Chapter 3

Slow viscous flow

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3.1 Stokes flow

For flows where the Reynolds number:

$$\mathrm{Re} = \frac{\rho U D}{\mu} \ll 1,$$

we can neglect the term $\rho \mathrm{D}\mathbf{u}/\mathrm{D}t$ in the Navier–Stokes equation so that the equations governing the fluid flow become the Stokes equation and the incompressibility condition:

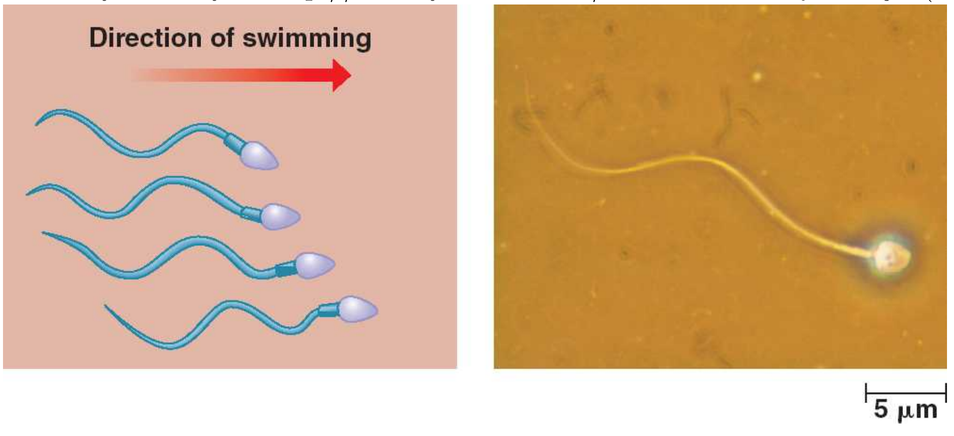
$$\mu \nabla^2 \mathbf{u} = \nabla p, \tag{3.1}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{3.2}$$

In the case of gravity driven flows, equation (3.1) is replaced by:

$$\mu \nabla^2 \mathbf{u} = \nabla P - \rho \mathbf{g}. \tag{3.3}$$

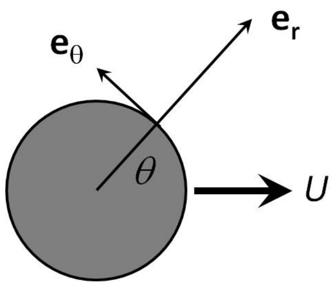
Unlike the Navier–Stokes equation, the Stokes equations are linear, which means that we can construct solutions using the principle of linear superposition. Furthermore, since there are no time derivatives the solutions are *time reversible* under the reflection $(\mathbf{u}, t) \to -(\mathbf{u}, t)$ (see demonstrations by G.I. Taylor http://www.youtube.com/watch?v=51-6QCJTAjU (t=13:13)).



This is bad news for small water-living creatures as it leads to the *scallop theorem*: if the swimming motion of a micro-organism is time reversible, it produces no net forward motion. Consequently, micro-organisms use wave motions along their flagella for propulsion, rather than the side-to-side motion employed by fish.
Videos: https://www.youtube.com/watch?v=NBH3UvlZo90, https://www.youtube.com/watch?v=LShQiec4

3.2 Translating sphere

Let us consider the Stokes flow generated by a sphere of radius a moving at speed U. Using spherical polar coordinates centred on the sphere, the velocity of the sphere is given by $U\mathbf{e_x} = (U\cos\theta, -U\sin\theta, 0)$, where θ represents the angle between the first vector of the coordinate frame and the direction of motion.



The potential flow solution for a sphere of radius a moving at speed U is given in spherical polar coordinates by:

$$\mathbf{u} = \nabla\phi = \left(U\frac{a^3}{r^3}\cos\theta, -U\frac{a^3}{2r^3}\sin\theta, 0\right).$$

However, although this satisfies the boundary condition for the r component of velocity, u, it does not satisfy the boundary condition for v. This solution also had the unrealistic property that the drag force on the sphere was identically zero: the *D'Alembert paradox*.
The flow is two-dimensional, so we may assume that the velocity is of the form:

$$\mathbf{u} = (u(r,\theta), v(r,\theta), 0)\,,$$

so that $\nabla\cdot\mathbf{u} = 0$ becomes:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v\sin\theta\right) = 0, \tag{3.4}$$

while equation (3.1) reduces (or expands) to:

$$\frac{\mu}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) - 2u - 2\frac{\partial v}{\partial\theta} - 2v\cot\theta\right] - \frac{\partial p}{\partial r} = 0, \tag{3.5}$$

$$\frac{\mu}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + 2\frac{\partial u}{\partial\theta} - \frac{v}{\sin^2\theta}\right] - \frac{1}{r}\frac{\partial p}{\partial\theta} = 0. \tag{3.6}$$

