Chapter 2

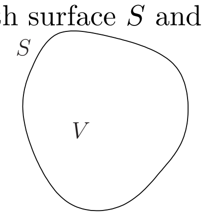
The Navier–Stokes equation

Contents

So far, we have considered mass conservation and the rate of deformation of the fluid. Now, we shall consider the forces acting on the fluid and how they relate to the velocity gradient.

2.1 Fluid momentum transport

Consider a fixed volume of fluid V with surface S and outward normal $\mathbf{n}$.



We consider the changes to the total fluid momentum contained within V. Applying Newton's law of motion to this volume we have:

$$\begin{pmatrix}\text{rate of increase}\\ \text{of momentum}\\ \text{in } V\end{pmatrix} = \begin{pmatrix}\text{net inward flow}\\ \text{of momentum}\\ \text{through } S\end{pmatrix} + \begin{pmatrix}\text{net force}\\ \text{acting}\\ \text{on } V\end{pmatrix} \tag{2.1}$$

The momentum density in the fluid is given by $\rho\mathbf{u}$ and so the first contribution to equation (2.1) is:

$$\begin{pmatrix}\text{rate of increase}\\ \text{of momentum}\\ \text{in } V\end{pmatrix} = \frac{d}{dt}\left(\int_V \rho\mathbf{u}dV\right) = \int_V \frac{\partial}{\partial t}\left(\rho\mathbf{u}\right)dV,$$

The net flow of momentum through the surface S is given by:

$$\begin{pmatrix}\text{net inward flow}\\ \text{of momentum}\\ \text{through } S\end{pmatrix} = -\int_S \rho\mathbf{u}\mathbf{u}\cdot\mathbf{n}dS,$$

which in suffix notation is written as:

$$-\int_S \rho u_i u_j n_j dS.$$

Now, to convert this to a volume integral, we apply the divergence theorem by replacing n_j with $\partial/\partial x_j$ so that:

$$-\int_S \rho u_i u_j n_j dS = -\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)\, dV.$$

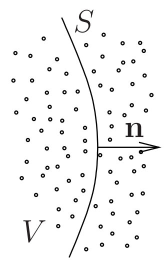
The forces acting on the fluid can be divided into two groups:

i. **Body forces:** These are external forces acting on the fluid, such as gravity or electro-magnetic forces (which are important in astrophysical fluids and also for flows within the Earth's core), however we shall only consider gravity which exerts a force:

$$\int_V \rho \mathbf{g} dV.$$

ii. **Molecular forces:** These are short-range due to interactions between fluid molecules on either side of the surface S, which exert a force:

$$\int_S \mathbf{f} dS.$$



Since we wish to use the divergence theorem to transform this into a volume integral we define the *total stress tensor*, $\boldsymbol{\tau}$, such that:

$$f_i = n_j \tau_{ji},$$

where $\mathbf{n}$ is the normal to the surface. With this definition, the molecular force acting on the fluid is given by:

$$\int_S \tau_{ji} n_j dS = \int_V \frac{\partial \tau_{ji}}{\partial x_j} dV.$$

Putting all these contributions back into equation (2.1), we obtain:

$$\int_V \left[\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) - \rho g_i - \frac{\partial \tau_{ji}}{\partial x_j}\right] dV = 0. \tag{2.2}$$

Since V is arbitrary, we obtain the momentum equation:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) - \rho g_i - \frac{\partial \tau_{ji}}{\partial x_j} = 0.$$

Note that the first two terms correspond to the conservation equation for the vector $\rho \mathbf{u}$. We can simplify this equation by noting from mass conservation (equation (1.6)) that:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0,$$

so that:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho \frac{Du_i}{Dt}.$$

Hence, the equation for fluid momentum transport is:

$$\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} = \rho g_i + \frac{\partial \tau_{ji}}{\partial x_j}, \tag{2.3}$$

or written in the usual vector notation:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \rho \mathbf{g} + \nabla \cdot \boldsymbol{\tau}. \tag{2.4}$$

Like the equation for mass conservation, this equation applies to all continuous materials. However, this is not sufficient to predict the motion of the fluid since we need an additional equation relating the molecular forces represented by the stress tensor $\boldsymbol{\tau}$ to the fluid motion.

2.2 Constitutive equations

The equation defining the stress tensor $\boldsymbol{\tau}$ is called the constitutive equation.

