{\eta}{2\nu}\frac{df}{d\eta}=0.$$

Substituting $v = df/d\eta$ we have:

$$\frac{dv}{d\eta}=-\frac{\eta}{2\nu}v,$$

which has general solution:

$$v=\frac{df}{d\eta}=A\exp\left(-\frac{\eta^2}{4\nu}\right).$$

Integrating again, we obtain:

$$f=A\int_0^{\eta}\exp\left(-\frac{\eta^2}{4\nu}\right)d\eta+B.$$

The above integral can be expressed in terms of the error function:

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x\exp\left(-x^2\right)dx.$$

Substituting $x=\eta/2\sqrt{\nu}$, we have:

$$f=A\sqrt{\nu\pi}\,\mathrm{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right)+B.$$

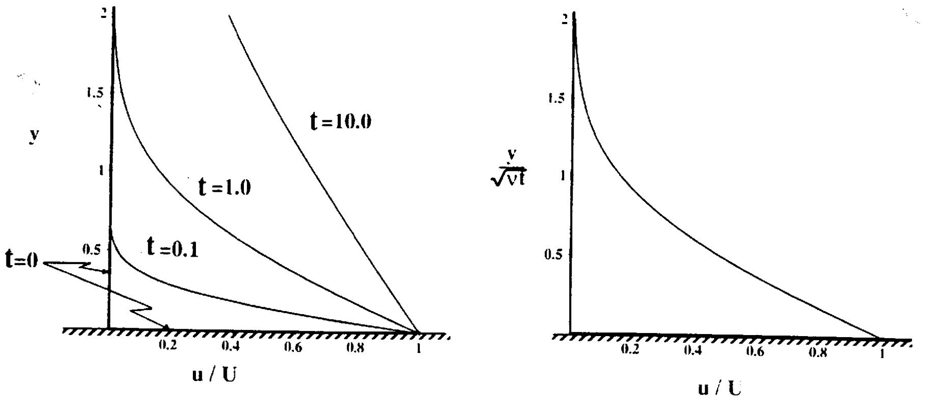
In terms of the original variables, this solution reads:

$$u(y,t) = A\sqrt{\nu\pi}\mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) + B.$$

The boundary conditions on $y = 0$ imposes $B = U$ and, since $\mathrm{erf}(x) \to 1$ as $x \to \infty$, the other boundary condition requires $A\sqrt{\nu\pi} = -U$. We eventually get:

$$u(y,t) = U\left[1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right]. \tag{4.2}$$

Finally, $u(y,0) = 0$ holds for all $y > 0$, so all the boundary conditions are satisfied.
The velocity $u(y,t)$ will be approximately zero wherever $y/2\sqrt{\nu t}$ is large. In addition, for a fixed value of y, the velocity will remain less than $0.01U$ until a time t such that $y \approx 4\sqrt{\nu t}$. Hence, at time t, the fluid is only moving within a narrow region of thickness $4\sqrt{\nu t}$. This narrow region is called the *viscous boundary layer*. Note that the boundary layer thickness is independent of U.



As an example, let us consider water for which $\nu \approx 10^{-6}\mathrm{m}^2 \cdot \mathrm{s}^{-1}$. After one second, the boundary layer thickness is around 4mm. After 100 seconds, it is still only 4cm in size. For lower viscosity fluids, the effects of the boundary are felt in a narrower region next to the boundary.

4.1.2 Start-up of shear flow

Let us now modify the previous problem by considering the start-up of a shear flow between two parallel plates located at $y = 0$ and $y = h$. Once again, we begin to move the lower plate with velocity U at $t = 0$. The problem is the same as that above except that the boundary condition at infinity is replaced by one at $y = h$. The velocity now satisfies:

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{4.3}$$

together with the boundary conditions: $u(0,t) = U$ and $u(h,t) = 0$, and the initial condition $u(y,0) = 0$.
First, we observe that the steady solution $u_s = U\,(1 - y/h)$ satisfies the equation at any $t \neq 0$ and the boundary conditions. We then write:

$$u(y,t) = u_s + v(y,t),$$

and seek a separable solution of the form:

$$v(y,t) = T(t)Y(y).$$

This gives:

$$YT' = \nu TY'',$$

so that:

$$\frac{Y''}{Y} = \frac{1}{\nu}\frac{T'}{T} = k,$$

where k is the constant of integration. Since u_s takes care of the moving boundary, we want to find solutions satisfying $Y(0) = Y(h) = 0$. We thus choose solutions of the form:

$$Y(y) = \sin\left(\frac{n\pi y}{h}\right),$$

so that:

$$\frac{Y''}{Y} = -\frac{n^2\pi^2}{h^2}.$$

It follows:

$$\frac{T'}{T} = -\frac{\nu n^2\pi^2}{h^2},$$

and so we have separable solutions of the form:

$$v_n = \exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right).$$

The general solution for u satisfying the boundary conditions is:

$$u(y,t) = U\left(1 - \frac{y}{h}\right) + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right).$$

