Deciphering Turbulent Music

MATH5000M Dissertation

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Abstract

Our aim of this study is to demonstrate there may exists some melodic patterns in some Western classical pieces that display the mathematical structure function of fluid turbulence corresponding to Kolmogorv's theory of turbulence and that found in by Aragón et al. in some of van Gogh's paintings. We begin with providing a brief definition of a turbulent flow in relation to it's Reynold number and list out some common characteristics. Furthermore we will describe turbulence as a subset of chaos theory. We define the structure functions found in Kolmogorov's theory of turbulence, therefore providing a mathematical definition of turbulence, and derive the equations to determine the scaling functions. Finally we will discuss the concept of the energy cascade in a turbulent flow.
The aim of our study is to replicate method used by Aragón et al. to show patterns of luminance exhibited the same mathematical structure function onto an alternative form of art, music. Therefore we will provide a brief overview of the methodology and results achieved in the study conducted by Aragón et al.
We begin our study by outlining how we would determine our variables, notes and note values, and then suggest some methods to analyse the data collected. We propose a hypothesis that the linear relation between the statistical moments of velocity difference and R values of a turbulent flow will be consistent to that of the relation between the statistical moments of pitch difference and time difference found in a piece of music.
Furthermore we will provide a definition of musical turbulence as defined by the melody. We will discuss the concept of musical expectation and the common melodic elements, which we will refer to when defining if a melodic line is turbulent.

Special thank you to my supervisor, Dr. Cédric Beaume for his enthusiasm for this topic, and especially for all his support, encouragement and patience throughout.

Contents

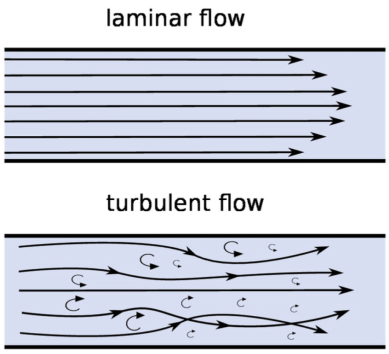


Figure 1: Illustration demonstrating the difference between a laminar and turbulent flow between two horizontal boundaries. Laminar flow is represented by parallel arrows to denote it's direction of flow, whereas turbulent flow consists of arrows which 'twists and turns' and are no longer parallel with one another.

1 What is Turbulence?

1.1 Laminar to Turbulent flow

We begin by giving a brief definition of a laminar and turbulent flow from simple observations.

1.1.1 Laminar Flow

Laminar flow is described as a flow that looks ordered and smooth [7, p. 8]. It is arguably the simplest state a flow can be! It can be visualised as each molecules in the flow to be travelling in straight parallel lines [2, Laminar] as represented in part (a) of figure 1 [15].

1.1.2 Turbulent Flow

Turbulent flow, on the other hand, is described to be the opposite to that of laminar flow, often being described to be chaotic and irregular, as represented by the 'twists and turns' of the flow in part (b) of figure 1. Additionally, in the turbulent flow case, the individual flows of each molecule are characterised to be unpredictable and random [26, Turbulence]. Further descriptions include the flows being disordered, with exchange of momentums between molecules, due to collisions occurring [2, Turbulence]. We note that the use of 'random' and 'chaotic' to describe a turbulent flow is in relation to it's definition in the literacy as oppose to that of the precise mathematical definition[1].

Narashmha agues that "the whole universe is full of turbulence"[24, p. 630], and therefore we can find examples in many fields of studies. Examples of turbulent flow can be found in most flows in nature and engineering [30, p. 1] on both small and large scales. For example turbulence in wind and water flow regulates the currents under the surface of the oceans [30, p. 1] which then in changes of weather [9, p. 300]. Examples can also be found in astrophysics, for example in turbulent interstellar gas clouds - for instance the Great Nebula in Orian is often interpreted compared to a large-scale turbulent puff of smoke [9, p. 300]. Some examples given in relation to engineering are of the following, the aerodynamic drag against a plane's wing or a car [10, p. 16] or in structural engineering when turbulent winds are taken into consideration in the structure of high-rise building [10, p. 16].

[1] This is explained in section 1.4

1.1.3 Viscosity

The **viscosity** μ can be thought of as the frictional force within a fluid, simply put as the measure of'thickness' of the fluid (i.e. the higher the viscosity, the 'thicker' the fluid and therefore poses more resistance). Example of a fluid with low viscosity is water, whereas one with high viscosity might be honey. Formally it is defined as"a measure of the resistance to flow that a fluid offers when it is subjected to shear stress"[2, Viscocity].
Further discussion on viscosity as a parameter and it's relation to categorising the behaviour of a flow is examined in subsequent sections.

Davidson argues that there are two types of transitions: (1) a transition in which the chaotic, turbulent flow first appear in patches throughout the overall flow or (2) the transition into turbulent flow develops uniformly through the overall flow [10, p. 10]. In the first transition, for a liquid with low viscosity, each individual patches of turbulence will grow until they merge to one another, and a turbulence flow is fully developed [10, p. 10]. Examples of this transition is evident in flows between boundary layers, for example a flow in a pipe. It is the collisions against these boundaries layers that creates the initial instability in some areas of the flow. The second type describes a transition which begins with a small instability in the mean flow, and from this each individual turbulent flow "break up into even more complex chaotic motion", as Davidson describes [10, p. 11]. Example of this transition is observed in the flume from a cigarette.

1.2 Defining Turbulence

In this section we will attempt to provide a more quantitative definition of turbulent flow in relation with Reynold numbers.
We begin with Navier-Stokes equation, as shown in equation 1.2.1, derived from considering an Eulerian representation of a fluid, that is homogeneous and incompressible, such that pressure, p($\mathbf{x}$,t), velocity, u($\mathbf{x}$,t), and density, $\rho(\mathbf{x}, t)$ at point $\mathbf{x} = (x_1, x_2, x_3)$. Additionally, it assumes that the mass conservation equation is satisfied [1, p. 3].

$$\rho\frac{Du}{Dt} = -\nabla p + \mu\nabla^2 u \tag{1.2.1}$$

Using the non-dimensionalising method proposed by Beaume [6, p. 29], we are able to extract the Reynold number from the Navier-Stokes equation. From this method we are able to determine the behaviour of a flow as a result of the Reynold number.

The initial step is to substitute the velocity, $\mathbf{u}$, x-position $\mathbf{x}$ and time, t with a characteristic unit, U, D, $\frac{D}{U}$ and their respected dimensionless vector quantity, $\mathbf{u*}$, $\mathbf{x*}$ and t*. Additionally we will take $\mu\frac{U}{D}$ as the characteristic unit for pressure.

$$\mathbf{u} = U\mathbf{u}*, \mathbf{x} = D\mathbf{x}*, t = \frac{D}{U}t*, p = \mu\frac{U}{D}p* \tag{1.2.2}$$

Substituting equation (2) into equation (1) and also dividing by $\mu\frac{U}{D^2}$ gives:

$$\overbrace{\underbrace{\frac{\rho UD}{\mu}}_{\text{Reynold Number, Re}} \qquad \frac{D\mathbf{u}^*}{Dt^*} = -\nabla^* p^* + \nabla^{*2}\mathbf{u}^*}^{\text{Navier-Stokes Equation}} \tag{1.2.3}$$

It is noted that the conservation of mass is preserved through the non-dimensionalising.

$$\nabla * \cdot \mathbf{u}* = 0$$

The key advantage of the non-dimensionalisation method is that the revised Navier-stokes equation, given by equation 1.2.3, is now universal and only dependent on the Reynolds number. To reiterate a previous statement, regardless whether it is a large scale or small scale flow, the units of measurements are now irrelevant in characterising the flow, one needs only calculate the Reynold number for that the specified scenario to determine if the flow is laminar or turbulent.

$$\frac{\rho U D}{\mu} \overbrace{\overbrace{\frac{D\mathbf{u}^*}{Dt^*}} = \overbrace{-\nabla^* p^*} + \overbrace{nabla^{*2}\mathbf{u}^*}}^{\text{Dimensionless}}$$