2.2.1 Ideal fluid

The simplest constitutive equation for a fluid is that of an ideal or inviscid fluid for which the only surface force is the pressure. For an incompressible fluid, pressure arises from the resistance to changes in volume and acts along the direction of the normal $\mathbf{n}$:

$$\mathbf{f} = -P\mathbf{n}.$$

Hence, for an ideal fluid: $n_j \tau_{ji} = -Pn_i$, yielding $\tau_{ji} = -P\delta_{ij}$. Substituting into the momentum transport equation, we obtain the Euler equation:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \rho \mathbf{g} - \nabla P. \tag{2.5}$$

2.2.2 Newtonian fluid

In most situations, additional forces come into play. In the example case of a shear flow, we observed a force that takes the following form per unit area: $\mathbf{f} = \mu\dot{\gamma}\mathbf{e}_x$, where we have defined the shear-rate $\dot{\gamma} = \partial u_1/\partial x_2$. In addition to the pressure, there is therefore a contribution to the stress caused by the shearing motion:

$$\tau_{ij} = -P\delta_{ij} + \sigma_{ij}, \tag{2.6}$$

where σ_{ij} is the viscous stress. The tensor $\boldsymbol{\sigma}$ is proportional to the velocity gradient. We are looking at forces and are hence interested in the deformations of the fluid. In chapter 1, we saw that only the symmetric part of the velocity gradient involves the deformation of the fluid. Considering an isotropic fluid, the stress must then be of the form:

$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = 2\mu E_{ij}, \tag{2.7}$$

where μ is the dynamic viscosity.
In the shear flow case, the only non-zero component of the velocity gradient is $\partial u_1/\partial x_2 = \dot{\gamma}$, so that $\sigma_{21} = \sigma_{12} = \mu\dot{\gamma}$. Notice that the stress tensor is symmetric.

Example: Let us consider the flow which occurs when we stretch out a cylinder of incompressible fluid: $w = \dot{\epsilon} z$.
Mass conservation requires that the volume of fluid in the column remains fixed:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

which can be achieved by setting $u = -\frac{1}{2}\dot{\epsilon} x$ and $v = -\frac{1}{2}\dot{\epsilon} y$. The velocity gradient tensor $\nabla \mathbf{u}$ is therefore given by:

$$\nabla \mathbf{u} = \begin{pmatrix} -\frac{1}{2}\dot{\epsilon} & 0 & 0 \\ 0 & -\frac{1}{2}\dot{\epsilon} & 0 \\ 0 & 0 & \dot{\epsilon} \end{pmatrix}.$$

Hence, the viscous stress has components: $\sigma_{xx} = \sigma_{yy} = -\mu\dot{\epsilon}$ and $\sigma_{zz} = 2\mu\dot{\epsilon}$.
For a Newtonian fluid:

$$\frac{\partial \tau_{ji}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[-P\delta_{ji} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] = -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j x_j} + \mu\frac{\partial^2 u_j}{\partial x_j x_i},$$

However, since $\nabla \cdot \mathbf{u} = 0$, the last term is zero and we obtain the governing equations for an incompressible Newtonian fluid:

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla P + \mu\nabla^2\mathbf{u} + \rho\mathbf{g}, \tag{2.8}$$

often solved together with the incompressibility constraint: $\nabla \cdot \mathbf{u} = 0$.

2.3 Hydrostatic and dynamic pressure

If there is no flow, the Navier–Stokes equation reduces to a balance between gravity and pressure:

$$-\nabla P + \rho\mathbf{g} = 0.$$

The resulting pressure solution:

$$P_H = P_0 + \rho\mathbf{g} \cdot \mathbf{x},$$

is referred to as *hydrostatic pressure*. Although gravity is responsible for driving some flows such as rivers or gravity waves, in many cases it is simply balanced by the hydrostatic pressure. As a consequence, it is often useful to subtract off the hydrostatic pressure by writing the pressure in the form:

$$P = P_H + p \tag{2.9}$$

where p is referred to as the *dynamic pressure*. This reduces the Navier–Stokes equation to:

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu\nabla^2\mathbf{u}. \tag{2.10}$$

2.4 Boundary conditions

In addition to the equation for the stress tensor, we need to know what boundary conditions to apply. In general, both the velocity and the forces must be continuous at a fluid boundary, however, the nature of the boundary impacts the way these laws are expressed.

2.4.1 Solid boundaries

Where a fluid is in contact with a solid surface moving at velocity $\mathbf{U}$, there is friction between the solid surface and the fluid: the velocity in the fluid $\mathbf{u}$ must be equal to the velocity of the solid surface:

$$\mathbf{u} = \mathbf{U},$$

on the boundary. As well as matching the velocity, we also have a boundary condition on the stress. By definition, the surface force density applied by the boundary on the fluid is equal to

$$\mathbf{f} = \mathbf{n} \cdot \boldsymbol{\tau}, \tag{2.11}$$

where $\mathbf{n}$ is the outward pointing normal to the surface. By Newton's third law, the fluid imposes an equal and opposite force density on the solid boundary.