The initial condition at $t = 0$ requires:

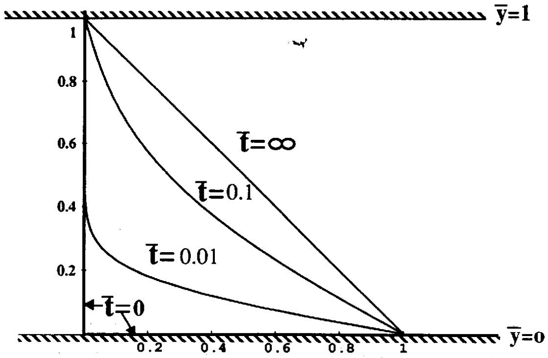
$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{h}\right) = -U\left(1 - \frac{y}{h}\right),$$

for $0 < y < h$. We can determine the a_n using Fourier series properties:

$$a_n = \frac{2U}{h}\int_0^h \left(\frac{y}{h} - 1\right)\sin\left(\frac{n\pi y}{h}\right)dy = -\frac{2U}{n\pi},$$

Hence, the solution is:

$$u(y,t) = U\left(1 - \frac{y}{h}\right) - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right). \qquad (4.4)$$



This flow resembles that of the unbounded plate until the boundary layer grows to the width of the channel. The solution then approaches the steady state u_s. Note that the slowest decaying exponential in the sum corresponds to $n = 1$. As a result, the flow reaches u_s on a time of order $h^2/(\nu\pi^2)$. For water in a 1cm channel, this time is about 10s and scales inversely with ν so that in a fluid of lower viscosity it becomes longer.

4.2 Vorticity dynamics

A more fundamental way of viewing these flows is to consider the vorticity of the flow: $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. Starting from the Navier–Stokes equation and dividing by ρ, we have:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}.$$

By using the vector identity:

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\,\mathbf{u},$$

we can rewrite this as:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\nabla \left(\frac{p}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \right) + \nu \nabla^2 \mathbf{u}.$$

Taking the curl of this equation yields:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = \nu \nabla^2 \boldsymbol{\omega}.$$

Using the vector identity:

$$\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = \mathbf{u} \nabla \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \mathbf{u} - \boldsymbol{\omega} \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \boldsymbol{\omega},$$

noting that $\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \mathbf{u}) = 0$ and considering the fluid incompressible, we have:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = \frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}. \tag{4.5}$$

The left-hand side of this equation represents the rate of change of vorticity of a fluid particle. The term $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$ corresponds to the vorticity enhancement by velocity gradients parallel to the direction of $\boldsymbol{\omega}$. It represents *vortex stretching*. Finally, $\nu \nabla^2 \boldsymbol{\omega}$ is the diffusion of vorticity due to viscosity.

For the special case of a two-dimensional planar flow:

$$\mathbf{u} = (u(x, y), v(x, y), 0),$$

the vorticity reads:

$$\boldsymbol{\omega} = (0, 0, \omega), \text{ where } \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

In this case, the vorticity equation reduces to:

$$\frac{D\omega}{Dt} = \nu \nabla^2 \omega. \tag{4.6}$$

Hence, in the limit $\nu \to 0$, the vorticity of a fluid particle remains constant. It is simply advected with the flow. When viscosity is considered, the vorticity obeys a diffusion equation (in the frame of the moving fluid).