Given the boundary conditions, it is natural to seek a solution for the velocity of the form:

$$u = f(r)\cos\theta, \qquad v = g(r)\sin\theta,$$

where $f(r)$ and $g(r)$ satisfy $f(a) = U$, $g(a) = -U$ and f and g both tend to zero as $r \to \infty$. Substituting into equation (3.4) gives:

$$f + \frac{r}{2}\frac{df}{dr} + g = 0. \tag{3.7}$$

Careful inspection of equation (3.5) shows that all the terms coming from the fluid velocity are proportional to $\cos\theta$, which suggests that the pressure is of the form:

$$p(r,\theta) = p_0 + h(r)\cos\theta,$$

where p_0 is a constant. Substituting into equations (3.5) and (3.6), we obtain:

$$\frac{\mu}{r^2}\left[\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - 4f - 4g\right] - \frac{dh}{dr} = 0, \tag{3.8}$$

$$\frac{\mu}{r}\left[\frac{d}{dr}\left(r^2\frac{dg}{dr}\right) - 2g - 2f\right] + h = 0. \tag{3.9}$$

Thus, we have three coupled linear ODE for f, g and h. Eliminating h by differentiating equation (3.9), adding equation (3.8) and substituting for g from equation (3.7), we obtain:

$$\frac{d^4f}{dr^4} + \frac{8}{r}\frac{d^3f}{dr^3} + \frac{8}{r^2}\frac{d^2f}{dr^2} - \frac{8}{r^3}\frac{df}{dr} = 0. \tag{3.10}$$

This is a fourth order Cauchy equation and has solutions of the form r^m where m are the roots of:

$$m^4 + 2m^3 - 5m^2 - 6m = 0,$$

that is, $m = -3, -1, 0$ and 2. The general solution for f is of the form:

$$f = Ar^2 + B + \frac{C}{r} + \frac{D}{r^3}. \tag{3.11}$$

The boundary conditions as $r \to \infty$ implies that we must have $A = B = 0$, hence f is of the form:

$$f = \frac{C}{r} + \frac{D}{r^3}.$$

Substituting into equation (3.7):

$$g = -f - \frac{r}{2}\frac{df}{dr} = -\frac{C}{2r} + \frac{D}{2r^3}.$$

Hence, applying the wall boundary conditions ($f = U$, $g = -U$ at $r = a$), we obtain:

$$C = \frac{3aU}{2}, \qquad D = -\frac{Ua^3}{2},$$

that is:

$$u = \left(\frac{3a}{2r} - \frac{a^3}{2r^3}\right)U\cos\theta, \tag{3.12}$$

$$v = -\left(\frac{3a}{4r} + \frac{a^3}{4r^3}\right)U\sin\theta. \tag{3.13}$$

$$\tag{3.14}$$

Finally, from equation (3.9), we obtain $h = 3\mu Ua/2r^2$, giving:

$$p = p_0 + \frac{3\mu Ua}{2r^2}\cos\theta. \tag{3.15}$$

From this solution, we can calculate the drag force acting on the sphere:

$$\mathbf{F} = -\int_S \mathbf{f}\,dS = -\int_S \mathbf{n}\cdot\boldsymbol{\tau}\,dS.$$

Note that the sign is negative because this is the force applied by the fluid onto the boundary. This force is directed along the axis, so we need only consider the component in the direction $(\cos\theta, -\sin\theta, 0)$. The normal to the surface is $-\mathbf{e}_r$, so:

$$F_D = \int_S (\tau_{rr}\cos\theta - \tau_{r\theta}\sin\theta)dS.$$

To find τ_{rr} and $\tau_{r\theta}$, we need the components of the strain-rate tensor E_{rr} and $E_{r\theta}$, which, in spherical polar coordinates, are given by:

$$E_{rr} = \frac{\partial u}{\partial r}, \qquad E_{r\theta} = \frac{1}{2}\left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta}\right).$$

Remember:

$$\tau_{rr} = -p + \mu E_{rr}, \qquad \tau_{r\theta} = 2\mu E_{r\theta}.$$

Hence, for $r = a$:

$$E_{rr} = 0, \qquad E_{r\theta} = \frac{3U}{4a}\sin\theta,$$

and therefore:

$$\tau_{rr}\cos\theta - \tau_{r\theta}\sin\theta = -p_0\cos\theta - \frac{3\mu U}{2a}\cos^2\theta - \frac{3\mu U}{2a}\sin^2\theta = -p_0\cos\theta - \frac{3\mu U}{2a}.$$

The term $p_0\cos\theta$ integrates to zero over the surface of the sphere, leaving:

$$F_D = -\frac{3\mu U}{2a}S,$$

where $S = 4\pi a^2$ is the surface area of the sphere. We obtain:

$$F_D = -6\pi\mu a U, \tag{3.16}$$

which is the *Stokes drag force* on a sphere, a classical result in fluid dynamics.
If the sphere is falling due to gravity, then the associated force is:

$$F_W = -\frac{4\pi a^3}{3}\Delta\rho g,$$

where $\Delta\rho$ is the difference in density between the material of the sphere and the surrounding fluid, giving a fall speed for the sphere equal to:

$$U = \frac{2\Delta\rho g a^2}{9\mu}.$$

It is tempting to try to perform the same calculation in two dimensions for a moving cylinder, however, the solution fails because, the equivalent general solution to equation (3.11) is:

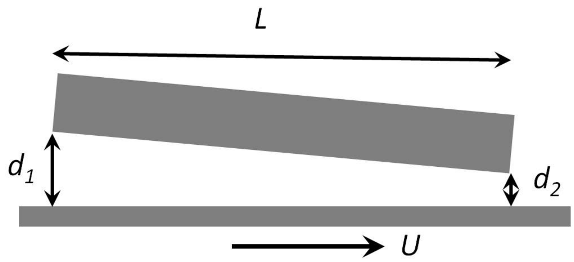
$$f = Ar^2 + B + C\log(r) + \frac{D}{r^2}.$$

and it is not possible to apply the boundary conditions at $r = a$ and $r \to \infty$ simultaneously. This result is known as *Stokes paradox* and was explained by Oseen who mentioned that the result was derived for small Reynolds numbers and that, in the case of the cylinder, large distances and therefore large Reynolds numbers were involved.

3.3 Lubrication flows

In many practical applications, the fluid flows in thin films where the boundaries are nearly parallel. These flows are "nearly" unidirectional and we can exploit this property to find approximate solutions.

3.3.1 The slider bearing flow



Consider a slider bearing where the fluid flows in the gap between the surface $y = 0$ moving at velocity $(U, 0, 0)$ and a fixed block at $y = h(x)$ of length L, where the two ends are at equal pressure p_0. We consider the case where the gap between the block and the moving surface varies slowly with x, so that $|dh/dx| \ll 1$. Specifically, we wish to consider the case:

$$h(x) = d_1 + \frac{(d_2 - d_1)}{L} x,$$

where $|h'| = |d_2 - d_1|/L \ll 1$. Since this is a two dimensional problem, we shall neglect z so that the velocity:

$$\mathbf{u} = (u(x,y), v(x,y)),$$

satisfies the following equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.17}$$

$$\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial p}{\partial x}, \tag{3.18}$$

$$\mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = \frac{\partial p}{\partial y}, \tag{3.19}$$

together with the boundary conditions: $u(x, 0) = U$, $u(x, h(x)) = 0$ and $v(x, 0) = v(x, h(x)) = 0$.

Constant gap

For a constant gap, $h' = 0$, we would set the x derivatives of the velocity to zero, so that:

$$\frac{dv}{dy} = 0,$$

$$\mu \frac{d^2 u}{dy^2} = \frac{\partial p}{\partial x},$$

$$\mu \frac{d^2 v}{dy^2} = \frac{\partial p}{\partial y}.$$

From the first of these equations, we deduce that $v(y) = 0$. The third equation gives $\partial p/\partial y = 0$ which implies that p is only a function of x:

$$\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx} = -G.$$

Hence, the pressure is given by

$$p = p_0 - Gx,$$

but, $p(L) = p_0$ implies $G = 0$ and hence:

$$\frac{d^2u}{dy^2} = 0.$$

The boundary conditions yields: $u = U(h - y)/h$.

Variable gap

Let us now consider $h' \ll 1$ and, for simplicity, rescale the equations (3.17) to (3.19) by choosing appropriate scales for variation in x and y. The natural choice for y is the average gap, $\bar{h} = \frac{1}{2}(d_1 + d_2)$, so we shall define:

$$y = \bar{h}y^*.$$

However, x derivatives only arise from changes in h so the appropriate scale is $\bar{h}/h'$:

$$x = \frac{\bar{h}}{h'}x^*.$$

We can additionally write the velocity and pressure in the form:

$$u(x, y) = Uu^*(x^*, y^*), \quad v(x, y) = Uv^*(x^*, y^*), \quad p(x, y) = \frac{\mu U}{\bar{h}}p^*(x^*, y^*).$$

Substituting these scalings into equations (3.17)–(3.19), we obtain:

$$h'\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{3.20}$$

$$h'^2\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} = h'\frac{\partial p^*}{\partial x^*}, \tag{3.21}$$

$$h'^2\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} = \frac{\partial p^*}{\partial y^*}. \tag{3.22}$$

The boundary conditions on u^* and v^* become $u^*(x^*, 0) = 1$, $u^*(x^*, h^*(x^*)) = 0$ and $v^*(x^*, 0) = v^*(x^*, h^*(x^*)) = 0$, where $h(x) = \bar{h}h^*(x^*)$. Noting that $|h'| \ll 1$, we seek solutions in the form:

$$u^* = u_0^* + h'u_1^* + \ldots.$$

Substituting in equation (3.20), we find:

$$\frac{\partial v^*}{\partial y^*} = -h'\frac{\partial u_0^*}{\partial x^*} + \ldots.$$