2.4.2 Free surfaces

If the fluid is in contact with air (or a fluid of much lower viscosity), the only force exerted by the air on the fluid results from the atmospheric pressure P_{atm}. In the absence of surface forces, the force applied by the air to the fluid is $-P_{\text{atm}}\mathbf{n}$. The force balance implies:

$$\mathbf{n} \cdot \boldsymbol{\tau} = -P\mathbf{n} + \mathbf{n} \cdot \boldsymbol{\sigma} = -P_{\text{atm}}\mathbf{n}, \tag{2.12}$$

known as the dynamic boundary condition. Consequently, there is no force parallel to the surface. A free surface cannot support shear.
We still require one additional boundary condition. Let the position of the surface be given by $f(\mathbf{x}, t) = h(x, y, t) - z = 0$. Since all points on the surface must remain on the surface:

$$\frac{Df}{Dt} = 0,$$

which is the kinematic boundary condition. In particular, if the surface remains fixed in time, we have:

$$\mathbf{u} \cdot \nabla f = 0,$$

where $\nabla f = \mathbf{n}$ is the normal to the surface. It follows:

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{0}.$$

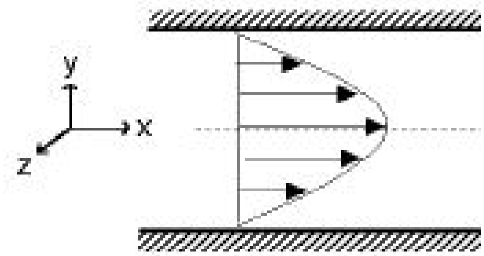
2.4.3 Boundary between immiscible fluids

At a boundary between two fluids of different viscosities, both the velocity $\mathbf{u}$ and force density $\mathbf{n} \cdot \boldsymbol{\tau}$ must be continuous.
Furthermore, if the surface between the fluids remains fixed in time then $\mathbf{u} \cdot \mathbf{n} = 0$. Consequently, the condition that $\mathbf{n} \cdot \boldsymbol{\tau}$ is continuous reduces to both P and $\mu\mathbf{n}\cdot\nabla\mathbf{u}$ being continuous.

2.5 One dimensional flow examples

2.5.1 Plane Poiseuille flow

Let us consider the stationary flow of fluid along a channel driven by a pressure gradient. We define Cartesian coordinates with x along the channel in the direction of flow and y across the channel, with boundaries at $y = \pm h$.



The fluid velocity, $\mathbf{u}$, satisfies no-slip boundary conditions at the walls: $\mathbf{u} = 0$ at $y = \pm h$. We wish to calculate the simplest flow realization, so we use the symmetries of the configuration to simplify the calculation:

- $\mathbf{u} \cdot \mathbf{e_y} = v = 0$: no wall-normal flow
- $\mathbf{u} \cdot \mathbf{e_z} = w = 0$: unidirectional velocity in the direction of the pressure gradient
- $\partial_z \equiv 0$: invariance in the spanwise direction

The first two hypotheses lead to the velocity having only one non-vanishing component while the third one indicates it only varies in the streamwise and wall-normal directions: $\mathbf{u} = u(x, y)\mathbf{e_x}$.
We recall the writing of the incompressibility constraint in Cartesian coordinates:

$$\partial_x u + \partial_y v + \partial_z w = 0, \tag{2.13}$$

and of the Navier–Stokes equation:

$$\rho \left[\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u\right] = -\partial_x p + \mu \left[\partial_x^2 u + \partial_y^2 u + \partial_z^2 u\right], \tag{2.14}$$
$$\rho \left[\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v\right] = -\partial_y p + \mu \left[\partial_x^2 v + \partial_y^2 v + \partial_z^2 v\right], \tag{2.15}$$
$$\rho \left[\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w\right] = -\partial_z p + \mu \left[\partial_x^2 w + \partial_y^2 w + \partial_z^2 w\right]. \tag{2.16}$$

The incompressibility constraint (2.13) yields:

$$\partial_x u = 0, \tag{2.17}$$

which, together with the starting hypotheses provides:

$$\mathbf{u} = u(y)\mathbf{e_x}. \tag{2.18}$$

The Navier–Stokes equation in the wall-normal and spanwise directions (2.15) and (2.16) give:

$$\partial_y p = \partial_z p = 0, \tag{2.19}$$

thus:

$$p = p(x). \tag{2.20}$$

Eventually, the Navier–Stokes equation in the streamwise direction (2.14) gives:

$$0 = -\partial_x p + \mu \partial_y^2 u. \tag{2.21}$$

The solution reads:

$$u = \frac{\partial_x p}{2\mu} y^2 + k_1 y + k_2, \tag{2.22}$$

where k_1 and k_2 are solved for using the boundary conditions:

