Chapter 1

Mathematical modelling of fluids

Contents

1.1 Introduction

Mechanics is the science concerned with behaviour of physical bodies subject to forces and displacements. It encompasses statics (study of forces and torques when there is no motion), kinematics (study of motion regardless of its causes) and dynamics (study of forces and torques when there is motion). Fluids are substances that flow or conform to the outline of their container (Merriam–Webster). Fluid dynamics is thus the study of the forces and torques where there is motion of liquids, gases and plasmas (e.g. water, air, interstellar plasma). While it is a relatively old subject (dating back to the 18th century with Newton, Euler and Lagrange for example), it is still a very active research area. Here are some currently very active research areas:

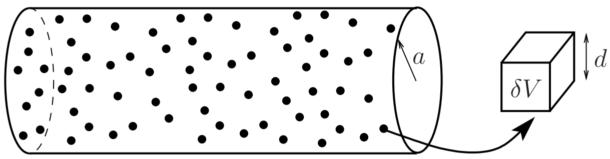
- biological fluids – blood and air flows, swimming organisms, drug delivery
- aerodynamics and hydrodynamics – aeroplanes, ships
- industrial fluids – casting, injection molding, mixing
- environmental fluids – pollution, water and wind power
- geophysical fluids – earth's core, atmosphere and ocean, weather
- astrophysical fluids – galaxies, stars, interstellar medium

Fundamentally all these fluids obey the same physical laws of motion, however, they differ widely in the lengthscale they display and in some of their physical properties (density, viscosity). As a consequence, fluids can show different types of motion depending upon what effects are dominant.

1.2 Continuum hypothesis

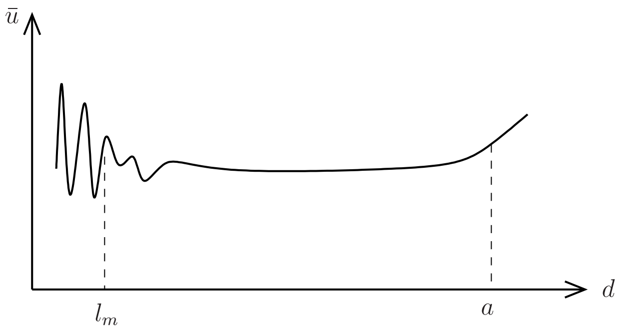
(Ref.: Paterson, §III.1)

In fluid dynamics, we do not attempt to calculate the motion of individual molecules — there are far too many of them, (a cubic centimetre of water contains of the order of 10^{23} molecules of typical size $l_m \simeq 1\mathring{A} = 10^{-10}$ m) and their individual motion is dominated by high frequency fluctuations caused by collisions with neighbouring molecules. Instead, we represent the average motion of a "blob" of fluid called a *fluid particle* of length d and volume δV.



We choose d so that:

- $d \gg l_m$ (molecular scale) – δV contains many molecules and the fluctuations due to individual motions are averaged out.
- $d \ll a$ (macroscopic scale) – δV is approximately a point in space.



Given this choice ($l_m \ll d \ll a$), the average velocity $\bar{u}$ is a smooth function of the variables and independent of d.

Continuum hypothesis: Molecular details can be smoothed out by assigning to the velocity at a point P the average velocity in a fluid element δV centred in P.

We can thus define the velocity field $\mathbf{u}(\mathbf{x}, t)$ as a smooth function of time and position, i.e., $\mathbf{u}$ is differentiable and integrable. Note that shock waves break this assumption.

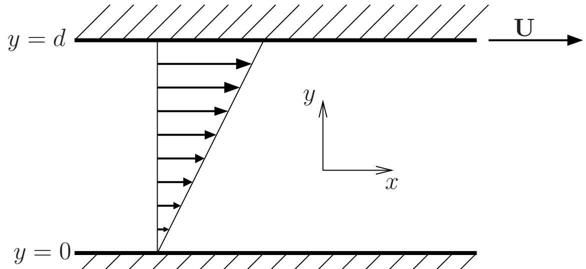
Similarly, $\rho(\mathbf{x}, t) = \dfrac{\text{mass in } \delta V}{\delta V}$ is the local density of mass.

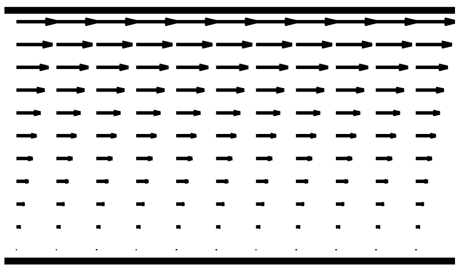
1.3 Velocity field

The fluid velocity is defined, within the continuum hypothesis, as the *vector field* $\mathbf{u}(\mathbf{x}, t)$, function of space and time.

Example 1.1
Shear flow: consider the flow between two parallel plates when one is moved relative to the other with a constant velocity U.

$$\mathbf{u} = \begin{pmatrix} \frac{Uy}{d} \\ 0 \\ 0 \end{pmatrix} = \frac{Uy}{d}\hat{\boldsymbol{e_x}}.$$



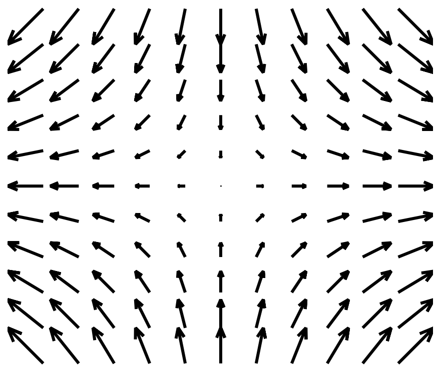


The direction of the flow is indicated by an arrow and its magnitude by the arrow length.