1.2.1 Reynold Number

A formal definition of the **Reynold number** can be found directly from the Navier-Stokes equation, equation 1.2.1: $\rho\frac{Du}{Dt} = -\nabla p + \mu\nabla^2 u$
Again we follow closely the method given by Beaume [6, p. 29]
By considering a flow past a boundary, we can give an estimate of either sides.

$$\rho\frac{D\mathbf{u}}{Dt} = \mid \rho\mathbf{u}\cdot\nabla\mathbf{u} \mid \sim \frac{\rho U^2}{D}$$

$$\mid \mu\nabla^2 u \mid \sim \frac{\mu D}{D^2}$$

As we can re-write $p = \mu\frac{U}{D}p* = constant$, then we find $-\nabla^* p = 0$, and therefore we can ignore this term. By taking a ratio of the two estimation above, we reach the Reynold number

$$\frac{\mid \rho\mathbf{u}\cdot\nabla\mathbf{u} \mid}{\mid \mu\nabla^2 u \mid} = \frac{\rho U^2}{D}x\frac{D^2}{\mu D} = \frac{\rho U D}{\mu} = Re \tag{1.2.4}$$

From this derivation of the Reynold number, we can define it as a ratio between the inertial force in the fluid and it's viscosity.

From this construction of the Reynold number, we can comment directly on the relation of inertial force against viscosity and it's effect on the behaviour of the flow. Fluids with high viscosity, and therefore a low Reynold number, tend to have a more stable structure, and therefore will often exhibit laminar flows.
When the viscosity is decreased to a point where the overall flow equilibrium becomes unstable, this increases the individual molecule flow velocity, namely each molecule has higher kinetic energy,as the resistant force acting on the molecule also decreases by definition. The increase in kinetic energy, increases the possibility of collisions, against molecules and boundary surfaces, resulting in a flow that becomes irregular and chaotic, thus a turbulent flow emerges [10, p. 4]. From equation 1.2.4, we can identify that a high Reynold number is required for a turbulent flow to emerge.
This can be summarised as below.

$$\text{Behaviour of a flow} = \begin{cases} \text{laminar} & \text{as } Re \to 0 \\ \text{turbulent} & \text{as } Re \to \infty \end{cases}$$

The following sections discuss some experiments to reiterate this relation between the Reynold number and classification of flow.

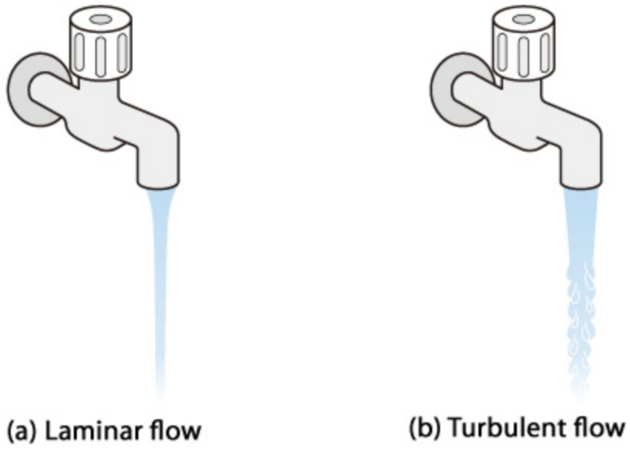


Figure 2: Image illustrating the difference between a laminar flow and a turbulent flow in a tap. The laminar flow (a) is represented as a streamline flow, whereas the in the turbulent flow, the flow's boundary has been broken and we observes 'wave'-like flows

1.2.2 Experiments

An example which the demonstrates the positive correlation between the Reynold number and chaos in a flow is looking at a flow from a tap. As shown in figure 2 [31] we can see that in the 'laminar' flow the stream of water from the tap is smooth, if we were to open the tap further (i.e. increasing the entry, D in the Reynold number's equation) the flow becomes turbulent. In this scenario we can see that the vortices begin to break through the surface, appearing as 'waves' and collect air bubbles resulting in the flow to change from transparent to cloudy and opaque.

A simple experiment that demonstrates the transition between laminar and turbulent flow in relations to increasing Reynold numbers is by injecting dye into a fluid. As shown in figure 3 [32, p. 61], we can observe three main stages [7, p. 10]:

1. laminar flow $0 \leq Re \lesssim 1000$
2. transition, $1000 \lesssim Re \lesssim 4000$
3. turbulent flow, $4000 \lesssim Re$

In the very first stage (the top image in figure 3), the injected dye is travelling in a straight, parallel line to the boundary surface, therefore exhibiting a laminar flow. In the next stage below, we can see that the dye is not longer 'smooth', whilst not being described as fully chaotic, does exhibits some disordered nature - this stage is the transition. The next two stages are examples of turbulent flow, in which the motion of the dye is observed to be unstructured and chaotic.

An example which examines the development of vortices in relation to the Reynold number is shown in figure 4 [10, p. 8]. We can see for a very low Reynold numbers, $Re < 1$ (part (a)), the overall flow maintains a laminar flow and simply 'bends' around the obstacle.At $5 < Re < 200$, the initial transition from laminar flow to turbulent can be seen, it is noted that within this transitional stage that the flow remains periodic [10, p. 9]. At $5 < Re < 40$ (part (b)), initial vortices appear isolated behind the obstacle, and at $100 < Re < 200$ (part (c)) these appearances of vortices are no longer isolated and are described to "peel off from the rear of the [obstacle] in a regular periodic manner" [10, p. 9]. Davidson notes that it is only for high Reynold numbers, for example $Re \sim 10^6$ (part (e)) [10, p. 9], that the turbulent flow manages to break out of the

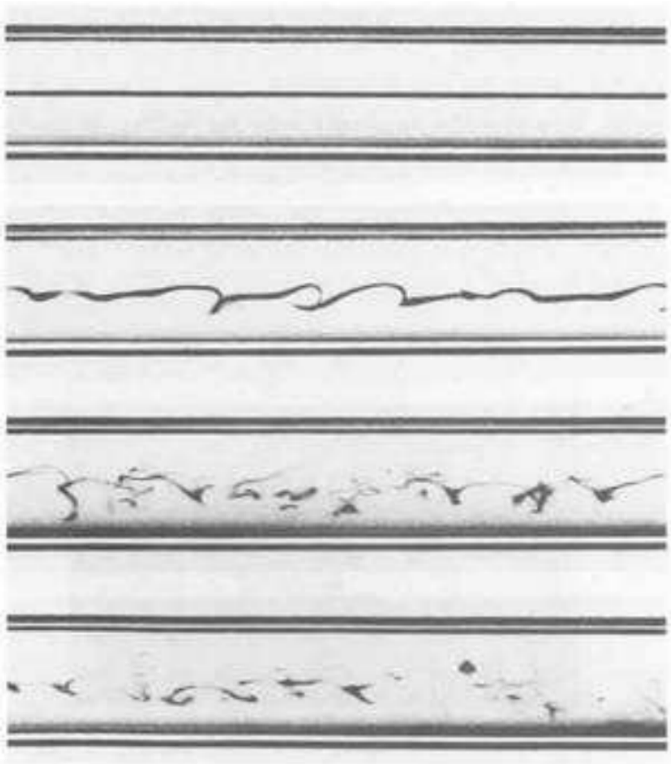


Figure 3: Image depicting four different stages of flow when an dye is injected. The laminar flow is depicted at the top, with the dye travelling parallel to that of the boundaries, and as we move down the stages, the flow becomes more turbulent.