$$\frac{\partial_x p}{2\mu} h^2 + k_1 h + k_2 = 0, \tag{2.23}$$
$$\frac{\partial_x p}{2\mu} h^2 - k_1 h + k_2 = 0, \tag{2.24}$$

yielding:

$$k_1 = 0, \qquad k_2 = -\frac{\partial_x p\, h^2}{2\mu}. \tag{2.25}$$

The trivial laminar flow in a channel, also called *plane Poiseuille flow*, then reads:

$$u = \frac{G\,h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right), \tag{2.26}$$

with $G = -\partial_x p$. Hence, the velocity has a parabolic profile, with a shear-stress at the wall given by: $\sigma_{yx} = \mu du/dy = \mp Gh$. The streamfunction is given by:

$$\frac{\partial \psi}{\partial y} = u = \frac{G}{2\mu}\left(h^2 - y^2\right),$$

so that:

$$\psi = \frac{Gy}{6\mu}\left(3h^2 - y^2\right).$$

The volume flow per unit depth is given by $\psi(h) - \psi(-h) = 2Gh^3/3\mu$.

2.5.2 Hagen–Poiseuille flow

The equivalent axisymmetric problem where a fluid flows along a cylindrical pipe is often referred to as Hagen–Poiseuille or Poiseuille flow in the name of the scientists who first derived and measured this flow.

We consider a pipe of radius a and use cylindrical polar coordinates based on the axis of the cylinder so that the fluid velocity is of the form: $\mathbf{u} = w(r)\hat{\mathbf{e}}_{\boldsymbol{z}}$. As was the case for channel flow, this flow automatically satisfies the incompressibility condition and, since we look for a steady flow: $D\mathbf{u}/Dt = 0$. The components of the Navier–Stokes equation reduce to:

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial r}, \\ 0 &= -\frac{1}{r}\frac{\partial p}{\partial \theta}, \\ 0 &= -\frac{\partial p}{\partial z} + \mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right). \end{aligned}$$

The pressure is a function of z alone and is of the form $p(z) = p_0 - Gz$ where G is the magnitude of the pressure gradient. We obtain:

$$\frac{d}{dr}\left(r\frac{dw}{dr}\right) = -\frac{Gr}{\mu}.$$

Integrating with respect to r yields:

$$\frac{dw}{dr} = \frac{A}{r} - \frac{Gr}{2\mu},$$

for some constant A. The velocity is smooth everywhere, so $A = 0$. Finally, integrating again and applying the boundary condition $w(a) = 0$, we obtain:

$$w(r) = \frac{G}{4\mu}\left(a^2 - r^2\right).$$

Hence, using the result from example 1.5, the volume flow through the pipe is equal to:

$$Q = \frac{G\pi a^4}{8\mu},$$

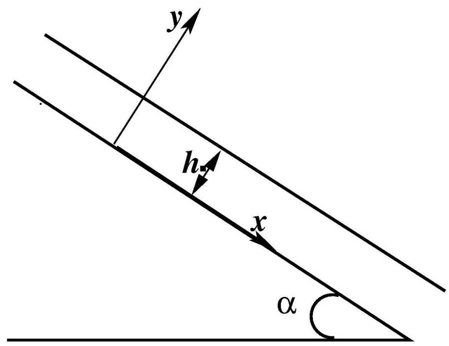
and so the pressure difference required to pump a fluid of viscosity μ at a volume flow rate Q along a pipe of radius a and length L is given by:

$$\Delta p = \frac{8\mu QL}{\pi a^4}.$$

This solution gives a good approximation provided that the pipe is long and straight and that the fluid is sufficiently viscous. It also provides a good method for measuring the viscosity of a fluid.

2.5.3 Flow down an inclined plane

A plane inclined at an angle α to the horizontal is coated with a layer of fluid of thickness h.
Let us define the Cartesian coordinates: x is directed down the slope and y perpendicular to the slope. Using the symmetries of the configuration to simplify the flow, we assume that the fluid velocity is of the form $\mathbf{u} = (u(y), 0, 0)$.