We thus write:

$$v^* = h'v_1^* + h'^2v_2^* + \ldots. \tag{3.23}$$

To find the size of the leading order pressure term, we need to look at equations (3.21) and (3.22). From equation (3.21), we have:

$$h'\frac{\partial p^*}{\partial x^*} = \frac{\partial^2 u_0^*}{\partial y^{*2}} + \ldots,$$

and from equation (3.22):

$$\frac{\partial p^*}{\partial y^*} = h' \frac{\partial^2 v_1^*}{\partial y^{*2}} + \dots.$$

The pressure p^* can be as large as $1/h'$ and thus takes the form:

$$p^* = \frac{1}{h'} p_{-1}^* + \dots. \tag{3.24}$$

Let us now consider the leading order terms in h' in each of the equations:

$$\frac{\partial u_0^*}{\partial x^*} + \frac{\partial v_1^*}{\partial y^*} = 0, \tag{3.25}$$

$$\frac{\partial^2 u_0^*}{\partial y^{*2}} = \frac{\partial p_{-1}^*}{\partial x^*}, \tag{3.26}$$

$$\frac{\partial p_{-1}^*}{\partial y^*} = 0. \tag{3.27}$$

The last two equations are analogous to those we found for $h' = 0$ the only change is that u_0^* is function of both x^* and y^*. Since the leading order gives $\partial p_{-1}^*/\partial y^* = 0$, we can integrate equation (3.26) to obtain:

$$u_0^* = \frac{dp_{-1}^*}{dx^*} \frac{y^{*2}}{2} + A(x^*)y^* + B(x^*),$$

and applying the boundary conditions, we obtain:

$$u_0^* = \frac{1}{2} \frac{dp_{-1}^*}{dx^*} y^*(y^* - h^*) + 1 - \frac{y^*}{h^*}.$$

To find v_1^*, we integrate equation (3.25):

$$\begin{aligned} v_1^* &= -\int_0^{y^*} \frac{\partial u_0^*}{\partial x^*} dy^* \\ &= \frac{1}{12} \frac{d^2 p_{-1}^*}{dx^{*2}} y^{*2}(3h^* - 2y^*) + \frac{1}{2} \frac{dh^*}{dx^*} \left(\frac{1}{2} \frac{dp_{-1}^*}{dx^*} - \frac{1}{h^{*2}} \right) y^{*2}. \end{aligned}$$

Hence, applying the boundary condition ($v^* = 0$ at $y^* = h^*$), we obtain:

$$\frac{1}{12} \frac{d^2 p_{-1}^*}{dx^{*2}} h^{*3} + \frac{1}{4} \frac{dp_{-1}^*}{dx^*} h^{*2} \frac{dh^*}{dx^*} = \frac{1}{2} \frac{dh^*}{dx^*},$$

which can be rewritten as:

$$\frac{d}{dx^*} \left(h^{*3} \frac{dp_{-1}^*}{dx^*} \right) = 6 \frac{dh^*}{dx^*}.$$

This equation is called the *Reynolds equation*. Upon integrating, we obtain:

$$\frac{dp_{-1}^*}{dx^*} = 6 \left(\frac{1}{h^{*2}} + \frac{A^*}{h^{*3}} \right),$$

for some constant A^*.

Now, let us go back to the original unstarred variables. Recall that:

$$p = \frac{\mu U}{\bar{h}} \left(\frac{1}{h'} p_{-1}^* + \dots \right),$$

giving:

$$\frac{dp}{dx} = \frac{\mu U}{\bar{h}}\left(\frac{1}{h'}\frac{dp^*_{-1}}{dx^*}\frac{dx^*}{dx} + \cdots\right),$$

so the leading order reads:

$$\frac{dp}{dx} \approx \frac{6\mu U}{h^3}\left(h + A\right), \tag{3.28}$$

where $A = \bar{h}A^*$.
For the specific case of the slider bearing $h(x) = d_1 + (d_2 - d_1)/Lx$:

$$\begin{aligned} p(L) &= p_0 - \frac{3\mu UL}{d_2 - d_1}\left[\frac{2}{h} + \frac{A}{h^2}\right]_0^L \\ &= p_0 + \frac{3\mu UL}{d_2^2 d_1^2}\left[2d_1 d_2 + A(d_1 + d_2)\right], \end{aligned}$$

and hence, to get $p_0 = p(L)$, it follows that $A = -2d_1 d_2/(d_1 + d_2)$, and:

$$\frac{dp}{dx} = \frac{6\mu U(d_2 - d_1)}{h^3}\left(\frac{x}{L} - \frac{d_1}{d_1 + d_2}\right).$$

To leading order, the fluid velocity reads:

$$u(x, y) = \frac{3U(d_2 - d_1)}{h^3}\left(\frac{x}{L} - \frac{d_1}{d_1 + d_2}\right) y(y - h) + U\left(1 - \frac{y}{h}\right).$$

Note that if $d_1 > d_2$ then $p(x) > p_0$ within the bearing and so there is a positive upward force on the bearing, whereas if $d_2 > d_1$ the force is negative.