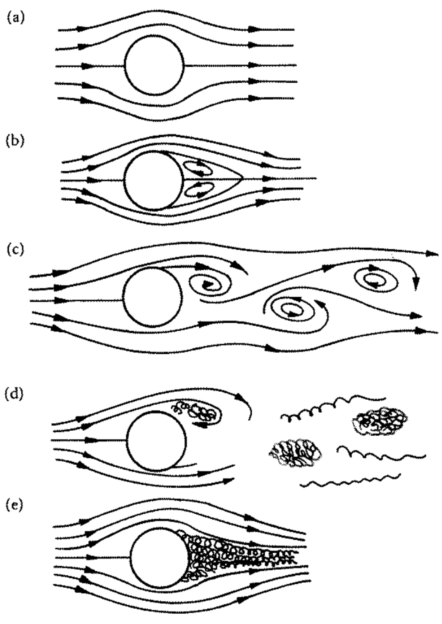


Figure 4: Image depicting the formation of eddies. We see in (a) that the flow remains laminar, and as we move down from (b) to (e), as the Reynold number increases, the flow gradually becomes more turbulent and eddies are formed.

vortices and a full turbulent wake is achieved.

In summary, in this section we have defined a behaviour of a flow through using the non-dimensionalisation method on the Navier-Stokes equation, and additionally derived a definition of the Reynolds number, providing some examples of values for each stage.

1.3 Characteristics of a turbulent flow

Whilst we have attempted to give a quantitative definition of turbulence, with respect to the Reynold numbers, it is still widely acknowledged that there is no universally accepted formal definition of turbulence. This is due to a number of reasons, one of which is whilst we have labelled a turbulent flow to be random and chaotic, there are arguably infinitely many 'levels' of 'randomness' and 'chaotic-ness', therefore should turbulent flow have in itself have subcategories? Davidson proposes a solution by avoiding a formal definition all together [10, p. 11] and instead group all flows described as turbulent and from this we can identify common characteristics. Tennekes and Lumley provides seven characteristics [30, p. 1-3]:

1. Irregularity
2. Diffusivity
3. Large Reynold number
4. Three-dimensional vorticity fluctuations
5. Dissipation
6. Continuum
7. Turbulent flows are flows

The following subsections will explain and discuss each characteristic in turn.

1.3.1 Irregularity

A noticeable characteristic of turbulent flow is the irregularity or 'randomness' aspect of turbulent flow [30, p. 1]. For example, if we examine a cigarette plume, we can observe it's turbulent flow through the unpredictable and unrepeated twist and turns of the smoke. Therefore the flow is defined as not periodic [10, p. 16].

Tennekes and Lumley argue that it is due to this characteristic that the study into turbulent flow relies on statistical methods and not deterministic ones [30, p. 1]. The statistical methods can be described to be predicting the "history of a random (velocity) field whose behaviour at each point in space-time is given by [the Newton's second law and the Continuity equation]"[9, p. 303]. Using these statistical methods, we are able to give statistical predictions on averaged functions, for example to average velocity distribution across a set time and probability density of occurrences of a chosen velocity throughout a flow [9, p. 304]. Furthermore Davidson suggests, using the cigarette plume example, we can use also statistical methods to approximate the "time-averaged concentration of smoke at a particular location, [or the] time-averaged width of the plume at averaged plume at a particular height"[10, p. 17].
It must be emphasized that it is not the purpose of the statistical method to 'piece together' the journey of a molecule through a flow. Instead, they are used to provide predictions for the overall flow as an averaged function[9, p. 303].

In summary, it is this reliance on using statistical methods to estimate solutions of the Navier-Stokes equation, that distinguishes turbulent from laminar flow analytically.

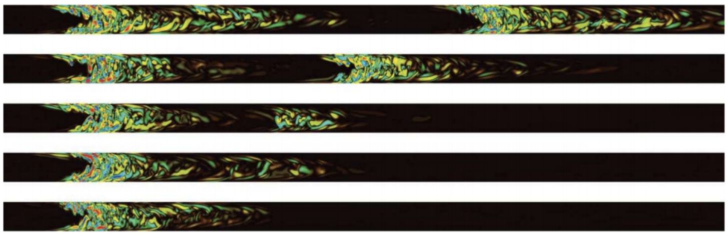


Figure 5: Image describing the spreading of turbulence. In the very bottom image the turbulent flow is contained, and as we move up we can see it beginning to break off until another turbulent flow is formed.

1.3.2 Diffusitivity

Diffusivity is the concept of natural interactions between molecules, for example exchange in momentum and kinetic energy. Tennekes and Lumley argue that if a flow is considered to be random, but does not "exhibit spreading of velocity fluctuations through the surround fluid" [30, p. 2], it cannot be categorized as turbulent. Therefore giving another differentiation between laminar and turbulent flow.

This characteristic can be seen in the transition period between a laminar and turbulent flow. Figure 5 [20, p. 193] shows a visualisation of the interaction between the vortices in the turbulent flow interacting with the laminar flow. There is an initial separation, visualised a the 'puff' of smoke's tail being drawn out, before the 'flow' in the tail interacts with the laminar flow and results in it's own turbulent vortex.

1.3.3 Large Reynold number

A simple defining characteristic of turbulent flow, already discussed, is the differing range of Reynold numbers for laminar and turbulent flows. In the most fundamental of relations, a large Reynold number often equates to a turbulent flow, whereas a low number will lean itself to a laminar flow.

Whilst this above statement is generally true, it is only applicable theoretical studies. Reynolds observed that with some adjustments, there is the ability to minimize disturbances, and therefore delay the transition to a turbulent flow, known as "relaminarized"[24, p. 635]. For example in an experiment, Reynolds remarked that the 'inlet disturbances were minimized' [10, p. 10], he was able to suppress the transition from $Re \sim 2000$ to $Re \sim 13000$ [10, p. 10]. From this experiment we can conclude the difficulty in categorically defining a turbulent flow.

1.3.4 Three-dimensional vorticity fluctuations

Another defining characteristic of turbulent flow is it's vorticity behaviour. In contrast, by definition, a laminar flow will not contain any vortices in it's flow. Tennekes and Lumley suggest that in order for a flow to be defined as turbulent, it needs to be three dimensional and that it is highly fluctuating, and therefore rotational [30, p. 2]. The vortices in a turbulent flow can initially appear as "islands of chaos in a laminar sea"[24, p.633] described as small scale, as the flow becomes more turbulent, the vortices will break out of these 'island' and grow until a flow is full of vortices, described as large scale.

This will be further explored in terms of energy exchange in later sections, specifically in section 1.4 *Energy Cascade*.

1.3.5 Dissipation

Following from the requirement that a turbulent flow needs to be diffusive, dissipation explores this further by focussing on loss of energy in the flow, namely kinetic energy. Whilst the focus is on energy loss, it is noted that the overall system requires exchanges in energy in both directions, gaining and losing energy. There is an additional requirement, the energy lost through dissipation of kinetic energy should be balanced with some supply of energy, if not the turbulent flow will decay quickly [30, p. 3]. Corrsin demonstrates this with the use Newton's law of conservation of energy to derive an equation demonstrating the change in 'turbulent energy' [9, p.305].

$$\frac{\partial}{\partial t}\varepsilon(k,t) = \underbrace{-2\mu k^2\varepsilon(k,t)}_{\text{(a)}} + \underbrace{\Gamma(k,t)}_{\text{(b)}} \tag{1.3.1}$$

The term (a) denotes the rate of loss dissipation in kinetic energy, in relation the the wave number, k and (b) denotes some rate of gain of energy [9, p. 306].

The mean energy dissipation rate per mean unit mass can be define as follows

$$\varepsilon = 2\mu S_{ij}S_{ij} \tag{1.3.2}$$

where $S_{ij} = \frac{\mu}{\eta}$ is defined as the rate of strain associated with the smallest eddies, and η is defined as one of *Kolmogorov's microscale* [2] [10, p. 18]. It is this rate of dissipation that differs turbulent flow with that of laminar flows.

1.3.6 Continuum

Tennekes and Lumley indicate that "turbulence is a continuum phenomenon" [30, p. 3], and argue that they are characterised by equations of fluid motions, namely the Navier-Stoke equations.

$$\rho\frac{Du}{Dt} = -\nabla p + \overbrace{\mu\nabla^2 u}^{\text{Nonlinear in u}}$$

It is from the non-linear term that stems the difficulty in finding general solutions to the Navier-Stokes equation, and therefore is the root of the complex, chaotic behaviour found in turbulent flow [10, p. 34]. It is noted that whilst the flow is described as random observationally, as it's behaviour can be described using the Navier-Stokes equation, it must be deterministic process [10, p. 84], as oppose to stochastic.