Since gravity is driving the flow, we consider the full pressure and include the body force due to gravity. The Navier–Stokes equation reduces to:

$$\nabla P = \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}.$$

The gravitational acceleration $\mathbf{g}$ is given by:

$$\mathbf{g} = (g \sin\alpha, -g\cos\alpha, 0).$$

Hence, the x and y components of the momentum equation reduce to:

$$\frac{\partial P}{\partial x} = \mu \frac{d^2 u}{dy^2} + \rho g \sin\alpha, \tag{2.27}$$

$$\frac{\partial P}{\partial y} = -\rho g \cos\alpha. \tag{2.28}$$

At the free surface $y = h$, the boundary condition reads: $\mathbf{n} \cdot \boldsymbol{\tau} = -P_{\text{atm}}\mathbf{n}$. In this case, $\mathbf{n} = (0, 1)$, which implies that:

$$\tau_{yy} = -P = -P_{\text{atm}}, \quad \tau_{yx} = \sigma_{yx} = \mu \frac{du}{dy} = 0.$$

From equation (2.28), the pressure is given by:

$$P = P_{\text{atm}} + \rho g \cos\alpha (h - y).$$

so that $\partial P/\partial x = 0$. Therefore, upon integrating equation (2.27), we get:

$$u(y) = -\frac{\rho g \sin\alpha}{2\mu} y^2 + Ay + B,$$

where A and B are integration constants determined by the boundary conditions: $u = 0$ at $y = 0$ and $du/dy = 0$ at $y = h$. Eventually, we obtain:

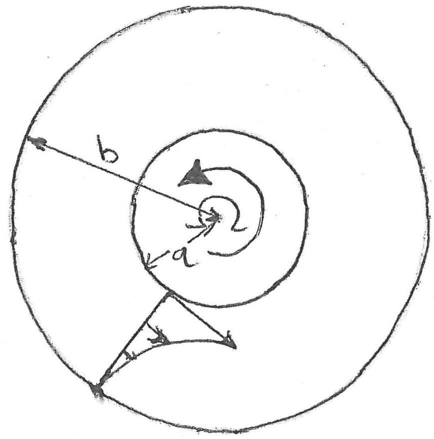
$$u(y) = \frac{\rho g \sin\alpha}{2\mu} y(2h - y).$$

Hence, the flow profile is again parabolic and corresponds to the flow in the bottom half of the channel.

2.5.4 Taylor–Couette flow

So far, we have only considered flows in which the fluid particles move in straight lines at constant speed so that the acceleration is equal to zero. Let us now consider a flow with curved streamlines so that $D\mathbf{u}/Dt \neq 0$.

Consider a fluid flow between two concentric cylinders of radii a and b respectively, where the inner cylinder is rotating at an angular velocity Ω. We define the cylindrical polar coordinates about the axis of the cylinders so that $\mathbf{u} = (0, v(r), 0)$ with boundary conditions $v(a) = a\Omega$ and $v(b) = 0$.



Again, this form of the fluid velocity automatically satisfies $\nabla \cdot \mathbf{u} = 0$. However, although the flow is steady, $D\mathbf{u}/Dt \neq 0$. Indeed, substituting the form of the fluid velocity, we find:

$$\mathbf{u} \cdot \nabla \mathbf{u} = \left(-\frac{v^2}{r}, 0, 0 \right).$$

Substituting into the Navier–Stokes equation, we obtain:

$$-\frac{\rho v^2}{r} = -\frac{\partial p}{\partial r}, \tag{2.29}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr} \right) - \frac{v}{r^2} \right], \tag{2.30}$$

$$0 = -\frac{\partial p}{\partial z}. \tag{2.31}$$

From equation (2.30), we see that $\partial p/\partial \theta$ is independent of θ. However, since p is periodic in θ, $p(\theta + 2\pi) = p(\theta)$ and so $\partial p/\partial \theta = 0$. Equation (2.30) reduces to:

$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = 0.$$

This is a Cauchy equation with general solution:

$$v = \frac{A}{r} + Br.$$

Applying the boundary conditions, we obtain:

$$v(r) = \frac{\Omega a^2}{b^2 - a^2}\left(\frac{b^2}{r} - r \right). \tag{2.32}$$

We can find the pressure by integrating equation (2.29):

$$p(r) = p(a) + \rho \int_a^r \frac{v^2(r')}{r'} dr'.$$

Since $v^2/r > 0$, the pressure increases with the distance to the center. This is the reason why the free-surface dips near a rotating rod.

In order to calculate the surface forces we need to obtain the velocity gradient, which is given in cylindrical polar coordinates by:

$$\nabla \mathbf{u} = \begin{pmatrix} \dfrac{\partial u}{\partial r} & \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial r} & \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{u}{r} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial r} & \dfrac{1}{r}\dfrac{\partial w}{\partial \theta} & \dfrac{\partial w}{\partial z} \end{pmatrix}.$$

Hence, in this flow:

$$\sigma_{r\theta} = \mu\left(\frac{dv}{dr} - \frac{v}{r}\right) = -\frac{2\mu\Omega b^2 a^2}{(b^2-a^2)r^2}.$$