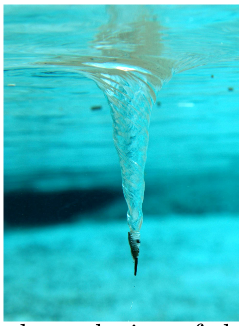
4.2.1 Impulsively moving boundary revisited

Let us now return to the case of the flow above a boundary that is set in motion at time $t = 0$. Initially, the vorticity is zero everywhere, except at $y = 0$ where the fluid velocity jumps from U to 0. At time t, the velocity is given by equation (4.2). The vorticity ω reads:

$$\omega = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi \nu t}} \exp\left(-\frac{y^2}{4\nu t} \right).$$

This is a Gaussian distribution of standard deviation $\sqrt{2\nu t}$. Hence, as times increases, the vorticity gradually spreads away from the boundary over a distance of order $\sqrt{2\nu t}$.

4.2.2 Decay of a line vortex



Let us consider the decay of a line vortex in which the fluid velocity is given in plane polar coordinates (r, θ) by:

$$\mathbf{u} = \frac{\Gamma}{2\pi r}\mathbf{e}_\theta.$$

This is an idealised version of the "bath plug" vortex.

If we assume that this flow remains axisymmetric, then we can seek a solution of the form:

$$\mathbf{u} = v(r,t)\mathbf{e}_\theta.$$

The θ component of the Navier–Stokes equation becomes:

$$\rho\frac{\partial v}{\partial t} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{v}{r^2}\right].$$

Since the pressure is periodic, $\partial p/\partial\theta = 0$, and thus:

$$\frac{\partial v}{\partial t} = \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right]. \tag{4.7}$$

At $t = 0$, $v(r, 0) = \Gamma/2\pi r$, so we seek a solution of the form:

$$v(r,t) = \frac{\Gamma}{2\pi r}f(\eta),$$

where $\eta = r/\sqrt{t}$, so that:

$$\begin{aligned}
\frac{\partial v}{\partial t} &= -\frac{\Gamma}{2\pi}\frac{1}{2t^{3/2}}\frac{df}{d\eta},\\
\frac{\partial v}{\partial r} &= \frac{\Gamma}{2\pi r^2}\left(\eta\frac{df}{d\eta} - f\right),\\
\frac{\partial^2 v}{\partial r^2} &= \frac{\Gamma}{2\pi r^3}\left(\eta^2\frac{d^2 f}{d\eta^2} - 2\eta\frac{df}{d\eta} + 2f\right).
\end{aligned}$$

Substituting into equation (4.7), we have:

$$-\frac{\Gamma}{2\pi r^3}\frac{\eta^3}{2}\frac{df}{d\eta} = \nu\frac{\Gamma}{2\pi r^3}\left(\eta^2\frac{d^2 f}{d\eta^2} - 2\eta\frac{df}{d\eta} + 2f + \eta\frac{df}{d\eta} - f - f\right),$$

which simplifies to:

$$-\frac{df}{d\eta} = 2\nu\left(\frac{1}{\eta}\frac{d^2 f}{d\eta^2} - \frac{1}{\eta^2}\frac{df}{d\eta}\right) = 2\nu\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df}{d\eta}\right).$$

Integrating both sides yields:

$$-f = \frac{2\nu}{\eta}\frac{df}{d\eta} + A.$$

We can thus write:

$$\frac{df}{d\eta} + \frac{\eta}{2\nu}f = -A\frac{\eta}{2\nu}.$$

And the solution is:

$$f = -A + B\exp\left(-\frac{\eta^2}{4\nu}\right),$$

giving:

$$v(r,t) = \frac{\Gamma}{2\pi r}\left[-A + B\exp\left(-\frac{r^2}{4\nu t}\right)\right].$$

Taking the limit $t \to 0$:

$$v(r,0) \to -A\frac{\Gamma}{2\pi r},$$

yielding $A = -1$. However, for the solution to be bounded at $r = 0$, we need $B = A = -1$. The velocity solution then reads:

$$v(r,t) = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right]. \tag{4.8}$$

We find that the viscosity only affects a region of size of order $2\sqrt{\nu t}$.

i. For $r \gg 2\sqrt{\nu t}$, the solution remains close to the initial irrotational flow with:

$$v \approx \frac{\Gamma}{2\pi r}.$$

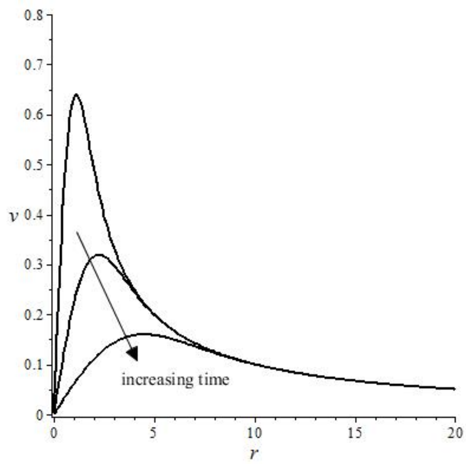
ii. For $r \ll 2\sqrt{\nu t}$, we can expand the exponential as:

$$v \approx \frac{\Gamma}{2\pi r}\left(1 - 1 + \frac{r^2}{4\nu t} + \dots\right),$$