1.3.7 Turbulent flows are flows

This characteristic is the re-emphasis that "[t]urbulence is not a feature of fluids but of fluid flows"[30, p. 3], and therefore the nature of turbulence is the same in liquids and gas. This is because the molecular properties in the fluid do not play a significant factor in the transition to turbulence [30, p. 3]. Furthermore, as previously mentioned, due to the presence of a non-linear term in the Navier-Stokes equations, there is a emphasizes on the importance of the initial boundary conditions in relation to an individual flow's pattern [30, p. 3]. Therefore as each flow is unique and it's flow are individually different, this reiterates the difficulty in providing a formal definition for a Turbulent flow.

The combination of the notable influence of initial condition and the unpredictability in the flow's motion opens the questions of whether we describe turbulence as a subset of chaos theory or if it is a stochastic process.

[2] The Kolmogorov's microscale will be further explained in subsequent sections

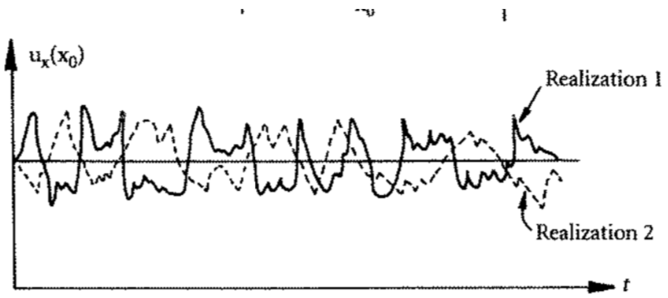


Figure 6: A graph depicting two realization of two scenarios which began with the same initial condition. We note that the graph plots the change in velocity in the x direction against time. From this figure we can see how the velocity at position x can vary drastically between realisation even if they identical experiments.

1.4 Turbulence: Subset of Chaos Theory or A Stochastic Process?

Davidson's summary of the common characteristics found in turbulent flow combines the two propositions:

> (1)the velocity field fluctuates randomly in time and is highly disordered in space, exhibiting a wide range of length scales;
> (2)the velocity field is unpredictable in the sense that a minute change to the initial conditions will produce a large change to the subsequent motion.[10, p. 12-13].

The first of which could be argued to exert a stochastic-like process, whereas the second characteristic is inclining towards chaos theory.

As discussed before, although the Navier-Stokes equation is deterministic, the velocity field, $\mathbf{u}(t)$ of a turbulent flow appears random [10, p. 84], and therefore requires statistical methods for analysis. Lorenz defines randomness to be a sequence which the probability of all outcomes are equal, essentially all outcomes are equally likely to happen [21, p. 6]. An example of a 'random sequence' is that of a coin toss, in which there is equal probability of the coin landing on heads or tails. A Dictionary of Physics defines a stochastic process as the following:

> Any process in which there is a random element. [...] In a **time-dependent stochastic process**, a variable that changes with time does so in such a way that there is no correlation between different time intervals. [...] It is necessary to use statistical methods and the theory of probability to analyse stochastic processes[...]. [2, Stochastic process]

Additionally, we can observe from figure 6 [10, p. 13] that the changes in velocity for each realization does corresponds to that described in a stochastic process. Therefore it can be argued that if a turbulent flow or a velocity field is described to be 'random', for example by Corrsin [9, p. 303], it is because it is believed to lend itself towards a stochastic process, as oppose to a subset of chaos theory.

On the other hand, Davidson argues that use of the term "chaotic" would be more appropriate when describing a turbulent field, implying that turbulence is a subset of chaos theory (specifically 'weak turbulence', the section which is not involved in the cascade [33, p.633]. In a very simplistic definition, chaos theory is the study in which small changes in initial conditions can result in a significant change in system behaviour[33, p. 633], known as the *Butterfly Effect* [11, p. 10]. Similarly to turbulence, chaos also does not have a universally accepted definition. However Lorentz argues that the term chaos is used to described systems in which "the variations

are *not random but look random*"[21, p. 4] and additionally governed by precise mathematical equations, in other words a chaotic behaviour is deterministic whilst posing some randomness, which results in it not 'looking determinstic"[21, p. 8]. much like the Navier-Stokes equation to describe the random motion within a turbulent flows.
Furthermore another argument for turbulence to be a subset of chaos theory, is it's sensitivity to initial condition, arguable a hallmark of chaos theory [10, p. 53]. For example, if we return to figure 6, we can see the two "nominally identical experiments", results in two very different descriptions of the velocity fields, labelled as realizations 1 and 2 [10, p. 13]. Whilst it can be argued that this supports labelling turbulence as a stochastic process, Davidson suggests that it is instead due to the highly sensitivity of the initial condition which affects the outcome so drastically. In the same concept as chaos, the minute inevitable changes in initial condition is the explanation to the 'stochastic-like' appearance of turbulence [10, p. 14]. Another aspect of turbulence is that it is non-period, consistent with the definition of (limited) chaos[21, p. 21].

We have now explained the dependence on initial conditions and the randomness aspect which characterises turbulence in terms of chaos. So why is it not universally accepted that turbulence is a subset of chaos theory?
The predicament we are faced with is that a a chaotic behaviour does not necessarily guarantee a turbulence [10, p. 53].. An example given by Davidson is that, "[s]imple non-random Eulerian velocity fields ([also known as] laminar flows) can cause fluid particles to follow complex trajectories which have certain chaotic properties."[10, p. 54][3]. Another problem we face is that whilst turbulence has already been defined to occur at high Reynold numbers, chaos can develop at low frequencies, and it is only at high frequencies chaos in which the energy cascade process begins [24, p. 644] [4]. A final complication is what Davidson calls the 'arrow of time' [10, p. 59]. This is concept that explains why some systems cannot be reversed. For turbulence, the 'arrow of time' does exist as a result of the Second Law of Thermodynamics and that 'entropy within a system must always increase'. Whilst in chaos theory the 'arrow of time' does not necessarily always exists, an example being our universe was created by the Big Bang, beginning as a black hole is hypothesized to vanishes back into black holes[12, p. 176-177].

1.5 Structure functions found in Kolmogorov's Theory of Turbulence

In his 1941 paper, Kolmogorov famously derives the n-order structure function.

1.5.1 n-th order structure function

The **n-th order structure function** is defined by equation 1.5.1. For a distance $R = \|R\|$, the change in velocity, Δv between two points of distance R, is dependent on the distance R to the power of $\frac{n}{3}$ [4, p. 276].

$$S_n(R) = \langle [\Delta v]^n \rangle = \beta \varepsilon^{\frac{n}{3}} R^{\frac{n}{3}} \tag{1.5.1a}$$

$$S_n(R) \propto R^{\xi_n} \tag{1.5.1b}$$

where $\xi_n = \frac{n}{3}$, and $\Delta v = v(r+R) - v(r)$

The significance of this law is the ability to demonstrate that the change in velocity in a flow that is described to be unpredictable and random is proportional to a simple power function (as shown in equation 1.5.1 (b))

[3]Further information on this can be found in The Essence of Chaos p 161 in which Lorenz discusses nonlinearity and chaos

[4]Further examples can be found in Narashimha Order and Chaos in Fluid Flows

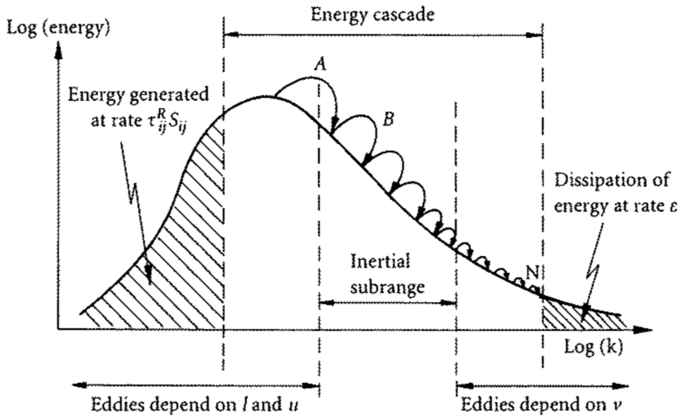


Figure 7: Image illustrating the Energy Cascade, demonstrating the relation between the natural logarithm of the wave number, k and the natural logarithm of the energy, in particular the kinetic energy. The energy cascade is divide into three sections, the first in which energy is generated, at small wave numbers, the cascade and finally the section at which energy is lost through dissipation as the wave number increases.