Stagnation-point flow: consider a flow where the velocity cancels at the origin.

$$\mathbf{u} = \begin{pmatrix} Ex \\ -Ey \\ 0 \end{pmatrix} = Ex\hat{\boldsymbol{e_x}} - Ey\hat{\boldsymbol{e_y}}.$$

The point $\mathbf{x} = 0$ where $\mathbf{u} = 0$ is called a *stagnation point*.



1.3.1 Particle paths

(Ref.: Paterson, §III.2)

One method for visualising fluid motion is to follow the motion of a "tracer" particle in the flow.
Let a particle be released at time t_0 and at position $\mathbf{x}_0$ within the velocity field. Since the particle moves with the fluid velocity:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}, t) \quad \text{such that} \quad \mathbf{x} = \mathbf{x}_0 \text{ at } t = t_0. \tag{1.1}$$

Example 1.2 (Stagnation point flow)

$$\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix} Ex \\ -Ey \\ 0 \end{pmatrix} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = Ex, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -Ey, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = 0$$

$$\Rightarrow x(t) = x_0\mathrm{e}^{E(t-t_0)}, \quad y(t) = y_0\mathrm{e}^{-E(t-t_0)}, \quad z(t) = z_0 \quad \text{if } \mathbf{x} = \mathbf{x}_0 = (x_0, y_0, z_0) \text{ at } t = t_0.$$

Note that particles at the stagnation point $x_0 = y_0 = 0$ do not move since $\mathbf{u} = 0$.

The time variable, t, can be eliminated to show that particle paths are hyperbolae of equation $y = \dfrac{x_0 y_0}{x}$.

1.3.2 Streamlines

A streamline is a line everywhere tangent to the local fluid velocity at time t. If the line is parametrised by a parameter s ("distance" along the streamline), then:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{u}(\mathbf{x}(s), t), \tag{1.2}$$

or, equivalently:

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w} \quad (= \mathrm{d}s), \tag{1.3}$$

since $\mathbf{u}(\mathbf{x}, t)$ is not explicitly function of s. If the flow is steady ($\partial \mathbf{u}/\partial t = 0$), then the streamlines are the same as the particle paths. Note that the converse is not necessarily true.

1.4 Time derivatives

The time derivative $\partial \mathbf{u}/\partial t$ measures the rate of change of velocity at the fixed position $\mathbf{x}$. This is referred to as the *Eulerian* time-derivative. However, this does not give the acceleration of a fluid particle at this point, since the particle is moving through this point along its particle path. Instead we require the *convective* derivative (also called Lagrangian derivative and material derivative) $\mathrm{D}\mathbf{u}(\mathbf{x}, t)/\mathrm{D}t$, which is the rate of change of $\mathbf{u}$ along the particle path $\mathbf{x}(t)$, i.e., moving with the fluid.

Using the chain rule:

$$\begin{aligned}
\frac{\mathrm{D}f}{\mathrm{D}t} &\equiv \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t), t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\
\Rightarrow \frac{\mathrm{D}f}{\mathrm{D}t} &= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} \\
\Rightarrow \frac{\mathrm{D}f}{\mathrm{D}t} &= \frac{\partial f}{\partial t} + (\mathbf{u} \cdot \nabla) f. \qquad (1.4)
\end{aligned}$$

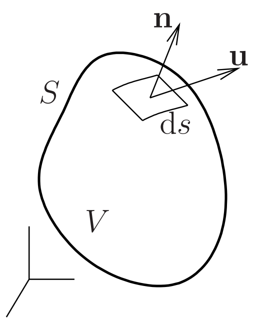
Hence the acceleration of a fluid particle is:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u}. \tag{1.5}$$

1.5 Conservation of mass

In any situation, the mass of a fluid must be conserved. For a continuous material, this principle is expressed in the form of the *continuity equation*.

Consider a volume V, fixed in space, with surface S and outward normal $\mathbf{n}$.



The total mass in V is:

$$M_V = \int_V \rho \, \mathrm{d}V,$$

where ρ is the density of mass (mass per unit volume).

M_V can only change if mass is carried inside or outside the volume by the fluid.

The mass flowing through the surface per unit time (i.e. the mass flux) is:

$$\frac{\mathrm{d}M_V}{\mathrm{d}t} = -\int_S \rho\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S,$$

and therefore:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = \int_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V = -\int_S \rho\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S, \quad \text{since } V \text{ is fixed.}$$

Applying the divergence theorem:

$$\int_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V = -\int_V \nabla\cdot(\rho\mathbf{u})\,\mathrm{d}V \Rightarrow \int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right]\mathrm{d}V = 0.$$

Since V is arbitrary, this equation must hold for all volume V. Thus, the *continuity equation*:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{1.6}$$

holds at all points in the fluid. Expanding the divergence as $\nabla\cdot(\rho\mathbf{u}) = \rho\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\rho$, we obtain the Lagrangian form of the continuity equation:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} = 0. \tag{1.7}$$

The density of a fluid particle moving with the fluid only changes if there is an expansion or a contraction of the flow.