The torque required to rotate the inner cylinder is given by:

$$\mathbf{T} = \int \mathbf{r} \times \mathbf{f} dS,$$

where $\mathbf{r}$ is the radial vector and:

$$\mathbf{f} = -\hat{\mathbf{r}} \cdot \boldsymbol{\tau},$$

since $\mathbf{n} = -\hat{\mathbf{r}}$. Thus, the magnitude of the torque T applied to the inner cylinder is given by:

$$T = -\int_S r\tau_{r\theta} dS = -L\int_0^{2\pi} a\tau_{r\theta} a d\theta = -2\pi a^2 L\sigma_{r\theta} = \frac{4\pi\mu L\Omega a^2 b^2}{(b^2-a^2)}, \tag{2.33}$$

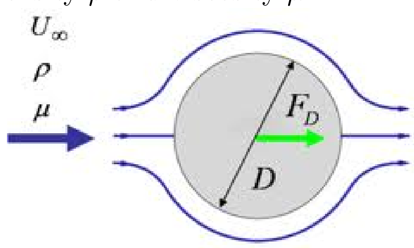
where L is the length of the Taylor–Couette cell. This experiment provides a practical method for measuring viscosity.

2.6 The Reynolds number

2.6.1 Dynamic similarity

A good starting point is to consider under what conditions are two flows "dynamically equivalent", by which we mean that they have the same flow pattern even though the scales and fluid properties may be different.

Let us consider the flow pattern generated by an obstacle (e.g. a sphere) of size D in a uniform flow of speed U in a fluid of density ρ and viscosity μ.



This problem has four dimensional parameters, D, U, ρ and μ. We can use these parameters to define a new system of units based upon independent units for mass (M), length (L) and time (T). Note that $[D] = L$, $[U] = L \cdot T^{-1}$, $[\rho] = M \cdot L^{-3}$ and $[\mu] = M \cdot L^{-1} \cdot T^{-1}$. It is thus logical to choose:

$$L = D, \quad \text{and } T = \frac{D}{U}.$$

Lastly, since both ρ and μ involve mass, we can choose either:

$$M = \rho D^3 \quad \text{or } M = \frac{\mu D^2}{U}.$$

As a consequence, there is a single independent dimensionless group that can formed from the combination of D, U, ρ and μ:

$$\mathrm{Re} = \frac{\rho U D}{\mu}. \tag{2.34}$$

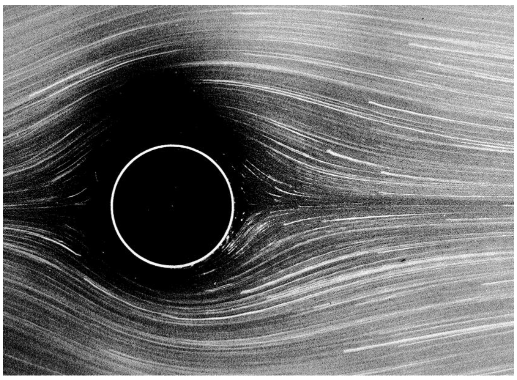
This number is called the *Reynolds number*. It indicates the balance between inertia and viscous forces. Flows with the same value of the Reynolds number (and dimensionless geometry) but different values of the dimensional parameters D, U, ρ and μ display the same flow pattern and are thus dynamically similar. By selecting fluid properties and flow rates appropriately, we can make smaller or larger scale models that give the same flow pattern.

2.6.2 Flow past a cylinder

To illustrate how the Reynolds number can be used to characterise a flow, let us consider the flow past a cylinder. Recall that for an inviscid fluid the potential flow pattern is fore-aft symmetric and produces zero drag, but that this flow is not seen in practice.

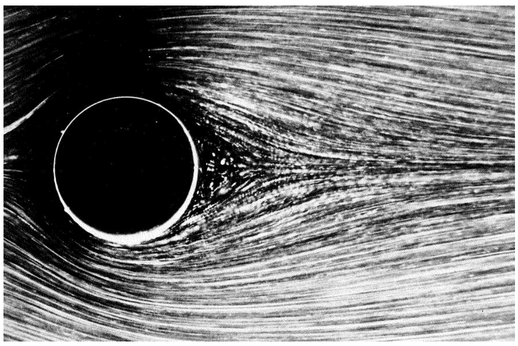
- $Re < 1$

  For small values of the Reynolds number the flow is nearly fore-aft symmetric. However, this flow pattern is distinct from the potential flow solution as it satisfies $\mathbf{u} = \mathbf{0}$ on the cylinder surface.

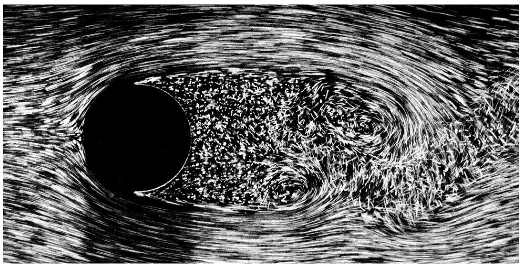
- $1 < Re < 46$

  As the Reynolds number increases, the flow loses its fore-aft symmetry and two recirculating eddies appear on the downstream side of the cylinder. These cells grow in size as the Reynolds number increases. Although the flow is no longer fore-aft symmetric it remains steady.