3.3.2 Alternative method for the slider bearing flow

In the above, we used a formal expansion to find the dominant terms in the governing equations by scaling and non-dimensionalising the equations. However, with sufficient experience it is possible to recognise the dominant terms in each equation without needing to rescale everything. Let us return to the original dimensional equations:

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) &= \frac{\partial p}{\partial x}, \\ \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) &= \frac{\partial p}{\partial y}, \end{aligned}$$

and seek out the largest terms in each equation by estimating the size of each term. Let us denote U the size of u, V the size of v and P the size of p. We also need estimates for the size of derivatives. Since variations in y occur over the gap h, let us denote $\partial\ /\partial y$ as being of size $1/h$, and since $h' = dh/dx$, we can estimate x derivatives as being of size h'/h. Let us now look at the terms in equation (3.17):

$$\begin{array}{ccc} \dfrac{\partial u}{\partial x} & + & \dfrac{\partial v}{\partial y} = 0, \\ \dfrac{Uh'}{h} & & \dfrac{V}{h} \end{array}$$

Since these two terms have to balance to yield a two-dimensional flow, we can deduce:

$$V \sim Uh'. \tag{3.29}$$

Notice that this is equivalent to the scaling we deduced in equation (3.23). Equation (3.18) has three terms,

$$\begin{array}{cccccc} \mu\Big(\dfrac{\partial^2 u}{\partial x^2} & + & \dfrac{\partial^2 u}{\partial y^2}\Big) & = & \dfrac{\partial p}{\partial x}. \\ \mu\dfrac{Uh'^2}{h^2} & & \mu\dfrac{U}{h^2} & & \dfrac{Ph'}{h} \end{array}$$

Since $|h'| \ll 1$, the first term is small compared with the second term, so we can deduce the scale for pressure:

$$P \sim \mu\frac{U}{hh'}, \tag{3.30}$$

the same scaling found in equation (3.24). Finally, equation (3.19) also has three terms:

$$\begin{array}{cccccc} \mu\Big(\dfrac{\partial^2 v}{\partial x^2} & + & \dfrac{\partial^2 v}{\partial y^2}\Big) & = & \dfrac{\partial p}{\partial y}. \\ \mu\dfrac{Vh'^2}{h^2} & & \mu\dfrac{V}{h^2} & & \dfrac{P}{h} \end{array}$$

We find that the scale of $\partial p/\partial y$ is larger than the terms on the left-hand side by a factor $1/h'^2$. Note that equation (3.19) yields a smaller scale for P.
By neglecting all but the dominant terms in equations (3.17)–(3.19), we obtain:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.31}$$

$$\mu\frac{\partial^2 u}{\partial y^2} \simeq \frac{\partial p}{\partial x}, \tag{3.32}$$

$$\frac{\partial p}{\partial y} \simeq 0. \tag{3.33}$$

Equation (3.33) shows that the pressure is approximately independent of y and so, integrating equation (3.32) and applying the boundary conditions at $y = 0$ and $y = h(x)$, we get:

$$u = \frac{1}{2\mu}\frac{dp}{dx}y(y-h) + U - \frac{Uy}{h},$$

so that:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{1}{2\mu}\frac{d^2p}{dx^2}y(y-h) + \left(\frac{1}{2\mu}\frac{dp}{dx} - \frac{U}{h^2}\right)\frac{dh}{dx}y.$$

Upon integration and application of the boundary condition ($v = 0$ at $y = 0$), we get:

$$v = \frac{1}{12\mu}\frac{d^2p}{dx^2}y^2(3h-2y) + \frac{1}{2}\frac{dh}{dx}\left(\frac{1}{2\mu}\frac{dp}{dx} - \frac{U}{h^2}\right)y^2.$$

The boundary condition ($v = 0$ on $y = h$) yields the Reynolds equation:

$$\frac{h^3}{12\mu}\frac{d^2p}{dx^2} + \frac{h^2}{4\mu}\frac{dh}{dx}\frac{dp}{dx} = \frac{U}{2}\frac{dh}{dx}.$$

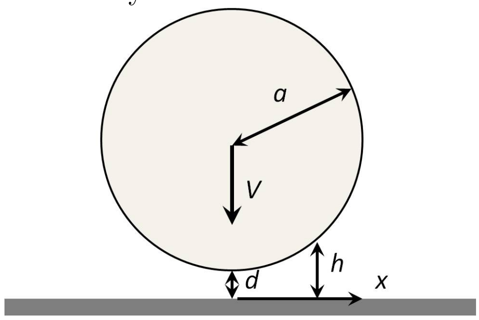
Integrating once returns:

$$\frac{dp}{dx} = \frac{6\mu U}{h^3}\left(h + A\right),$$

as found in equation (3.28) above.

3.3.3 Cylinder approaching a wall

Consider the motion of a cylinder towards a wall in the limit where the minimum gap d is small compared to the radius of the cylinder a.