Furthermore from this law, we can see by taking the logarithmic function of both sides, that this simply becomes an equation for a straight line, in which the gradient is ξ_n. In other words, we can determine a linear relation between the logarithm of the change in velocity and the logarithm of the distance between the two points.

$$\langle [\Delta v]^n \rangle = \beta \varepsilon^{\frac{n}{3}} R^{\frac{n}{3}} \tag{1.5.2}$$
$$= C_1 R^{\xi_n} \tag{1.5.3}$$

$$\log(\langle [\Delta v]^n \rangle) = \log(C_1 R^{\xi_n}) \tag{1.5.4}$$
$$= \log(R^{\xi_n}) + \log(C_1) \tag{1.5.5}$$
$$= \xi_n \log(R) + C_2 \tag{1.5.6}$$

where $\xi_n = \frac{n}{3}$ and C_1 and C_2 are constants.

A development of the n-th order structure function is the **Two-thirds law**, also known as the variance.

$$\langle [\Delta v]^2 \rangle = \beta \varepsilon^{\frac{2}{3}} r^{\frac{2}{3}} \tag{1.5.7}$$

We can rewrite equation 1.5.7 in terms of kinetic energy, $E(k)$ and it's wave number, k.

$$E(k) = \alpha \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \tag{1.5.8}$$

Equation 1.5.8 is known as the **Five-thirds law**.

1.5.2 Energy Cascade

Figure 7 [10, p. 205] demonstrates what is known as the **Energy Cascade**. It depicts the change in kinetic energy in relation to the wave number. In general we can see there is an initial peak of energy, followed decrease as the wave number increases (or the sizes of the vortices decreases), and finally a sudden decay at a set wave number.
An illustrated depiction of this energy cascade, in terms of vortices is show in figure 8 [10, p. 19]. We can interpret similar to a family tree, in which the two 'blobs' or vortices at the very

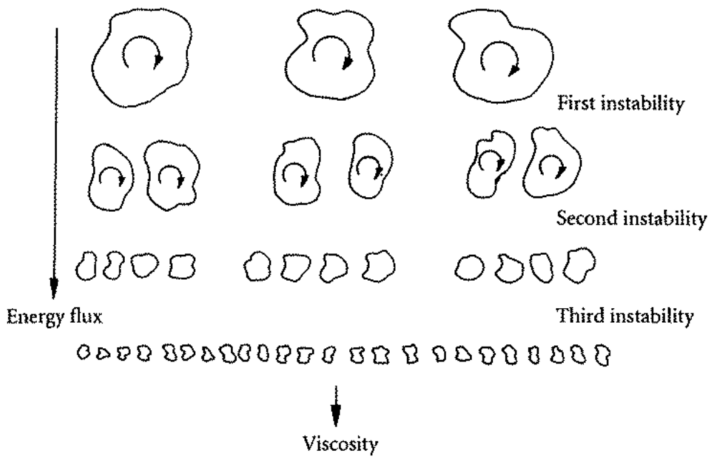


Figure 8: Image illustrating the division of eddies into smaller ones as time increases down the image.

top, each splits into smaller vortices, and in this example splits into two more and this continue until the vortices become so small they fully dissipate. From the figure we can see three zones: (a) Forcing range, (b) Inertial range and (c) Dissipation range.

The first section called the **Forcing range** due to the presence of a forcing force, at wave number k_f, for example this could be from blowing into a pipe. The force injects some energy which causes the peak at k_f, it is this force which 'forces' the curve to peak. We can observe the effect of the forcing force onto the length scale (also known as the wavelength) as shown in figure 7. Due to the condition that $\mathbf{u} = 0$ at stationary boundaries (known as 'no-slip' condition [10, p. 33], the flow evolves into a parabolic. We can derive the length scale as denoted, by definition of a wave number:

$$\lambda_f = \frac{2\Pi}{k_f} = O(h)$$

Where λ_f is the wave length at the point of when the forcing force in injected.

The **Inertial range** is described as the range in which the "the largest eddies, which [were] created by instabilities in the mean flow, are themselves subject to inertial instabilities in the mean flow and rapidly break up or evolve into yet smaller vortices."[10, p. 18]. Davidson further explains that the term 'break-up' is used to symbolize the transferring of energy from larger vortices to smaller ones [10, p. 18] - it is this transfer of energy that results in the linear decrease in the'Energy cascade'. In this we range, we can determine the rate of this decrease by calculating the gradient. To do this we turn to the Five-thirds law and use the same method as we did for the two-thirds law, to arrive at:

$$\log(E) = -\frac{5}{3}\log(k) + C_3$$

where C_3 is a constant.
From this manipulation, we can conclude that the relation between the energy and wave number is determined as an equation for a straight line with a gradient of $-\frac{5}{3}$.

To explain the "transferring" of energy, we return to the Navier-Stokes equation.

$$\partial_x u = -(u \cdot \nabla)u - \cancel{\nabla p} - \frac{1}{Re}\nabla^2 u$$

Note that we are able to ignore the ∇p as it is only included to ensure that flow is incompressible (i.e. $\nabla \cdot u = 0$).
We begin by rewriting u in terms using the 1-dimensional Fourier Transform:

$$u = \sum_{k=0}^{\infty} u_k e^{ikx}$$

Taking 'del squared' of u, we get:

$$\nabla^2 u = \partial_x^2 u = -k^2 u$$

At small wave numbers, k, $k \to 0$, we can see that $\nabla^2 u$ is also small, and therefore the non-linear term is dominant and so we can simplify the Navier-Stokes equation.

$$\partial_x u \approx -(u \cdot \nabla) u$$

Let us rewrite u as:

$$\begin{aligned} u &= u_0 e^{ikx} \\ u^2 &= (u_0^2) e^{2ikx} && \text{(from squaring both sides)} \end{aligned}$$

We can observe by increasing the velocity by a power of two, the wave number doubles, and by definition the length scale halves. Going back to figure 8, we can see as each vortex splits, it's length scale decreases as expected. Therefore the kinetic energy is passed down as each vortex 'breaks apart', creating what labelled as the cascade.
From this we can conclude that is this non-linear term that continually shifts the energy to the right of the cascade, and therefore signifies the transfer of energy from large to small scales.

In the **Dissipation range**, we reach a section of small-scale structures. It is within this section that for a set wave number, η (known as the Kolmogorov scale) that the dissipation suddenly becomes significant, as shown by the sudden decay. From here Davidson argues that viscosity plays a dominant part in the decay by "mopping up whatever energy cascades down from above"[10, p. 19]. To explain why this is, we return once again to the Navier-Stokes equation and the Fourier transform of u, except in this range the wave number is large, $k \to \infty$ (again, we ignore ∇p for the same reason as mentioned before)

$$\partial_x u = -(u \cdot \nabla) u - \cancel{\nabla p} - \frac{1}{Re} \nabla^2 u$$

If u is large, we can see that $\nabla^2 u$ is dominant, and therefore we can simplify the Navier-stokes:

$$\partial_x u \approx -\frac{1}{Re} \nabla^2 u$$

Substituting the Fourier transform of $\nabla^2 u$, we arrive at:

$$\partial_x u \approx -\frac{1}{Re} \nabla^2 u \approx -\frac{k^2}{Re} u$$

From which we can analyse that due to the sign, that the relation between the two terms can be described as 'bringing it the system back to zero'. For example if the rate of change of velocity, $\partial_x u$ is positive, in other words there is an increasing change in velocity, there is a requirement that $\frac{k^2}{Re} u$ is negative. There are two elements to this observation:
(1) As both k and Re constants, it follows that $\partial_x u \propto u = u_0 e^{-\frac{k^2}{Re} t}$, as $k \to \infty$,$e^{-\frac{k^2}{Re} t} \to 0$, and therefore $u \to 0$. This term signifies the dampening effect, and the significant effect of the dissipation. (2) In this stage, we have large wave numbers (i.e. large k, known as the Kolmogorov

scale, η), therefore $\frac{k^2}{Re}$ is large. This factor results in the large dissipative rate, and therefore the sudden decay in the energy cascade.