- $46 < Re$

Above Reynolds numbers of around 46, the flow is no longer steady. The eddies behind the cylinder become unsteady and are shed alternately from the two sides, forming a double line of eddies known as a von Kármán vortex street.



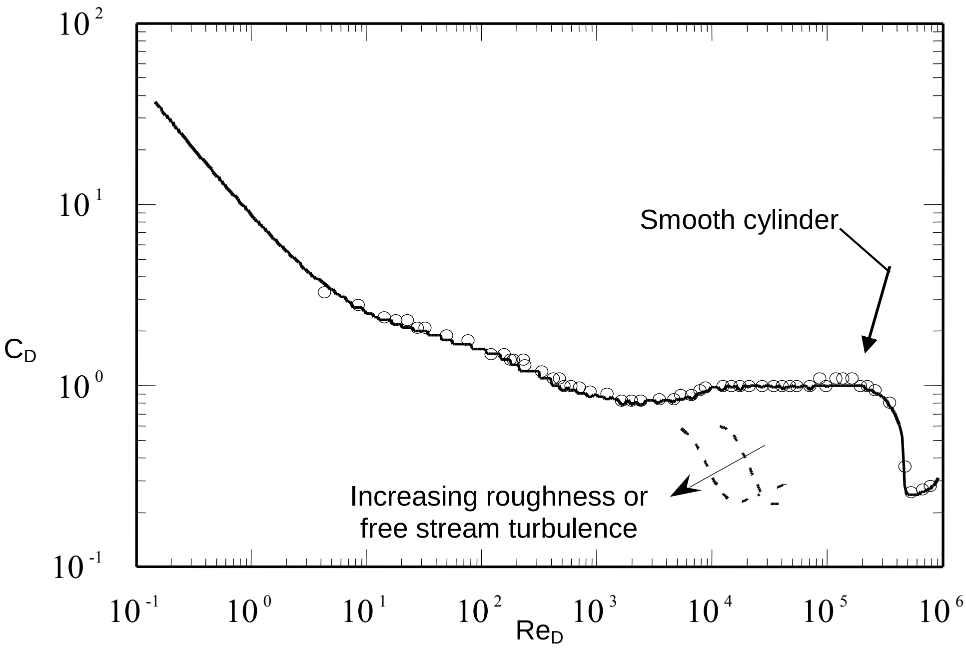
As the Reynolds number increases further the flow in this wake region behind the cylinder becomes chaotic.

Simulations at $Re = 25$, $Re = 50$, $Re = 100$ and $Re = 220$:
https://www.youtube.com/watch?v=8WtEuw0GLg0.

As well as looking at the flow pattern, we can also measure the drag force on the cylinder. Forces have units of $M \cdot L \cdot T^{-2}$ so, if we use ρ to define the unit of mass, we can write the drag force in the form:

$$F = \frac{1}{2}\rho U^2 A C_D(\mathrm{R}e),$$

where A is the cross-sectional area and C_D is a dimensionless number that is a function of the object shape and the Reynolds number. Note that the factor $\frac{1}{2}$ is introduced by convention. The graph below shows how the drag coefficient on a cylinder varies with the Reynolds number.



For low Reynolds numbers, the drag coefficient decreases roughly as $1/\mathrm{R}e$ and levels out to an $O(1)$ value for Reynolds numbers above 200. A sharp drop occurs at Reynolds numbers between 100,000 and 1,000,000.

The fact that the drag coefficient remains approximately constant over a wide range of Reynolds numbers makes it useful for defining how the shape of an object affects the drag force. For example, cars typically have a drag coefficient in the range 0.25 to 0.5. The boxy shapes, such as Range Rovers tend to be at the high end, whereas the best energy efficient designs have drag coefficients around 0.25.