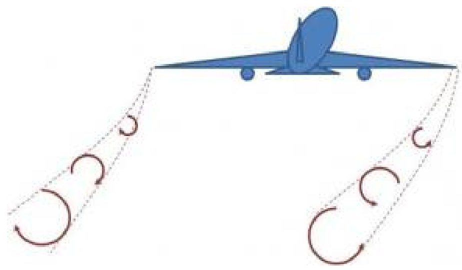
so that:

$$v \approx \frac{\Gamma r}{8\pi\nu t}.$$

This corresponds to solid body rotation at an angular velocity of $\Gamma/8\pi\nu t$.



The intensity of the vortex decreases with time as the "core" spreads out radially. This process is quite slow for low viscosity fluids, which is why vortex lines persist for a large distance/long time behind an aircraft.



4.3 High Reynolds number flows

In the above examples, the flow is two dimensional, so $\mathbf{u} \cdot \nabla\omega = 0$ and the vorticity equation (4.5) reduces to the diffusion equation. By balancing the terms on either side of this equation, we can obtain an estimate for the thickness δ of the boundary layer. If $\varOmega$ is the approximate magnitude of the vorticity then:

$$|\nu\nabla^2\omega| \sim \nu\frac{\varOmega}{\delta^2},$$

while:

$$\left|\frac{\partial\omega}{\partial t}\right| \sim \frac{\varOmega}{t}.$$

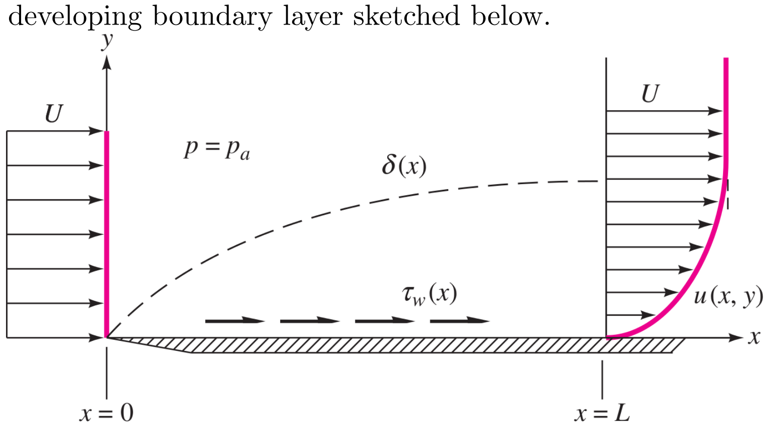
Combining both expressions, we get:

$$\frac{\varOmega}{t} \sim \nu\frac{\varOmega}{\delta^2},$$

giving $\delta^2 \sim \nu t$. The boundary layer thickness is $O(\sqrt{\nu t})$, as we have seen in the previous examples.

4.3.1 Blasius boundary layer

We consider the developing boundary layer sketched below.



Such a two-dimensional stationary flow is governed by the following Navier–Stokes equations:

$$\rho\left(u\partial_x u + v\partial_y u\right) = -\partial_x p + \mu\left(\partial_x^2 u + \partial_y^2 u\right), \tag{4.9}$$

$$\rho\left(u\partial_x v + v\partial_y v\right) = -\partial_y p + \mu\left(\partial_x^2 v + \partial_y^2 v\right), \tag{4.10}$$

where ρ is the fluid density, u is the streamwise (x-) velocity, v the wall-normal (y-) velocity, p is the pressure and μ is the fluid dynamic viscosity. The flow is incompressible, so we pose:

$$\partial_x u + \partial_y v = 0. \tag{4.11}$$

Lastly, the boundary conditions are:

$$u = v = 0 \qquad \text{at } y = 0, \tag{4.12}$$

$$(u, v) \to (U, 0) \qquad \text{at } y \to \infty. \tag{4.13}$$

Asymptotically reduced equations

Dynamics occur on a(n arbitrary) length scale L in the streamwise direction and δ, the boundary layer thickness, in the wall-normal direction. The spatial derivatives then follow the scalings:

$$\partial_x \sim \frac{1}{L}, \qquad \partial_y \sim \frac{1}{\delta}. \tag{4.14}$$