We use Cartesian coordinates with the origin located at the position on the wall nearest to the cylinder with the y axis directed normal to the wall towards the cylinder and x directed in the plane of the wall perpendicular to the cylinder axis. The velocity takes the form:

$$\mathbf{u} = (u(x, y), v(x, y)).$$

In these coordinates, the position of the cylinder surface is given by:

$$h(x) = d + a - \sqrt{a^2 - x^2}.$$

However, we are interested in the flow in the region $|x| \ll a$, so:

$$h(x) \approx d + \frac{x^2}{2a}.$$

Note that $dh/dx = x/a$ is small provided that $|x| \ll a$. Let us choose the minimum gap d as the scale for y. The scale for x is less obvious. Let us write:

$$h(x) = d\left(1 + \frac{x^2}{2ad}\right).$$

We can see that variations in h are felt for $x \sim \sqrt{ad}$, so we shall rescale x and y as follows:

$$x = \sqrt{ad}x^* = \frac{d}{\epsilon}x^*, \quad y = dy^*,$$

where $\epsilon = \sqrt{d/a} \ll 1$. With this scaling, $h(x) = dh^*(x^*)$, where:

$$h^*(x^*) = 1 + \frac{x^{*2}}{2}. \tag{3.34}$$

The governing equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.35}$$

$$\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial p}{\partial x}, \tag{3.36}$$

$$\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = \frac{\partial p}{\partial y}, \tag{3.37}$$

with boundary conditions: $u(x, 0) = v(x, 0) = 0$, $u(x, h(x)) = 0$ and $v(x, h(x)) = -V$.

The velocity v being of size V, we can expand:

$$v(x,y) = V\left(v_0(x^*,y^*) + \epsilon v_1(x^*,y^*) + \dots\right),$$

and therefore from conservation of mass:

$$\frac{\partial u}{\partial x} \sim \frac{V}{d},$$

so that u must be of size $\epsilon^{-1}V$:

$$u(x,y) = V\left(\epsilon^{-1}u_{-1}(x^*,y^*) + \dots\right).$$

Thus, the flow mainly goes in the x direction. Turning to equation (3.36), we have:

$$\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 u}{\partial x^2} = \mu\frac{V}{d^2\epsilon}\frac{\partial^2 u^*_{-1}}{\partial y^{*2}} + \dots,$$

and hence:

$$p \sim \frac{\mu V}{d\epsilon^2}.$$

Therefore, the pressure can be expanded in the following fashion:

$$p(x,y) = \frac{\mu V}{d}\left(\epsilon^{-2}p^*_{-2}(x^*,y^*) + \dots\right).$$

Substituting into equations (3.35) to (3.37) and keeping only the leading order terms in ϵ, we have:

$$\frac{\partial u^*_{-1}}{\partial x^*} + \frac{\partial v^*_0}{\partial y^*} = 0, \tag{3.38}$$

$$\frac{\partial^2 u^*_{-1}}{\partial y^{*2}} = \frac{\partial p^*_{-2}}{\partial x^*}, \tag{3.39}$$

$$\frac{\partial p^*_{-2}}{\partial y^*} = 0, \tag{3.40}$$

with boundary conditions $u^*_{-1}(x^*,0) = u^*_{-1}(x^*,h^*) = 0$, $v^*_0(x^*,0) = 0$ and $v^*_0(x^*,h^*) = -1$. As before, $\partial p^*_{-2}/\partial y^* = 0$, so that the solution of equation (3.39), satisfying the boundary conditions, is given by:

$$u^*_{-1} = \frac{1}{2}\frac{dp^*_{-2}}{dx^*}y^*(y^* - h^*). \tag{3.41}$$

From equation (3.38), we get:

$$v^*_0 = \frac{1}{12}\frac{d^2p^*_{-2}}{dx^{*2}}y^{*2}(3h^* - 2y^*) + \frac{1}{4}\frac{dp^*_{-2}}{dx^*}\frac{dh^*}{dx^*}y^{*2}.$$

Applying the boundary condition at $y^* = h^*$, we obtain:

$$-1 = \frac{h^{*3}}{12}\frac{d^2p^*_{-2}}{dx^{*2}} + \frac{h^{*2}}{4}\frac{dp^*_{-2}}{dx^*}\frac{dh^*}{dx^*}.$$