2.6.3 The Reynolds number and the Navier–Stokes equation

We can obtain the Reynolds number from the Navier–Stokes equation by non-dimensionalising it, i.e. by choosing units based upon the natural length and time scales. We substitute:

$$\mathbf{u} = U\mathbf{u}^*, \qquad \mathbf{x} = D\mathbf{x}^*, \qquad t = \frac{D}{U}t^*,$$

where $\mathbf{u}^*$ and $\mathbf{x}^*$ are now dimensionless vector quantities, and choose $\mu U/D$ as the unit for pressure:

$$p = \mu\frac{U}{D}p^*.$$

The Navier–Stokes equation becomes:

$$\frac{\rho U^2}{D}\frac{\mathrm{D}\mathbf{u}^*}{\mathrm{D}t^*} = -\mu\frac{U}{D^2}\nabla^* p^* + \mu\frac{U}{D^2}\nabla^{*2}\mathbf{u}^*,$$

and, dividing by $\mu U/D^2$, we have:

$$\mathrm{Re}\frac{\mathrm{D}\mathbf{u}^*}{\mathrm{D}t^*} = -\nabla^* p^* + \nabla^{*2}\mathbf{u}^*.$$

Conservation of mass remains:

$$\nabla^* \cdot \mathbf{u}^* = 0,$$

and so the only parameter in the governing equations is the Reynolds number. An alternative choice for the pressure is $p = \rho U^2 p^*$, yielding:

$$\frac{\mathrm{D}\mathbf{u}^*}{\mathrm{D}t^*} = -\nabla^* p^* + \frac{1}{\mathrm{Re}}\nabla^{*2}\mathbf{u}^*.$$

The Reynolds number arises naturally from consideration of the terms in the Navier–Stokes equation (2.10):

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu\nabla^2\mathbf{u}.$$

For a steady flow past an obstacle, the size of the left-hand side can be estimated as:

$$|\rho\mathbf{u}\cdot\nabla\mathbf{u}| \sim \frac{\rho U^2}{D},$$

while that of the viscous term as:

$$|\mu\nabla^2\mathbf{u}| \sim \frac{\mu U}{D^2}.$$

Hence, if we take the ratio of these two terms, we get:

$$\frac{|\rho\mathbf{u}\cdot\nabla\mathbf{u}|}{|\mu\nabla^2\mathbf{u}|} \sim \frac{\rho U^2}{D} \times \frac{D^2}{\mu U} = \frac{\rho U D}{\mu} = Re.$$

The Reynolds number can be thought of as the ratio of the relative sizes of the terms governing fluid inertia and viscosity.

If the Reynolds number is large then $|\rho\mathrm{D}\mathbf{u}/\mathrm{D}t| \gg |\mu\nabla^2\mathbf{u}|$. The pressure gradient balances $\rho\mathbf{u}\cdot\nabla\mathbf{u}$ and the pressure differences over the obstacle are of size ρU^2. The drag force is thus roughly of the size of $\rho U^2 A$, so that the drag coefficient is of order unity.

Conversely, if the Reynolds number is small then $|\mu\nabla^2\mathbf{u}| \gg |\rho\mathrm{D}\mathbf{u}/\mathrm{D}t|$ and the pressure gradient balances $|\mu\nabla^2\mathbf{u}|$. This gives a pressure difference of the size of $\mu U/D$ and hence the magnitude of the force arising from viscous drag scales as:

$$\mu\frac{U}{D}A = \rho U^2 A\left(\frac{\mu}{\rho U D}\right) = \frac{\rho U^2 A}{Re},$$

so that the drag coefficient scales as $1/\mathrm{R}e$, as was found for the case of the cylinder.

2.6.4 Flow at low and high Reynolds numbers

A small (large) Reynolds number suggests that the inertia (viscosity) terms are small compared to the other terms in the Navier–Stokes equation and might be neglected. We do however need to be careful as the correct scales for U and D are not always obvious. The small Reynolds number case describes slow viscous flows where we can neglect $\rho \mathrm{D}\mathbf{u}/\mathrm{D}t$. The resulting equations:

$$-\nabla p + \mu \nabla^2 \mathbf{u} = 0, \qquad \nabla \cdot \mathbf{u} = 0, \tag{2.35}$$

are called *Stokes equations* and the corresponding solutions *Stokes flows*. As the Stokes equations do not contain $\mathrm{D}\mathbf{u}/\mathrm{D}t$, they are linear and not directly dependent on time. They are considerably easier to solve than the full Navier–Stokes equations. Indeed, the exact solutions given in the previous chapter are in fact solutions of the Stokes equations.

The opposite limit of high Reynolds number flows is more complicated. Excluding the viscous term from the Navier–Stokes equation reduces it to the Euler equation for an ideal fluid:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p.$$

However, in doing this, we removed the term with the highest spatial derivative, $\nabla^2 \mathbf{u}$, which is mathematically dangerous since it means that we cannot impose the full no-slip boundary condition. Therefore there must be a layer of fluid, called a *boundary layer* near the surface where the shear-rates are sufficiently high that viscosity cannot be neglected. In many cases however, these layers are sufficiently thin that we can neglect them and in these cases the Euler equation (and hence the Bernoulli equation) gives a good approximation to the flow. In other flows, such as flow past a cylinder, these boundary layers can grow in size and affect large regions of the flow.