1.6 Incompressibility

In an *incompressible* fluid, the density of each fluid particle remains constant and the continuity equation (1.7) reduces to:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \Leftrightarrow \rho\nabla\cdot\mathbf{u} = 0.$$

So, for an incompressible flow:

$$\nabla\cdot\mathbf{u} = 0. \tag{1.8}$$

This places a restriction on the form of the fluid velocity:

$$\mathbf{u} = \nabla\times\mathbf{\Psi}, \tag{1.9}$$

for some vector field $\mathbf{\Psi}$. Since $\nabla\cdot(\nabla\times\mathbf{\Psi}) = 0$ for any vector field $\mathbf{\Psi}$, this automatically satisfies the continuity equation.

1.6.1 Two dimensional flows

Let us consider the incompressible flow:

$$\mathbf{u} = \begin{pmatrix} u(x,y) \\ v(x,y) \\ 0 \end{pmatrix} = u(x,y)\,\hat{\boldsymbol{e_x}} + v(x,y)\,\hat{\boldsymbol{e_y}}.$$

We can introduce $\mathbf{\Psi} = \psi(x,y)\boldsymbol{e_z}$, where $\psi(x,y)$ is a scalar function such that:

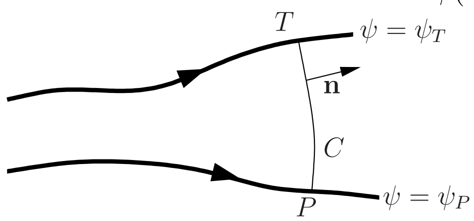
$$u = \frac{\partial\psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial\psi}{\partial x}. \tag{1.10}$$

The function $\psi(x,y)$ is called the streamfunction and has a number of important properties.

Streamlines: On a streamline: $d\psi = 0 \Rightarrow \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -v dx + u dy = 0$. So, $\mathbf{u} \times \mathbf{dl} = 0$, i.e., the streamline element $\mathbf{dl} = (dx, dy)$ is parallel to $\mathbf{u}$. A streamline is a line on which the streamfunction is constant. The gradient of the streamfunction is orthogonal to the velocity field:

$$\mathbf{u} \cdot \nabla \psi = u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} = 0.$$

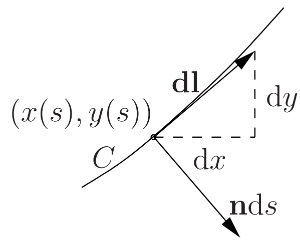
Flux between streamlines: Consider the two streamlines: $\psi(x, y) = \psi_P$ and $\psi(x, y) = \psi_T$.



The fluid flow or *flux* through $C : \{x(s), y(s)\}$, an arbitrary curve connecting P and T, is:

$$Q = \int_P^T \mathbf{u} \cdot \mathbf{n}\, ds. \tag{1.11}$$

Let $\mathbf{dl} = dx\boldsymbol{e_x} + dy\boldsymbol{e_y} = ds\left(\frac{dx}{ds}\boldsymbol{e_x} + \frac{dy}{ds}\boldsymbol{e_y}\right)$ be an infinitesimal displacement along the curve C.



We define the infinitesimal vector normal to $\mathbf{dl}$:

$$\mathbf{n}\, ds = dy\boldsymbol{e_x} - dx\boldsymbol{e_y} = ds\left(\frac{dy}{ds}\boldsymbol{e_x} - \frac{dx}{ds}\boldsymbol{e_y}\right).$$

$$\text{So,} \quad Q = \int_P^T \left(\frac{\partial \psi}{\partial y}\frac{dy}{ds} + \frac{\partial \psi}{\partial x}\frac{dx}{ds}\right) ds = \int_P^T \frac{d\psi}{ds} ds = \int_{\psi_P}^{\psi_T} d\psi = \psi_T - \psi_P.$$

Hence, the flux between two streamlines is equal to the streamfunction difference between the two streamlines. Consequently, the flow is faster when the streamlines are close together.

Following from the definition of the streamfunction: $\|\mathbf{u}\| = \|\nabla \psi\|$, which shows that the speed of the flow increases with the gradient of the streamfunction.

Example 1.3
A bath-plug vortex can be defined as $\mathbf{u} = \begin{pmatrix} \frac{y}{x^2+y^2} \\ \frac{-x}{x^2+y^2} \end{pmatrix}$. It is incompressible since:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} + \frac{(-2y)(-x)}{(x^2+y^2)^2} = 0.$$

So, $\dfrac{\partial\psi}{\partial y} = u = \dfrac{y}{x^2+y^2} = \dfrac{1}{2}\dfrac{2y}{x^2+y^2} \Rightarrow \psi(x,y) = \dfrac{1}{2}\ln\left(x^2+y^2\right) + \alpha(x)$.
Then $v = -\dfrac{\partial\psi}{\partial x} = -\dfrac{x}{x^2+y^2} + \dfrac{\mathrm{d}\alpha}{\mathrm{d}x} \Rightarrow \dfrac{\mathrm{d}\alpha}{\mathrm{d}x} = 0$. So, α is constant and:

$$\psi(x,y) = \frac{1}{2}\ln\left(x^2+y^2\right) \quad \text{(choosing } \alpha = 0\text{)}.$$