To summarise, Tennekes and Lumley reiterates observations made from the energy cascade by stating "[...] most of the energy is associated with large-scale motions, most of the vorticity is associated with small-scale motions"[30, p. 23]. Furthermore we can indicate the above observations are supported with Davidson's arrangement of the Reynolds number.

$$Re = \frac{ul}{\mu}$$

where l is the size of the vortices. In the forcing range, k is small, length scale is large, and therefore the Reynold number is also large, which is to expected at an turbulent flow. In the inertial range, as k increases and the length scale decreases, and therefore a decrease in Reynolds number, this results in the flow becoming less turbulent, which is expected when the vortices are continually decreasing in size. Finally, in the dissipative range, when k is the largest and length scale is at its smallest, the Reynold number is tiny, resulting in potentially a laminar flow. Additionally we can observe, as k increases, viscosity does indeed become more significant, supporting Davidson's earlier statement.

2 Turbulent Luminance in Impassioned van Gogh Paintings

Aragón, Naumis, Bai, Torres and Maini hypothesized that some patterns of luminance in a select number of Van Gogh's paintings correlated with the structures functions found in Kolmogorov's turbulence theory. They further argue that the analysis Van Gogh's painting, as oppose to another artist, was appropriate due the paintings often described as "turbulent" and "chaotic"[4, p. 275]. Furthermore they chose to analyse paintings that were painted in 'van Gogh's last period', the period in which he would often be exhibit "episodes of prolonged psychotic agitation"[4, p. 275]. Further links with Van Gogh paintings and turbulence found in nature can be found in *Starry Night.* The vortices in the sky of the painting were commented to be similar to that of eddies in dust and gas turbulence found in a picture of a distant star [4, p. 275] Additionaly Aragón et al. separated *Starry Night* into the top half, the 'sky' and the bottom half, the'building' and also a 'blurred' version was taken into account, to argue that turbulent characteristics were present in all sections and it was not the effect of one section dominating the entire painting.. Further Van Gogh's paintings that were chosen to analyse included *Two Peasant Women Digging in Field with Snow*, *Road with Cypress and Star*, and *Wheat Field with Crows.*

The main hypothesis of the paper was to analyse each van Gogh painting, initially focussing on *Starry Night* by comparing the probability distribution function (PDF) of the change in luminance, against the change in distance within the painting, and the probability distribution function of velocity differences found in turbulent flows. It is noted that such correlations with structures in fluid turbulence, has also been concluded in other unexpected areas, for example in the changes in the foreign exchange markets [13]. A similar conclusion was concluded in the foreign exchange markets as in luminance, in this paper. Breymann, Peinke, Talkner and Dodge concluded that specifically the relationship between the probability densities of FX price changes and time delay corresponded to relationship to the velocity change and spatial separation found in turbulent flow. Furthermore the study determined that the scaling exponent found for the FX probabilities densities were similar to that of the Kolmogorov's scaling exponent found in turbulent flow [13, p. 769] .

It was concluded that the artwork analysed did exhibit characteristics of 'turbulent luminance',

Difference in luminance \ Distance between pixels	1	2	3	4	5	6	7
-7	0	0	1	0	0	0	0
-6	1	3	1	1	0	0	0
-5	2	4	4	3	0	0	0
-4	9	4	4	6	0	0	0
-3	7	9	7	5	2	1	0
-2	3	11	6	6	2	0	0
-1	11	5	5	12	2	1	0
0	12	6	10	12	0	0	0
1	9	9	11	15	1	0	0
2	5	7	6	6	1	0	0
3	4	5	4	2	0	0	0
4	5	8	8	1	0	0	0
5	4	6	2	1	0	0	0
6	0	1	1	0	0	0	0
7	0	0	0	0	0	0	0

Figure 9: Matrix created by Aragón et al. denoting the number of occurrences of a difference of luminance in a distance between pixels.

and additionally these characteristics were consistent with ones founded in Kolmogorov theory of turbulent flow. The authors comment that perhaps it is this correlation to turbulence that explains the "disturbing feeling [that] transmit the sense of turbulence"[4, p. 276].

2.1 Methodology and Results

We shall now discuss the methodology adopted. The paper begins with defining luminance of a pixel as a combination of it's red, blue and green (known as the RBG) constitutes, given by the equation:

$$0.299R + 0.587G + 0.114B \tag{2.1.1}$$

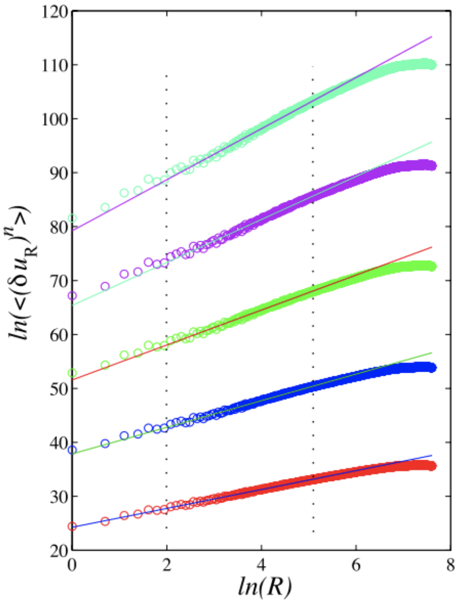
The methodology can be summarised in three steps [4, p. 277]:

1. The luminance of each pixel is obtained from the digital image by using the formula (2.1.1)
2. The PDF of pixel luminance fluctuations is calculated by building up a matrix L whose rows contain the difference in luminance δu_R and columns contain separation between pixels R (Note that the pixel separation is scalar i.e., $R = \|R\|$, furthermore it is noted that diagonal distances were approximated by the nearest integer function [4, p., 278]).
3. From the above matrix, we determine the normalized PDF of the luminance differences:
$$P(\delta u_R) = (\delta u_R - \langle \delta u_R \rangle)/(\langle (\delta u_R)^2 \rangle)^{\frac{1}{2}}$$

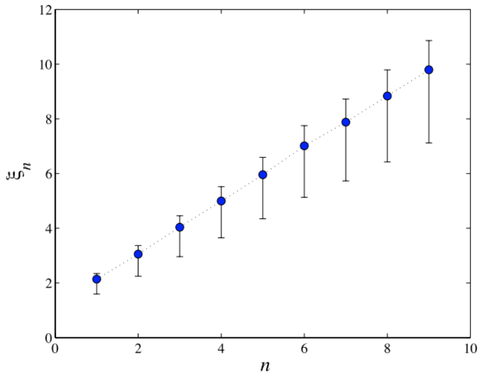
The matrix that was created through analysing Van Gogh *Starry Night* is shown in figure 9[4, p. 278]. We can observe from the matrix that Aragón et al. took both positive and negative luminance difference, therefore specifying if the pixel in comparison becomes brighter and more saturated, or darker. A semi-log graph is plotted of varying pixel separations and the figure are fitted to a symmetric distribution with variance [4, p. 277]

$$P_\lambda(\delta \nu_R) = \frac{1}{2\pi\lambda} \int_0^\infty \exp(-\frac{(\delta \nu_R)^2}{2\sigma^2}) \times \exp(-\frac{\ln^2(\frac{\sigma}{\sigma_0})}{2\lambda^2}) \frac{d\sigma}{\sigma^2}$$

with $\sigma_0 = 1.0$
The use of the parameter suggests that the probability densities of the luminance difference can be fitted into a Gaussian distribution. However the method used to choose this parameter in this paper is unclear, as it is stated that a "trial and error method was [used to find] the value



(a) Log-Log plots of the statistical moments for n=1, 2, 3, 4, and 5 against the pixel difference as found in van Gogh *Starry Night* and a straight line is fitted using the a least-square fit.