The associated *Reynolds equation* is:

$$\frac{d}{dx^*}\left(h^{*3}\frac{dp^*_{-2}}{dx^*}\right) = -12. \tag{3.42}$$

Integrating once, we find:

$$\frac{dp^*_{-2}}{dx^*} = -12\frac{x^*}{h^{*3}} + \frac{A^*}{h^{*3}}.$$

Since this flow is symmetric about $x = 0$, $dp^*_{-2}/dx^* = 0$ at $x^* = 0$ and so $A^* = 0$. It follows:

$$u^*_{-1} = -\frac{6x^*}{h^{*3}} y^*(y^* - h^*).$$

Recall from equation (3.34) that $dh^*/dx^* = x^*$ which yields:

$$p^*_{-2}(x^*) = p^*_\infty + \frac{6}{h^{*2}},$$

where p^*_∞ is the dimensionless pressure at infinity.
Having found the pressure, we can now calculate the force that the cylinder exerts on the fluid. This force is oriented in the y direction and its amplitude given by:

$$F_y = L\int_{-a}^{a} f_y dx,$$

where L is the length of the cylinder and $\mathbf{f} = \mathbf{n}\cdot\boldsymbol{\tau}$. Here:

$$\mathbf{n} = \frac{1}{\sqrt{1+h'^2}}\Big(-h', 1\Big),$$

where $h' = \partial h/\partial x$. For $|x| \ll a$, $\mathbf{n} \approx (-h', 1)$ and hence:

$$f_y = -p - \mu\frac{dh}{dx}\left(\frac{du}{dy} + \frac{dv}{dx}\right) + 2\mu\frac{dv}{dy}.$$

Since the pressure is of size $\epsilon^{-2}\mu V/d$, it constitutes the dominant contribution to f_y:

$$F_y = -L\int_{-a}^{a}(p(x) - p_\infty)dx.$$

Changing variables to $x^* = \epsilon d^{-1}x = \epsilon^{-1}a^{-1}x$, we get:

$$\begin{aligned} F_y &= -\frac{\epsilon^{-2}\mu V L}{d}\int_{-\epsilon^{-1}}^{\epsilon^{-1}}\left(p^*_{-2}(x^*) - p^*_\infty\right) d\epsilon^{-1}dx^* \\ &= -\epsilon^{-3} 6\mu V L\int_{-\epsilon^{-1}}^{\epsilon^{-1}}\frac{1}{(1+\frac{x^{*2}}{2})^2}dx^*. \end{aligned}$$

To perform this integral, we substitute $x^* = \sqrt{2}\tan u$ and obtain:

$$\begin{aligned} F_y &= -\epsilon^{-3} 6\sqrt{2}\mu V L\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4 u du \\ &= -\frac{9\sqrt{2}}{4}\pi\mu V L\left(\frac{a}{d}\right)^{3/2}. \end{aligned}$$

If the cylinder is falling towards the wall under a constant force $F = -Mg$:

$$V = \frac{4Mgd^{3/2}}{9\sqrt{2}\pi\mu L a^{3/2}} = -\frac{d}{dt}d,$$

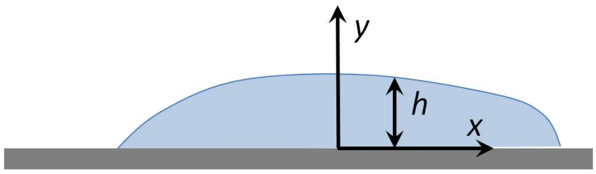
and therefore:

$$\frac{1}{\sqrt{d(t)}} = \frac{1}{\sqrt{d_0}} + \frac{2Mg}{9\sqrt{2}\pi\mu L a^{3/2}}t,$$

where d_0 is the gap at $t = 0$. The cylinder will not contact the wall in finite time.

3.3.4 Free surface flows

Let us now consider the gravitational spreading of a blob of viscous fluid on a surface.



For simplicity, we shall consider a two-dimensional blob whose height is given by $y = h(x,t)$. In order to use lubrication theory, we shall assume that $|dh/dx \ll 1|$ so that the normal to the surface is approximately $(-h', 1)$. At leading order, the boundary conditions at $y = h$ are:

$$-P + 2\mu\frac{\partial v}{\partial y} = -P_{atm}, \qquad \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0,$$

together with $u = v = 0$ at $y = 0$. The kinematic boundary condition at the free surface requires that:

$$v = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}. \tag{3.43}$$

The Stokes equations read:

$$\nabla P = \rho\mathbf{g} + \mu\nabla^2\mathbf{u},$$
$$\nabla\cdot\mathbf{u} = 0,$$

and reduce to:

$$\frac{\partial P}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{3.44}$$

$$\frac{\partial P}{\partial y} = -\rho g + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{3.45}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{3.46}$$

Rather than scale these equations, let us look for the dominant terms. We shall use U and V to denote the size of the velocity components u and v respectively, and take $1/h$ as the size of $\partial\ /\partial y$, and since $h' = dh/dx$ we can estimate x derivatives as being of size h'/h.
Since this flow is driven by gravity, P and ρg must balance so that $P \sim \rho gh$. Equation (3.44) gives:

$$\frac{\partial P}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$
$$h'\rho g \qquad \frac{\mu U}{h^2}h'^2 \qquad \frac{\mu U}{h^2}$$

Hence, we can estimate that:

$$U \sim \frac{\rho g h^2}{\mu}h',$$

and neglect the x derivatives of u so that:

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{1}{\mu}\frac{\partial P}{\partial x}.$$