This flow is easier to visualise if we use polar coordinates (r,θ) in the (x,y)-plane, so that the streamfunction becomes $\psi = \dfrac{1}{2}\ln\left(x^2+y^2\right) = \ln r$. The streamfunction is independent of θ which shows that the streamlines are circles about the origin.
In polar coordinates, the velocity field is:

$$\mathbf{u} = \nabla\times\left(\psi(r,\theta)\hat{\mathbf{e}}_{\boldsymbol{z}}\right) \quad \Rightarrow u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{\partial\psi}{\partial r} \tag{1.12}$$

$$\Rightarrow u_r = 0, \quad u_\theta = -\frac{1}{r}. \tag{1.13}$$

1.6.2 Axisymmetric flows

We consider:

$$\mathbf{u} = u(r,z)\,\hat{\mathbf{e}}_{\boldsymbol{r}} + w(r,z)\,\hat{\mathbf{e}}_{\boldsymbol{z}}.$$

Examples include flows in a circular pipe or past a sphere. In this case, $\boldsymbol{\Psi}$ is in the $\hat{\mathbf{e}}_{\boldsymbol{\theta}}$ direction and we can define:

$$\boldsymbol{\Psi} = \frac{1}{r}\Psi(r,z)\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}.$$

where $\Psi(r,z)$ is the *Stokes streamfunction* (using Ψ to distinguish from planar streamfunction ψ). Note the prefactor $1/r$ in the definition.
The fluid velocity is given by:

$$\mathbf{u} = \nabla\times\left(\frac{1}{r}\Psi(r,z)\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}\right), \tag{1.14}$$

so, in cylindrical polar coordinates:

$$w(r,z) = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\Psi r\right)\right) = \frac{1}{r}\frac{\partial\Psi}{\partial r} \quad \text{and} \quad u(r,z) = -\frac{\partial}{\partial z}\left(\frac{1}{r}\Psi\right) = -\frac{1}{r}\frac{\partial\Psi}{\partial z}. \tag{1.15}$$

Stokes streamfunctions have properties analogous to planar streamfunctions.

i. Ψ is constant on streamlines

$$\begin{aligned}\mathbf{u}\cdot\nabla\Psi = u\frac{\partial\Psi}{\partial r} + w\frac{\partial\Psi}{\partial z} &= \frac{1}{r}\left(-\frac{\partial\Psi}{\partial z}\frac{\partial\Psi}{\partial r} + \frac{\partial\Psi}{\partial r}\frac{\partial\Psi}{\partial z}\right)\\ &= 0.\end{aligned}$$

Thus, the gradient of Ψ is orthogonal to the velocity field. Moreover:

$$\begin{aligned}d\Psi = 0 &\Rightarrow \frac{\partial\Psi}{\partial r}dr + \frac{\partial\Psi}{\partial z}dz = 0\\ &\Rightarrow rwdr - rudz = 0\\ &\Rightarrow wdr - udz = 0\\ &\Rightarrow \mathbf{dl}\times\mathbf{u} = 0,\end{aligned}$$

implying that Ψ is constant in the direction of the flow.

For axisymmetric flows it is useful to think of *streamtubes*: surface of revolution spanned by all the streamlines through a circle about the axis of symmetry.

ii. Relation between volume flux and streamtubes
The volume flux, or fluid flow, between two streamtubes with $\Psi = \Psi_i$ and $\Psi = \Psi_o$ is:

$$Q = \int_S \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = 2\pi(\Psi_o - \Psi_i). \tag{1.16}$$

Proof

$$\begin{aligned}
Q = \int_S \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S &= \int_S \nabla\times\left(\frac{1}{r}\Psi\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}\right)\cdot\mathbf{n}\,\mathrm{d}S, \quad \text{(definition of }\Psi) \\
&= \oint_{C_o}\frac{1}{r}\Psi\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}\cdot\mathrm{d}\mathbf{l} + \oint_{C_i}\frac{1}{r}\Psi\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}\cdot\mathrm{d}\mathbf{l}, \quad \text{(Stokes theorem)} \\
&= \Psi_o\oint_{C_o}\frac{1}{r}\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}\cdot\mathrm{d}\mathbf{l} + \Psi_i\oint_{C_i}\frac{1}{r}\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}\cdot\mathrm{d}\mathbf{l}, \quad (\Psi\equiv\Psi_{\{o,i\}}\text{ onto }C_{\{o,i\}}) \\
&= \Psi_o\int_0^{2\pi}\mathrm{d}\theta + \Psi_i\int_{2\pi}^{0}\mathrm{d}\theta = 2\pi(\Psi_o-\Psi_i) \quad (\text{Note, }\mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{\mathbf{e}}_{\boldsymbol{r}} + r\mathrm{d}\theta\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}).
\end{aligned}$$

□

Example 1.4
For a uniform flow parallel to the axis, $u = 0$ and $w = U$,

$$\frac{1}{r}\frac{\partial\Psi}{\partial r} = U \quad\text{and}\quad \frac{\partial\Psi}{\partial z} = 0 \Rightarrow \Psi(r) = \frac{1}{2}Ur^2.$$

(We choose the integration constant such that $\Psi = 0$ on the axis, at $r = 0$).