In addition, these length scales are not comparable: $L \gg \delta$. We can thus introduce a small parameter $\epsilon \ll 1$ such that:

$$\frac{\delta}{L} = \epsilon. \tag{4.15}$$

The streamwise velocity u is of the same order of magnitude as the velocity infinitely far away from the plate U but the order of magnitude of the wall-normal velocity v is yet to be determined. To that end, we assume that both terms in the incompressibility constraint are of the same order of magnitude:

$$\frac{U}{L} \sim \frac{v}{\delta}, \tag{4.16}$$

which gives:

$$\begin{aligned} & v \sim \frac{U\delta}{L} && (4.17)\\ \Rightarrow \quad & v \sim \epsilon\, U, && (4.18) \end{aligned}$$

implying that the wall-normal velocity is smaller than the streamwise one.
We can now rescale the wall-normal quantities according to the streamwise quantities. We define the rescaled quantities by:

$$\begin{aligned} x^* &= \frac{x}{L}, && (4.19)\\ y^* &= \frac{y}{\delta} = \frac{y}{\epsilon\, L}, && (4.20)\\ u^* &= \frac{u}{U}, && (4.21)\\ v^* &= \frac{v}{\epsilon\, U}, && (4.22)\\ p^* &= \frac{p}{\rho U^2}, && (4.23) \end{aligned}$$

where the pressure is nondimensionalised in such a way that it remains of the same order of magnitude as the other terms.
Remark: in the case of an incompressible flow, the pressure can be thought of as a mathematical function whose role is to ensure incompressibility.
Using the dimensionless variables, the incompressibility constraint reads:

$$\frac{U}{L}\partial_{x^*} u^* + \frac{\epsilon U}{\epsilon L}\partial_{y^*} v^* = 0, \tag{4.24}$$

yielding:

$$\partial_{x^*} u^* + \partial_{y^*} v^* = 0. \tag{4.25}$$

Similarly, the Navier–Stokes equation becomes:

$$\begin{aligned} \rho\left(\tfrac{U^2}{L} u^*\partial_{x^*} u^* + \tfrac{U^2}{L} v^*\partial_{y^*} u^*\right) &= -\frac{\rho U^2}{L}\partial_{x^*} p^* + \mu\left(\frac{U}{L^2}\partial^2_{x^*} u^* + \frac{U}{\epsilon^2 L^2}\partial^2_{y^*} u^*\right), && (4.26)\\ \rho\left(\tfrac{\epsilon U^2}{L} u^*\partial_{x^*} v^* + \tfrac{\epsilon U^2}{L} v^*\partial_{y^*} v^*\right) &= -\frac{\rho U^2}{\epsilon L}\partial_{y^*} p^* + \mu\left(\frac{\epsilon U}{L^2}\partial^2_{x^*} v^* + \frac{U}{\epsilon L^2}\partial^2_{y^*} v^*\right), && (4.27) \end{aligned}$$

and simplifies into:

$$\begin{aligned} u^*\partial_{x^*} u^* + v^*\partial_{y^*} u^* &= -\partial_{x^*} p^* + \frac{1}{Re_L}\partial^2_{x^*} u^* + \frac{1}{\epsilon^2 Re_L}\partial^2_{y^*} u^*, && (4.28)\\ u^*\partial_{x^*} v^* + v^*\partial_{y^*} v^* &= -\frac{1}{\epsilon^2}\partial_{y^*} p^* + \frac{1}{Re_L}\partial^2_{x^*} v^* + \frac{1}{\epsilon^2 Re_L}\partial^2_{y^*} v^*, && (4.29) \end{aligned}$$

where we have introduced the Reynolds number $Re_L = \rho UL/\mu$.
We are interested in $Re_L \gg 1$. As $\epsilon \ll 1$, the term $\partial^2_{x^*} u^*/Re_L$ is the smallest term in equation (4.28) and $\partial^2_{x^*} v^*/Re_L$ is the smallest term in equation (4.29). We drop them.
To keep a balance between advection (left-hand-side) and diffusion (right-hand-side), and therefore retain sensible physics, we impose $\epsilon^2 Re_L = 1$. As a result, the small quantity we have introduced is now related to the Reynolds number:

$$\epsilon = \frac{\delta}{L} = Re_L^{-1/2}. \tag{4.30}$$

Consequently, the leading order of system (4.28), (4.29) is:

$$u^*\partial_{x^*}u^* + v^*\partial_{y^*}u^* = -\partial_{x^*}p^* + \partial^2_{y^*}u^*, \tag{4.31}$$
$$\partial_{y^*}p^* = 0, \tag{4.32}$$

where equation (4.32) implies that the pressure does not vary across the boundary layer, but only along it.
We can then express the pressure at any point in the critical layer by applying the Bernoulli equation on a streamline away from the boundary layer:

$$p + \frac{\rho u^2}{2} = \text{cst}, \tag{4.33}$$

which gives in dimensionless form:

$$p^* + \frac{{u^*}^2}{2} = \text{cst}. \tag{4.34}$$

Outside the boundary layer, the velocity is $u^* = 1$, so we have:

$$p^* + \frac{1}{2} = \text{cst}, \tag{4.35}$$
$$\Rightarrow \qquad \partial_{x^*}p^* = 0, \tag{4.36}$$

a relation valid for all y^*.
The resulting set of reduced equations for the boundary layer writes:

$$\partial_{x^*}u^* + \partial_{y^*}v^* = 0, \tag{4.37}$$
$$u^*\partial_{x^*}u^* + v^*\partial_{y^*}u^* = \partial^2_{y^*}u^*, \tag{4.38}$$

and is to be solved together with the following boundary conditions:

$$u^* = v^* = 0 \qquad \text{at } y^* = 0, \tag{4.39}$$
$$u^* \to 1 \qquad \text{at } y^* \to \infty. \tag{4.40}$$

A second boundary condition for v is not necessary as v is only derived once with respect to y in the above equations.

The Blasius equation

While studying the dimensional boundary layer equations:

$$\partial_x u + \partial_y v = 0, \tag{4.41}$$
$$u\partial_x u + v\partial_y u = \nu\partial^2_y u, \tag{4.42}$$

where $\nu = \mu/\rho$, Blasius conjectured that the boundary layer is self-similar, i.e., that at any given point along the boundary layer, the velocity profile is the same to a stretching factor on the spatial dimension. He wrote:

$$\eta = y\left(\frac{U}{\nu x}\right)^{1/2}, \qquad \frac{u(x,y)}{U} = f'(\eta), \tag{4.43}$$

where $f(\eta)$ is a dimensionless quantity and $f'(\eta)$ denotes its derivative with respect to the dimensionless wall-normal coordinate η.
One of the virtues of this rescaling is that the wall-normal direction is rescaled by a quantity proportional to the laminar boundary layer thickness. In other words, the laminar boundary layer is mapped onto a rectangle.
As the flow is incompressible and two-dimensional, we introduce a streamfunction ψ such that:

$$u = \partial_y \psi, \qquad v = -\partial_x \psi. \tag{4.44}$$

The incompressibility constraint is automatically verified:

$$\partial_x u + \partial_y v = \partial_x(\partial_y \psi) + \partial_y(-\partial_x \psi) = 0. \tag{4.45}$$

Using the streamfunction, equation (4.42) reduces down to:

$$\partial_y \psi\, \partial_x \partial_y \psi - \partial_x \psi\, \partial_y^2 \psi = \nu \partial_y^3 \psi. \tag{4.46}$$

Equation (4.46) has yet to be written in terms of Blasius's variables (4.43). To do so, we note that the definition of the streamfunction (4.44) implies:

$$U f'(\eta) = \partial_\eta \psi \partial_y \eta \tag{4.47}$$

$$\Rightarrow \quad \psi = U \int_0^\eta \frac{1}{\partial_y \eta} f'(\eta) d\eta, \tag{4.48}$$

providing the new definitions:

$$\psi = U\gamma(x) f(\eta), \qquad \gamma(x) = (\nu x/U)^{1/2}. \tag{4.49}$$

With these new variables, the spatial derivatives of ψ become:

$$\partial_x \psi = U\left(\partial_x \gamma f + \gamma \partial_\eta f \partial_x \eta\right) \tag{4.50}$$

$$= U\left(\gamma' f - \gamma f' \frac{y\gamma'}{\gamma^2}\right) \tag{4.51}$$

$$= U\gamma' f - U f' \frac{y\gamma'}{\gamma}, \tag{4.52}$$

and:

$$\partial_y \psi = U\gamma \partial_\eta f \partial_y \eta \tag{4.53}$$

$$= U\gamma f' \frac{1}{\gamma} \tag{4.54}$$

$$= U f', \tag{4.55}$$

where $\gamma' = \partial_x \gamma$ and $f' = \partial_\eta f$.