Furthermore, from equation (3.46):

$$V \sim h'U \sim \frac{\rho g h^2}{\mu} h'^2.$$

Hence, the sizes of the terms in equation (3.45) are:

$$\begin{array}{ccccccc} \dfrac{\partial P}{\partial y} & = & -\rho g & + & \mu\left(\dfrac{\partial^2 v}{\partial x^2}\right. & + & \left.\dfrac{\partial^2 v}{\partial y^2}\right). \\ \rho g & & \rho g & & h'^4\rho g & & h'^2\rho g \end{array}$$

Equation (3.45) reduces to:

$$\frac{\partial P}{\partial y} = -\rho g,$$

so that

$$P = -\rho g y + A(x).$$

Furthermore, since $|dv/dy| \sim h'^2(\rho g h)/\mu$, we can also neglect this term in the boundary condition. As $P = P_{atm}$ at $y = h$:

$$P = P_{atm} + \rho g(h - y), \tag{3.47}$$

and therefore $\partial P/\partial x = \rho g \partial h/\partial x$. It follows:

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{\rho g}{\mu}\frac{\partial h}{\partial x}.$$

Again since $|dv/dx| \sim h'^3(\rho g h)/\mu$, we can neglect its contribution to the boundary conditions and the boundary conditions on u simplify to $u = 0$ on $y = 0$ and $du/dy \approx 0$ on $y = h$. The leading order solution for $|h'| \ll 1$ is:

$$u = -\frac{\rho g}{2\mu}\frac{\partial h}{\partial x} y(2h - y). \tag{3.48}$$

From equation (3.46):

$$v = \frac{\rho g}{6\mu}\left[\frac{\partial^2 h}{\partial x^2} y^2(3h - y) + 3\left(\frac{\partial h}{\partial x}\right)^2 y^2\right],$$

and so, at $y = h$:

$$v = \frac{\rho g}{6\mu}\left[2\frac{\partial^2 h}{\partial x^2} h^3 + 3\left(\frac{\partial h}{\partial x}\right)^2 h^2\right].$$

Substituting into the kinematic boundary condition gives:

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu}\left[\frac{\partial^2 h}{\partial x^2} h^3 + 3\left(\frac{\partial h}{\partial x}\right)^2 h^2\right] = \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right). \tag{3.49}$$

This is a non-linear diffusion equation for $h(x, t)$.

3.4 Hele-Shaw flow

Let us consider now the case of the general flow in the gap between two parallel plates separated by a distance h.

At the boundaries $z = 0$ and $z = h$, we impose no slip boundary conditions: $\mathbf{u} = 0$. We assume that the variations of $\mathbf{u}$ in the (x, y) plane are slow so that $\nabla^2\mathbf{u} \approx \partial^2\mathbf{u}/\partial z^2$.
The Stokes equations reduce to:

$$\frac{\partial p}{\partial x} \approx \mu\frac{\partial^2 u}{\partial z^2},$$
$$\frac{\partial p}{\partial y} \approx \mu\frac{\partial^2 v}{\partial z^2},$$
$$\frac{\partial p}{\partial z} \approx 0,$$

and the mass conservation equation reads:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

The velocity in the (x, y) plane is given by:

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}z(h - z),$$
$$v = -\frac{1}{2\mu}\frac{\partial p}{\partial y}z(h - z).$$

Integrating over z, we obtain the average velocity in the (x, y) plane:

$$\bar{u} = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{1}{h}\int_0^h z(h - z)dz = -\frac{h^2}{12\mu}\frac{\partial p}{\partial x},$$
$$\bar{v} = -\frac{1}{2\mu}\frac{\partial p}{\partial y}\frac{1}{h}\int_0^h z(h - z)dz = -\frac{h^2}{12\mu}\frac{\partial p}{\partial y},$$

which we can write as:

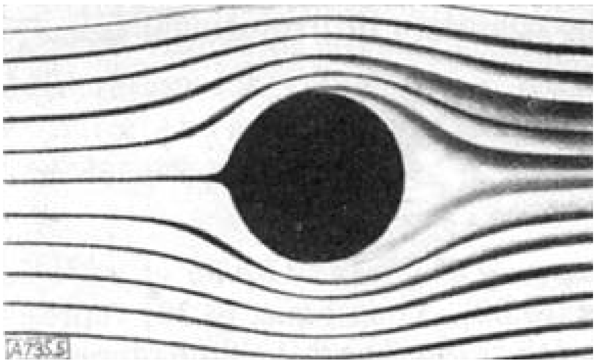
$$\bar{\mathbf{u}} = -\frac{h^2}{12\mu}\nabla_H p, \tag{3.50}$$

where ∇_H is the restriction of the gradient operator to the (x, y) plane.
Hence, the two-dimensional flow in the (x, y) plane is a potential flow with the velocity potential $\Phi = -h^2/12\mu p$. Furthermore, integrating the continuity equation between 0 and h, we obtain:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0,$$

and so this flow is incompressible and hence:

$$\nabla_H \cdot \bar{\mathbf{u}} = 0. \tag{3.51}$$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