Now consider a streamtube of radius a.

The volume flux

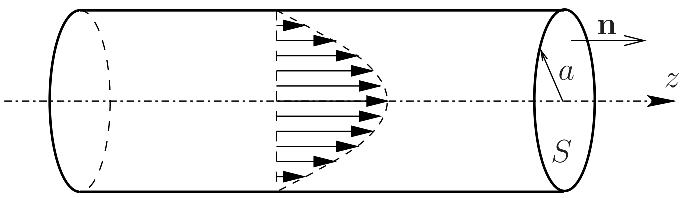
$$Q = \int_S \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = \int_S \mathbf{u}\cdot\hat{\mathbf{e}}_{\boldsymbol{z}}\,\mathrm{d}S = \int_S w\,\mathrm{d}S = U\int_S \mathrm{d}S = \pi U a^2.$$

Also,

$$2\pi(\Psi_o - \Psi_i) = 2\pi(\Psi(a) - \Psi(0)) = 2\pi\left(\frac{1}{2}Ua^2 - 0\right) = \pi U a^2 \quad \text{as required.}$$

Example 1.5
Consider a flow in a long pipe of radius a:

$$u = 0, \quad w = \frac{U}{a^2}(a^2 - r^2) \quad \text{with} \quad \begin{cases} w = 0 \text{ on } r = a, \\ w = U \text{ on } r = 0. \end{cases}$$



$$\frac{\partial\Psi}{\partial z} = -ru = 0 \Rightarrow \Psi \equiv \Psi(r),$$

$$\text{and } \frac{\mathrm{d}\Psi}{\mathrm{d}r} = rw = \frac{U}{a^2}(a^2 r - r^3) \Rightarrow \Psi(r) = \frac{U}{a^2}\left(\frac{a^2r^2}{2} - \frac{r^4}{4}\right) + C.$$

Hence,

$$\Psi(r) = \frac{Ur^2}{4a^2}(2a^2 - r^2) \qquad \text{(choose } C \text{ such that } \Psi(0) = 0).$$

So, $\Psi(0) = 0$ and $\Psi(a) = Ua^2/4$, and the volume flux

$$Q = \int_S \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S = 2\pi\left(\Psi(a) - \Psi(0)\right) = \frac{\pi}{2}\, Ua^2.$$

Indeed,

$$\begin{aligned}\int_S \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S &= \int_0^{2\pi}\int_0^a wr\, \mathrm{d}r\mathrm{d}\theta = \frac{2\pi U}{a^2}\int_0^a (a^2 r - r^3)\, \mathrm{d}r, \quad (\text{since } \mathrm{d}S = r\mathrm{d}\theta\mathrm{d}r)\\ &= \frac{2\pi U}{a^2}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^a = \frac{\pi U a^2}{2}, \quad \text{as required.}\end{aligned}$$

1.7 Viscosity

Let us return to the shear-flow between two parallel plates when one is moved relative to the other, with a constant velocity U: $\mathbf{u} = Uy/d\hat{\boldsymbol{\imath}}$. The motion of the upper plate requires a force, $\mathbf{F}$, that is proportional to its surface area, A. Since this force results from the rate at which the fluid is being deformed, it should be proportional to U/d so that:

$$F = \mu\frac{AU}{d}.$$

Here, μ is a constant that depends only on the properties of the fluid and is called *dynamic viscosity*. It is convenient to define two new quantities: the *stress*, $\sigma = F/A$, and the *shear rate*, $\dot{\gamma} = U/d$, so that the relation becomes:

$$\sigma = \mu\dot{\gamma}. \tag{1.17}$$

Fluids that obey equation (1.17) are referred to as *Newtonian fluids*. In practice, most fluids, including air, water and even sticky fluids like golden syrup, obey this relationship to a high degree of accuracy within a wide range of viscosities. Dynamic viscosities have dimension $M \cdot L^{-1} \cdot T^{-1}$ and their S.I. unit is the Pa·s (Pascal second).
The dynamic viscosity of air is of the order of 10^{-5} Pa·s, that of water is approximately 10^{-3} Pa·s, that of golden syrup around 10^2 Pa·s, while magma in the Earth's interior has dynamic viscosities of around 10^{22} Pa·s. Note that some fluids, such as those containing polymers, do not obey this law. These fluids are out of the scope of the present course.

1.8 Cartesian tensors

The pressure, p, is a scalar quantity and can be represented by a function equal to its value at each point in space.
The fluid velocity, $\mathbf{u}$, is a vector field and can be represented by its coefficients (u_1, u_2, u_3) with respect to a set of Cartesian axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as a column vector:

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

This can be represented more compactly as u_i, where $i = 1, 2, 3$.
The pressure gradient ∇p is another vector quantity and is given by:

$$\nabla p = \begin{pmatrix} \frac{\partial p}{\partial x_1} \\ \\ \frac{\partial p}{\partial x_2} \\ \\ \frac{\partial p}{\partial x_3} \end{pmatrix},$$

or more compactly in suffix notation as $\partial p/\partial x_j$, where $j = 1, 2, 3$.
The scalar product of these two vectors $\mathbf{u} \cdot \nabla p$ is given by:

$$\mathbf{u} \cdot \nabla p = u_1 \frac{\partial p}{\partial x_1} + u_2 \frac{\partial p}{\partial x_2} + u_3 \frac{\partial p}{\partial x_3},$$

and in suffix notation:

$$\mathbf{u} \cdot \nabla p = u_i \frac{\partial p}{\partial x_i}.$$

Here, we are using the Einstein convention: repeated suffixes denote summations.
The basic rules of suffix notation are:

i. A suffix that appears once is called a *free* index. The number of free indices denote the type of quantity in question. A scalar quantity has no free index, a vector has one and an n-th rank tensor has n. Terms that are added or equated must have the same free indices.

ii. If a suffix appears twice, it is called a *dummy* index. Since we sum over dummy indices, the number of pairs of dummy indices does not affect the type of the quantity being described. It is also possible to change the index name without affecting the result. However, it is important not to use a letter already in use as a free index.

As we have already seen, taking the gradient of a scalar produces a vector quantity and so taking the gradient of vector produces a quantity with two associated dimensions, called a second rank tensor. Since there are three components of velocity and three coordinate directions, the velocity gradient $\nabla \mathbf{u}$ has 9 components. It can be represented in the form of a matrix as:

$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix},$$

but it is much more convenient to write this in suffix notation as:

$$\frac{\partial u_i}{\partial x_j},$$

where $(i, j) = (1, 2, 3)^2$. In terms of the matrix representation, i denotes the row and j the column of the entry.

1.8.1 Scalar and vector products

The Kronecker delta δ_{ij} is another example of a second rank tensor:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \tag{1.18}$$

which, in matrix representation, is the identity matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have already seen that calculating the scalar product of two vectors comes down to summing over a pair of indices:

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i.$$

This operation is equivalent to the action of the Kronecker delta on the two vectors **a** and **b** since

$$\mathbf{a} \cdot \mathbf{b} = a_i \delta_{ij} b_j = a_i b_i \qquad \text{since } \delta_{ij} b_j = b_i.$$

The vector product can be represented in Einstein notation by introducing the alternating tensor ϵ_{ijk}:

$$\epsilon_{ijk} = \begin{cases} 1 & ijk = \text{even, i.e., 123, 231 or 312} \\ -1 & ijk = \text{odd, i.e., 132, 213 or 321} \\ 0 & i = j,\ j = k \text{ or } k = i \end{cases}, \tag{1.19}$$

which is a third rank tensor. Since ϵ_{ijk} has 3 free indices, the resulting quantity $\epsilon_{ijk} a_j b_k$ is a vector with index i:

$$c_i = \epsilon_{ijk} a_j b_k,$$

which has the following components:

$$c_1 = a_2 b_3 - a_3 b_2, \qquad c_2 = a_3 b_1 - a_1 b_3, \qquad c_3 = a_1 b_2 - a_2 b_1,$$

and so represents the product of the vectors **a** and **b**.
We can extend these products to tensors. For example, $\mathbf{a} \cdot \mathbf{A} = a_i A_{ij}$ is a vector formed from the scalar product of the vector **a** with the first index of the tensor **A**. Note that, by convention, the dot signifies scalar product of the two neighbouring indices. This product may be performed using the matrix notation by writing **a** as a row vector and then multiplying it by the matrix **A**,

$$\mathbf{a} \cdot \mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Similarly, $\mathbf{A} \cdot \mathbf{a} = A_{ij} a_j$ may be performed in matrix notation as

$$\mathbf{A} \cdot \mathbf{a} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Note that these two scalar products give different results unless **A** is symmetric (i.e., $A_{ij} = A_{ji}$). For example, if we use $K_{ij} = \dfrac{\partial u_i}{\partial x_j}$ to denote the velocity gradient, then:

$$[\mathbf{K} \cdot \mathbf{u}]_j = K_{ji} u_i = u_i K_{ji} = u_i \frac{\partial u_j}{\partial x_i} = [\mathbf{u} \cdot \nabla \mathbf{u}]_j,$$

whereas:

$$[\mathbf{u}\cdot\mathbf{K}]_j = u_i K_{ij} = u_i \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\frac{\partial}{\partial x_j}(u_i u_i) = \frac{1}{2}[\nabla \mathbf{u}^2]_j.$$

We can also form products between two indices on the same tensor. For example

$$\delta_{ij}A_{ij} = A_{ii} = A_{11} + A_{22} + A_{33} = \mathrm{Tr}\mathbf{A},$$

the trace of matrix **A**, which is a scalar quantity.
The scalar product of two second rank tensors **A** and **B** is another second rank tensor $\mathbf{C} = \mathbf{A}\cdot\mathbf{B}$ where

$$C_{ij} = A_{ik}B_{kj}.$$

This is equivalent to matrix multiplication. We can also form the double dot product $\mathbf{A} : \mathbf{B}$, which is the scalar formed by contracting i with j.

$$\mathbf{A} : \mathbf{B} = \delta_{ij}A_{ik}B_{kj} = A_{ik}B_{ki},$$

which is equal to the trace of **C**.
We can also apply cross-products between components of a tensor. For example:

$$c_i = \epsilon_{ijk}A_{jk},$$

is a vector with components:

$$c_1 = A_{23} - A_{32}, \qquad c_2 = A_{31} - A_{13}, \qquad c_3 = A_{12} - A_{21}.$$

Finally, we have the triple product rule:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$

which results from the following relationship between ϵ_{ijk} and δ_{ij}:

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}. \tag{1.20}$$

One way to remember this rule is: second-with-second × third-with-third - alternative pairings.