The terms of equation (4.46) therefore become:

$$\partial_y\psi\,\partial_x\partial_y\psi = \left(Uf'\right)\,\partial_x\left(Uf'\right) \tag{4.56}$$
$$= \left(Uf'\right)\,Uf''\left(-\frac{y\gamma'}{\gamma^2}\right) \tag{4.57}$$
$$= -U^2\frac{y\gamma'}{\gamma^2}f'f'', \tag{4.58}$$

$$\partial_x\psi\,\partial_y^2\psi = \left(U\gamma' f - Uf'\frac{y\gamma'}{\gamma}\right)\,\partial_y\left(Uf'\right) \tag{4.59}$$
$$= \left(U\gamma' f - Uf'\frac{y\gamma'}{\gamma}\right)\,U\frac{1}{\gamma}f'' \tag{4.60}$$
$$= U^2\frac{\gamma'}{\gamma}ff'' - U^2\frac{y\gamma'}{\gamma^2}f'f'', \tag{4.61}$$

$$\nu\partial_y^3\psi = \nu\partial_y^2\left(Uf'\right) \tag{4.62}$$
$$= \nu U\partial_y\left(\frac{1}{\gamma}f''\right) \tag{4.63}$$
$$= \nu U\frac{1}{\gamma^2}f'''. \tag{4.64}$$

In the end, equation (4.46) simplifies into:

$$-U^2\frac{y\gamma'}{\gamma^2}f'f'' - \left(U^2\frac{\gamma'}{\gamma}ff'' - U^2\frac{y\gamma'}{\gamma^2}f'f''\right) = \nu U\frac{1}{\gamma^2}f''', \tag{4.65}$$
$$\nu U\frac{1}{\gamma^2}f''' + U^2\frac{\gamma'}{\gamma}ff'' = 0, \tag{4.66}$$
$$f''' + \frac{U\gamma'\gamma}{\nu}ff'' = 0. \tag{4.67}$$

The variable γ is easily obtained from definition (4.49) and yields $U\gamma'\gamma/\nu = 1/2$. The equation Blasius obtained for the dimensionless quantity f is then:

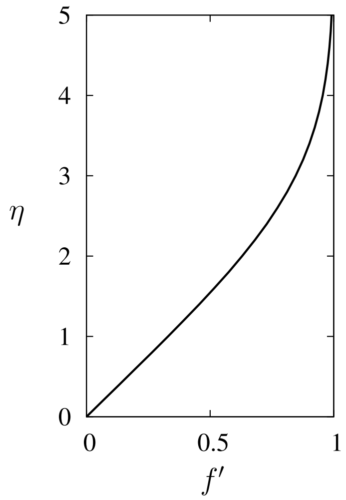
$$f''' + \frac{1}{2}ff'' = 0. \tag{4.68}$$

This equation is complemented with the boundary conditions (4.39) and (4.40) that now write:

$$f' = f = 0, \qquad \eta = 0, \tag{4.69}$$
$$f' \to 1, \qquad \eta \to \infty. \tag{4.70}$$

Solution

Equation (4.68) is generally solved numerically, as below.