(b) A plot of the scaling function found from figure 10a against varying n-values. Again a straight line is fitted using the least-square fit.

Figure 10: Plots which supports that van Gogh's *Starry Night* does exhibit similar characteristics as that of Kolmogorov's structure function.

of λ that yields the best fit to the measured PDF"[4, p. 277]. From there, Aragón et al. plotted log-log plots of pixel separation, R against the statistical moments, $\langle(\delta u_R)^n\rangle$, as shown in figure 10a[4, p. 279], for $n = 1, 2, 3, 4, 5$, and plotted a least-square fit to determine a linear relation. From this by calculating the gradient of each n, the scaling exponent ξ_n was determined, and plotted figure 10b [4, p. 297] against n: the derivation is shown below[5]: We begin with an equation of a straight line, letting $\langle(\delta u_R)^n\rangle = y$ and $R = x$

$$\ln\, y = \alpha \ln\, x + \beta$$

We take the natural logarithm of both sides

$$\begin{aligned} y &= e^{\alpha \ln x + \beta} \\ &= e^{\alpha \ln x} e^{\beta} \\ &= \gamma e^{\alpha \ln x} \text{ (Letting } \gamma = e^{\beta}) \\ &= \gamma x^{\alpha} \end{aligned}$$

By substituting values of $y = \langle(\delta u_R)^n\rangle$ and $x = R$ back in, we can conclude:

$$\langle(\delta u_R)^n\rangle = \gamma R^{\alpha} \tag{2.1.2}$$

Comparing this equation with that of Kolmogorov's nth order structure functions (equation 1.5.1 (b)), we can see that $\alpha = \xi_n$. In other words, the gradient of the linear relation found when comparing the luminance difference and pixel separation, is equal to that of the scaling function in turbulent flow. This result supports the hypothesis that patterns of luminance

[5]The derivation is identical to that of ξ_n in section 1.5.1

found in Van Gogh's *Starry Night* exerts characteristics to that of Kolmogorov's turbulent flow. Furthermore when the scaling exponent was plotted against n, it again can be fitted with great accuracy to that of a straight line [4, p. 279], therefore we can write

$$\xi_n = pn + q \tag{2.1.3}$$

From which we can conclude that the scaling function, with a factor of p and shifted by q, found in this study is consistent to that in turbulent flow, $\xi_n = \frac{n}{3}$.

However Aragón et al. does not specify what the values of p and q are in *Starry Night*, and so we cannot comment on how closely related this scaling function is to that of Kolmogorov's, and cannot evaluate the validity of the hypothesis. Furthermore they argue that this relation is consistent with other van Gogh's paintings as the probability densities against pixel separation plots are similar to that of the *Starry Night*, retaining a reasonable bell-shape curve for varying pixel separation and with the use of the parameter, λ can be fitted to resemble a Gaussian distribution. The slight deviations in the other paintings results in Aragón et al. suggesting that there are indeed other factors that influences the degree of turbulence, for example the size of the eddies present in the paintings or if it was a case of smooth versus sharp transitions of luminance [4, p. 281].

2.2 Discussion of determining a relation between luminance in paintings and turbulent flow

The significance of this study is the ability to not only be able to successfully identify a mathematical relation within something is often described to not be systematic and governed by creativity and emotion, but also to provide a quantitative analysis to an abstract piece of work. Whether Van Gogh had even intended for this painting to be 'turbulent' or not, there seems to be a correlation, arguably a strong correlation, between the literature definition of turbulence within a painting and that of Kolomogorov's mathematical one. It is concluded that "the distribution of luminance on a given painting determines if it is turbulent"[4, p. 208]. Furthermore it was suggested that these laws that governs mathematical turbulence found in this painting could be transferred to other artwork and art representations [4, p. 282], perhaps to musical pieces.

Aragón et al. argue that the approach formulated in this paper allows for further research into developing quantitative analysis for art representation. In the next section, we test this argument by analysing a different art form, music using similar methods outlined in the paper.

3 Analysing Music

The methodology we propose to determine if the similar turbulent characteristic could be found in a musical piece follows that used by Aragón et al. We focussed on investigating the relationship between intervals between two pitches and varying time differences. The main goal of this study is to determine if the probability density function (PDF) of pitch differences, known as intervals, separated by a time reference is consistent with that of velocity difference and the PDF of fluctuations in luminance in turbulent flow and paintings. The variables that are compared are summarised in table 1.Therefore we argue that a large change in velocity is similar to a large pitch difference or interval. Following the layout of Aragón et al. we label the interval difference as δu_R and the time difference to be R.

Turbulent Flow	**Painting**	**Music**
Velocity	Luminance	Pitch
Distance	Pixel separation	Time

Table 1: Table comparing the variables use when comparing 'turbulence' in a velocity field, paintings and music.The table is laid out such that the row of each columns are can be directly compared.

Frequency (Hz)	220.00	233.08	246.94	261.63	277.18	293.66	311.13
Pitch	A	A$\sharp$ \ B$\flat$	B	(middle) C	C$\sharp$ \ D$\flat$	D	D$\sharp$ \ E$\flat$
Note number	37	38	39	40	41	42	43

Table 2: Table denoting the note's frequency and it's assigned number according to the 88-key piano, with the lowest key labelled as 1 and the highest 88. As demonstrated by the table, we take enharmonic notes to be the 'same note' each with the same frequency and the same note number, even though harmonically they would regarded as different notes

3.1 Quantifying Music Notes

We begin by explaining the method we have chosen to quantify a musical note. We do this simply by labelling each note with it's respective piano key number. An alternative description is that we choose our pitch difference unit to be a semitone. For example we took the standard 88-key piano, assuming that it is a equal temperament tuning, we label the very bottom (or bass) note,C_8, as our note number 1 and the highest note on the piano, A_0 as note number 88. An example is shown in Table 2, and full reference that was referred to while conducting this study is shown in appendix as figure 41.
Note we include rests when inputting of our data, as it then ensured the duration of notes values were an accurate representation to what we would hear. We decided to treat rests as if it was a 'silent note' and give it a label of -1. From this we reach an assumption that for each time value there is a note number that is played at that time. We distinguish the difference between 'pitch' and 'notes'. The term 'notes' can be ambiguous, as it is not always interpreted to mean a sound with specific frequency. For example referring to Note C, it is not always clear if we mean middle C or the C an octave above or below. However, the term 'pitch' has some associations with a set frequency and therefore each pitch is unique, excluding enharmonics[6]. For the purpose of this study, when referring to intervallic differences, the terms 'notes' and 'pitch' are used interchangeably.

From quantifying the musical notes, we are then able to determine the interval difference simply by subtracting the two note numbers from one another. Table 3 lists the intervals against the pitch difference.We note that this it not a rigorous labelling of intervals, for example for a pitch difference of 4, this is labelled as an interval difference of a perfect fourth, however it can also be a interval difference of an augmented third or a diminished fifth. For simplicity, we take the assumption the melodies of pieces analysed generally are made up of conventional intervals, as listed in table 3. Additionally at pitch differences larger than twelve, we can determine the interval using the equation: interval = $x$mod12 where x = pitch difference. By convention, musically we would calculate an interval between two notes by 'counting up', beginning at the lower note and counting up towards the higher note. However we can see by quantifying the note, we need not to ensure that we always count upwards between two pitches, the numeric pitch difference is equal to the interval regardless if the interval is ascending or descending.

[6]For example C$\sharp$ and D$\flat$ will have the same frequency even though musically they are

Pitch Difference	Interval
0	(Same note)
1	Minor 2nd
2	Major 2nd
3	Minor 3rd
4	Major 3rd
5	Perfect 4th
6	Augmented 4th/Diminished 5th
7	Perfect 5th
8	Minor 6th
9	Major 6th
10	Minor 7th
11	Major 7th
12	Octave (same note)

Table 3: Table listing the labelling the numeric pitch difference and it's musical interval name. We note that this is a simplified table, consisting of the most common intervals, and that for each pitch difference it can be labelled as other name.