1.8.2 ∇, $\nabla\cdot$ and $\nabla\times$

We have already seen that we can write the gradient of a scalar and of a vector as

$$\frac{\partial p}{\partial x_j} \quad \text{and} \quad \frac{\partial u_i}{\partial x_j} \quad \text{respectively.}$$

Taking the gradient increases the rank of a tensor by one, from scalar to vector, vector to second rank tensor, etc.
The divergence is obtained by taking the dot product between nabla and one of the indices of the tensor. For a vector **u**:

$$\nabla\cdot\mathbf{u} = \frac{\partial u_i}{\partial x_i}.$$

Note that this is simply the product between the gradient operator, δ_{ij} and the vector **u**. Similarly, we can define the divergence of a tensor A_{ij} as:

$$\frac{\partial}{\partial x_i}A_{ij}.$$

This is a vector quantity. Note that the summation can be over either of the two indices, so we can obtain a second vector using the second index:

$$\frac{\partial}{\partial x_j} A_{ij}.$$

By convention, the notation $\nabla \cdot \mathbf{A}$ is taken to mean summation over the first index (the one closest to the dot). The potential for ambiguities in this formulation means that it is better to stick to suffix notation when dealing with tensors.
Finally we can obtain the curl of a vector or tensor by the action of ϵ_{ijk} on the gradient:

$$[\nabla \times \mathbf{u}]_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}.$$

Example 1.6
Let us consider the conservation equation:

$$\frac{\partial c}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

where c is scalar quantity and $\mathbf{j} = c\mathbf{u}$ is the vector flux of c. We write the divergence term in Cartesian coordinates:

$$\nabla \cdot \mathbf{j} = \frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial j_3}{\partial x_3},$$

so, in suffix notation, we have:

$$\nabla \cdot \mathbf{j} = \frac{\partial j_k}{\partial x_k},$$

where $k = 1, 2, 3$. The flux $\mathbf{j}$ is the product of the scalar c with the vector $\mathbf{u}$, so:

$$\mathbf{j} = (j_1, j_2, j_3) = (cu_1, cu_2, cu_3)\,,$$

which is written in index notation as:

$$j_k = cu_k.$$

Now substituting into the conservation equation we obtain,

$$\frac{\partial c}{\partial t} + \nabla \cdot \mathbf{j} = \frac{\partial c}{\partial t} + \frac{\partial j_k}{\partial x_k} = \frac{\partial c}{\partial t} + \frac{\partial}{\partial x_k}(cu_k) = 0.$$

Finally we can apply the product rule to the differential to give:

$$\frac{\partial}{\partial x_k}(cu_k) = u_k \frac{\partial c}{\partial x_k} + c\frac{\partial u_k}{\partial x_k},$$

so that the equation becomes:

$$\frac{\partial c}{\partial t} + u_k \frac{\partial c}{\partial x_k} + c\frac{\partial u_k}{\partial x_k} = 0. \tag{1.21}$$

In the usual (Gibbs) vector notation, this writes:

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c + c\nabla \cdot \mathbf{u} = 0. \tag{1.22}$$

Example 1.7
Now suppose the quantity c is replaced by a vector $\mathbf{v}$. The equivalent conservation law would be of the form:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

where the flux $\mathbf{J}$ is a second rank tensor. By replacing c with v_i in equation (1.21), we obtain:

$$\frac{\partial v_i}{\partial t} + u_k \frac{\partial v_i}{\partial x_k} + v_i \frac{\partial u_k}{\partial x_k} = 0, \tag{1.23}$$

which is clear and unambiguous. Written out in full, this represents the equations:

$$\begin{aligned}
\frac{\partial v_1}{\partial t} + \left(u_1 \frac{\partial v_1}{\partial x_1} + u_2 \frac{\partial v_1}{\partial x_2} + u_3 \frac{\partial v_1}{\partial x_3} \right) + v_1 \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) = 0, \\
\frac{\partial v_2}{\partial t} + \left(u_1 \frac{\partial v_2}{\partial x_1} + u_2 \frac{\partial v_2}{\partial x_2} + u_3 \frac{\partial v_2}{\partial x_3} \right) + v_2 \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) = 0, \\
\frac{\partial v_3}{\partial t} + \left(u_1 \frac{\partial v_3}{\partial x_1} + u_2 \frac{\partial v_3}{\partial x_2} + u_3 \frac{\partial v_3}{\partial x_3} \right) + v_3 \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) = 0.
\end{aligned}$$

In this case, $J_{ki} = u_k v_i$ and $[\nabla \cdot \mathbf{J}]_i = \frac{\partial}{\partial x_k}(u_k v_i)$.