This laminar boundary layer solution is self-similar: the same profile holds at any given position x along the boundary layer, the only change being a stretching coefficient in the wall-normal direction as η is a linear function of y and depends on x.
The boundary layer equation (4.68) and this solution are valid for $x \in [0; L_{max}]$ and $\eta \in \mathcal{R}^+$, where L_{max} represents the point at which a change in dynamics occurs that violates one of the hypotheses made. This point can arise due to a transition where the boundary layer becomes turbulent. There, wall-normal velocities become of the same order as streamwise velocities because of the creation and advection of eddies and the whole analysis carried out here breaks down.
Since $u/U \to 1$ as $\eta \to \infty$, we can define the boundary layer thickness as the region in which $u/U \leq 0.99$, or in other words $f' \leq 0.99$. The data from the figure above provides a boundary layer thickness of $\eta \approx 5.0$. Therefore:

$$\delta \left(\frac{U}{\nu x}\right)^{1/2} \approx 5.0, \tag{4.71}$$

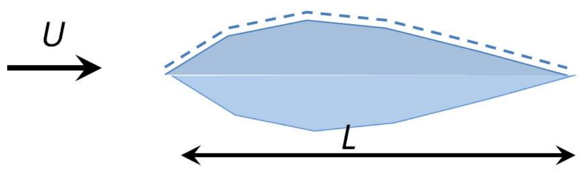
$$\Rightarrow \quad \delta \approx 5.0 \left(\frac{\nu x}{U}\right)^{1/2}, \tag{4.72}$$

$$\Rightarrow \quad \frac{\delta}{x} \approx 5.0 \left(\frac{\nu}{Ux}\right)^{1/2}, \tag{4.73}$$

$$\Rightarrow \quad \frac{\delta}{x} \approx 5.0 Re_x^{-1/2}. \tag{4.74}$$

4.3.2 Boundary layer on a solid boundary

Now, let us consider a steady flow of magnitude U past a streamlined body of length L.



The advection term, $\mathbf{u} \cdot \nabla\omega$, balances diffusion:

$$\mathbf{u} \cdot \nabla\omega = \nu\nabla^2\omega.$$

We expect the vorticity to vary over the length of the body, so that:

$$|\mathbf{u} \cdot \nabla\omega| \sim \frac{U\Omega}{L}.$$

Balancing this with the diffusion term, we have:

$$\frac{U\Omega}{L} \sim \nu \frac{\Omega}{\delta^2},$$

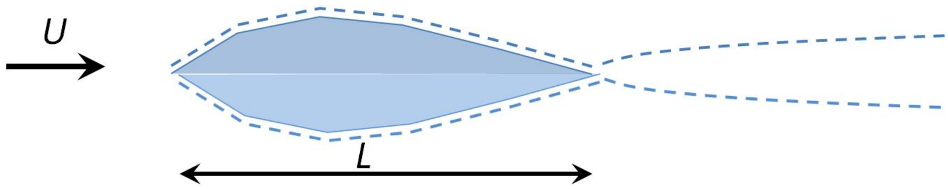
and hence:

$$\delta^2 = \frac{\nu L}{U} = L^2 \left(\frac{\nu}{UL}\right).$$

Thus, the vorticity is confined to a boundary layer of thickness of the order of $L\mathrm{Re}^{-1/2}$, where $\mathrm{Re} = UL/\nu$ is the Reynolds number. Provided the Reynolds number is large, the boundary layer thickness is small compared with the size of the body.

4.3.3 Wake behind a streamlined body

So far, we have argued that there exists a thin boundary layer on the surface of the body in which the vorticity is concentrated. The fluid in this boundary layer is advected around the surface of the body until the point where the flow separates behind it. This creates a region of vorticity behind the body which is referred to as the wake.



We can estimate the structure of the wake from the form of the flow above an impulsively started plate. In the frame of the fluid particle, the thickness of the wake grows as $\delta = \sqrt{\nu t}$ while the fluid is being carried away from a fixed point on the boundary at a speed U. Hence the thickness of the wake at a distance x behind the fixed point is approximately:

$$\delta = \sqrt{\frac{\nu x}{U}}.$$

It is parabolic, however, by the time the wake has grown to the size of the body, it has diffused so much that it is no longer detectable.

4.3.4 Separation from a bluff body

The above arguments are based on the case when the body has a streamlined shape so that its surface can be approximated by a flat plate. However, in the case of a bluff body such as a sphere or a cylinder, the boundary layer detaches from the body before it reaches its end. The reason that this happens is the presence of an opposing pressure gradient in the boundary layer. In the case of flow past a cylinder, the potential flow solution has high pressure at the front and back of cylinder, but low pressure at the top and bottom. Thus the pressure gradient changes sign at the top of cylinder. Provided $u > 0$ away from the boundary layer, $\partial p/\partial x > 0$ past the top of the cylinder. This drives the inner flow in the opposite direction to the flow outside the boundary layer and lifts the boundary layer away from the surface, causing a large wake behind the body.