1.9 Strain-rate and vorticity tensors

Let us now examine the velocity gradient $\partial u_i / \partial x_j$. For an incompressible flow, $\nabla \cdot \mathbf{u} = 0$ and so this tensor has zero trace. There are still 8 remaining components. A useful simplification is to decompose the velocity gradient into the sum of a symmetric and an antisymmetric tensor:

$$\frac{\partial u_i}{\partial x_j} = E_{ij} + \Omega_{ij}, \quad \text{where} \quad E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{and} \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \tag{1.24}$$

It is easily verified that $E_{ij} = E_{ji}$ and $\Omega_{ij} = -\Omega_{ji}$.
The symmetric tensor, $\mathbf{E}$, is called the *strain-rate tensor* and the antisymmetric tensor, $\boldsymbol{\Omega}$, is called the *vorticity tensor*.
Recall that the vorticity $\omega = \nabla \times \mathbf{u}$. In suffix notation:

$$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}. \tag{1.25}$$

Multiplying this equation by ϵ_{ilm} and using the triple product rule, we obtain:

$$\epsilon_{ilm}\omega_i = \epsilon_{ijk}\epsilon_{ilm} \frac{\partial u_k}{\partial x_j} = \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \right) \frac{\partial u_k}{\partial x_j} = \frac{\partial u_m}{\partial x_l} - \frac{\partial u_l}{\partial x_m} = 2\Omega_{ml},$$

so that:

$$\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k. \tag{1.26}$$

This result is clear if we write $\boldsymbol{\Omega}$ in matrix notation:

$$\boldsymbol{\Omega} = \frac{1}{2} \begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} & \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} & 0 & \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} & \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Example 1.8
Consider the simple shear flow:

$$\mathbf{u} = \begin{pmatrix} \dot{\gamma} y \\ 0 \\ 0 \end{pmatrix},$$

using (x, y, z) rather than (x_1, x_2, x_3) for Cartesian coordinates. The gradient of velocity writes:

$$\nabla \mathbf{u} = \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

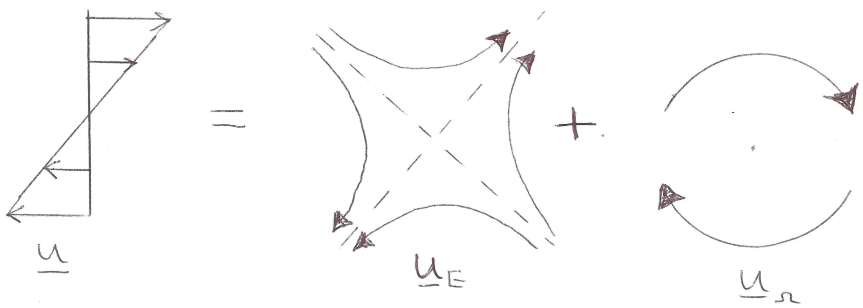
Therefore, the strain-rate and vorticity tensors for this flow are:

$$\mathbf{E} = \frac{1}{2} \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{\Omega} = \frac{1}{2} \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ -\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us now reconstrust the linear flows corresponding to the symmetric and antisymmetric parts of the velocity gradient tensor, $\mathbf{u_E} = \mathbf{E} \cdot \mathbf{x}$ and $\mathbf{u_\Omega} = \mathbf{\Omega} \cdot \mathbf{x}$:

$$\mathbf{u_E} = \begin{pmatrix} \frac{\dot{\gamma}}{2} y \\ \\ \frac{\dot{\gamma}}{2} x \\ \\ 0 \end{pmatrix}, \qquad \mathbf{u_\Omega} = \begin{pmatrix} \frac{\dot{\gamma}}{2} y \\ \\ -\frac{\dot{\gamma}}{2} x \\ \\ 0 \end{pmatrix}.$$

The streamlines of $\mathbf{u_E}$ are given by $x^2 - y^2 = \text{cst}$, while the streamlines of $\mathbf{u_\Omega}$ are circles $x^2 + y^2 = \text{cst}$.



Thus $\mathbf{u_E}$ is a hyperbolic flow with extension along the line $y = x$ (and contraction along $y = -x$), while $\mathbf{u_\Omega}$ is a clockwise rotation.

Furthermore, since $\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k$, it follows that:

$$u_{\Omega i} = \Omega_{ij} x_j = -\frac{1}{2}\epsilon_{ijk}\omega_k x_j = \frac{1}{2}\epsilon_{ikj}\omega_k x_j = \frac{1}{2}[\omega \times \mathbf{x}]_i,$$

i.e., $\mathbf{u_\Omega}$ is a solid body rotation at an angular velocity of $\omega/2$. It is the strain-rate $\mathbf{E}$ that produces a deformation of the fluid.

1.10 Polar tensors

So far we have only discussed tensors with respect to fixed Cartesian coordinates, however, in many problems it is often more convenient to work in polar coordinates. Tensor calculus in polar coordinates tends to be more complicated because of the rotation of the base vectors. For example in cylindrical polar coordinates (r, θ, z) the gradient of the velocity (u_r, u_θ, u_z) is given by

$$\begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{1}{r}\frac{\partial u_z}{\partial \theta} & \frac{\partial u_z}{\partial z} \end{pmatrix}.$$

Note that when we take the trace we recover the formula for $\nabla \cdot \mathbf{u}$ in cylindrical polar coordinates

$$\nabla \cdot \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}.$$

As this is not a course in tensor calculus we shall not attempt to derive these formulae. Instead we will refer to a formula sheet when using cylindrical or spherical polar coordinates.