Note Value	Time duration
Semiquaver	1
Quaver ♪	2
Crotchet ♩	4
Minim 𝅗𝅥	8
Semibreve 𝅝	16

Table 4: An example depicting the different note value in reference to the chosen time unit. Here the time unit is a semiquaver, and the table demonstrate the time value (i.e. how long each note will last when struck) of a quaver, crotchet, mimum and semibreve in relation to the time unit.

3.2 Determining a time reference

We determine the time reference by locating the smallest note value in the piece and assigning this as our time unit, in other words a time value of 1. For example if the smallest note value is a semiquaver, we can construct the other note values in terms of this time reference, as shown in table 4. We generally begin counting time at the first note in the melody is struck, and therefore it is not assumed that time will start in the first bar. Furthermore the counting of time will be begin at zero, for example the first note of the melody will be recorded at t=0, then the second at t=1, and so on. As a consequence of the method we have chosen quantify notes and also rests, we can add that there exists a note value for every given time value.

We note the main difference between choosing time as our independent variable as oppose to using distance, as was used in turbulent flow and when analysing paintings, is that when analysing the distance between two spatially free points, there are arguably infinite many directions, dependent on the parameters. However by choosing time we are limited to only taking the positive linear direction.

3.3 Method

Once we quantify the notes and determine the time reference for a set piece, we input this into an excel spreadsheet of two columns, (1) time at which the note is struck and (2) the note

Excel spreadsheet number	Piece
0	Bach Minuet in G Major, Full
1	Bach Minuet in G Major, section A
2	Bach Minuet in G Major, section B
3	Mozart Sonata in C Major, Full
4	Mozart Sonata in C Major, Section A (Exposition)
5	Mozart Sonata in C Major, Exposition First theme
6	Mozart Sonata in C Major, Exposition Transition
7	Mozart Sonata in C Major, Exposition Second Theme
8	Mozart Sonata in C Major, Exposition Codetta
9	Mozart Sonata in C Major, Section B (Dev and Recap)
10	Mozart Sonata in C Major, Development
11	Mozart Sonata in C Major, Recapitulation
12	Mozart Sonata in C Major, Recapitulation First theme
13	Mozart Sonata in C Major, Recapitulation Transition
14	Mozart Sonata in C Major, Recapitulation Second theme
15	Mozart Sonata in C Major, Recapitulation Codetta
16	Beethoven Für Elise
17	Satie Gymnopédie No 1

Table 5: Table listing the excel spreadsheet numbering created when conducting the study

number.
This datafile is then imported into Spyder at which various analysis can be applied using python coding. Table 5 displays the excel spreadsheet numbering in reference to the piece inputted. The methodology carried out follows closely to that proposed by Aragón et al., and so a matrix is created in the similar manner. A pseudocode can be found in figure 42, which provides a brief description in what the code should do. The initial 'matrix creating' code, shown in figure 43 takes two rows of a datafile set by the 'i value' corresponding to the excel spreadsheet number as denoted in table 5, and calculates the time difference, moving in a positive direction (i.e. going down the table), and the respective interval. From which the corresponding matrix element, (m,n)=(time difference-1, interval) is appended. The matrix would be formatted such that each row was a set time difference, starting at $R = 1$ for the first row, as we ignore comparing notes played at the same time, and each column was an interval (this is dependent on the maximum and minimum intervals within a piece).

We note that there are two options for taking the interval between two notes for this study. The first, is similar to that in Aragón et al., where the difference can be positive or negative, meaning there is an ascending or descending interval. The second, is taking the absolute value of the interval and therefore focussing on the sizes of the interval only. In reference to the layout of the matrix, if we were to take the absolute value, the first column would correspond to an interval of zero - meaning the same note is struck again within a time difference of R. However if we were to take positive and negative values, to find an interval of zero, this would be middle column.

The following graphs can then coded, and each are given a brief description to what we might expect to analyse from them.

3.3.1 Heat Map

The advantage of representing the intervals against time difference as a heat map, is that it gives a visual representation of the most common interval for all time differences, with the most occurrences assigned a brighter colour of yellow, and decreasing in luminance as the occurrences decrease to navy and so on. The code used to create this representation is shown in figure 49Additionally by choosing to take including both positive and negative intervals, we can determine easily whether the overall melodic structure of the piece is ascending or descending. Generally we would expect to see the highest occurrences to be at small time differences and small interval sizes, in other words to have the highest luminance.

3.3.2 Bar Chart

1. Note number against occurrences
For a piece, by plotting the pitch against the number of occurrences, we can determine which the most 'common' notes, from there we can predict the tonality of the piece. The code used to created such a representation is shown in figure 47. Additionally from this representation we can determine the common distribution of the pitches within the piece and comment on the the register in which most of the notes lie in. Therefore observing if we expect the piece to sound 'high' or 'low'. Furthermore we are able to comment on density of the melodic lines, explaining if the majority of the melodic line is compacted into a set of notes, or if the melody is predominantly made up of excursions across a large range of notes.

Figure 11 is an example plot of the 'expected' histogram, from which we can observe that the expected shape is similar to that of a positively skew Gaussian distribution. The mode of the distribution (i.e. the peak of the curve) is at the pitch with the highest number of occurrences. Furthermore, as denoted by the purple bars, we propose that pitches with higher occurrences are ones that are diatonic with the key, and suggest the pitch at the mode of the distribution is either the tonic, mediant or dominant of the scale degrees (For further information on this, refer to table 6). Therefore from this proposition, the pitches with significantly less occurrences are ones that are dissonant with the tonality, and therefore are only used to add some 'colour' and embellishment to the melodic line.
Furthermore, the reason for the use of a positively skewed distribution is based on the assumption that the melodic line is predominantly in the upper line, or the right hand for piano pieces. We predict that in the direction of descending pitches, there will be a sudden decrease in occurrences if the lower line, or left hand providing a physical lower limit. Additionally towards higher pitches, we propose a slow decay, as representation of some excursions deviating away from the region at which the melody is concentrated at.

2. Frequency against interval
This representation shows similar results to that of the heat map, however in this analysis we restrict the data by specifying a time difference, R, for example the interval between crotchets or bars, therefore presenting a more in depth analysis. The code used to create this representation can be found in figure 48. Additionally, from this we can hypothesise an ability to recreate the melodic line, therefore, by comparing back to the score, we comment on the accuracy of using statistical models to represent an abstract piece of art. We can link this hypothesis with the theory of turbulence. By suggesting a statistical method can be used to predict a melodic line that is often described to be random or does not follow a sequence, corresponds to that of describing turbulence to be a subset of deterministic chaos.

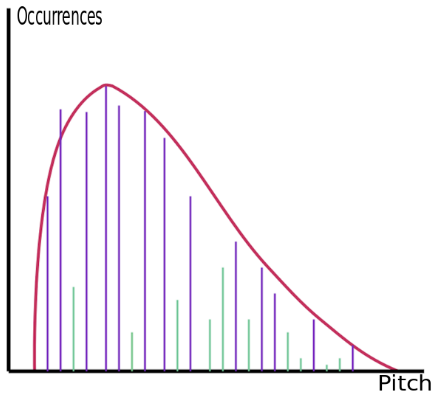


Figure 11: A plot of the expected shape for the pitch against occurrences. We fit the shape to a positively skewed Gaussian Distribution. The vertical bars are to denote the number of occurrences of the set pitch within the extract or piece that we have extracted the data from.

3.3.3 Line graph

1. Pitch number against time reference
By plotting the pitch against the time, using the code demonstrated in figure 45 at which the note was struck, we can get a graphical representation of the melodic line. From this we can comment if the melodic line is predominantly ascending or descending. Additionally, from the gradients of the line, we can determine if it was made up of large leaps, therefore being described as disjoint, or if the melody was mostly scalic, and therefore exhibits small gradients in the line graph. Furthermore, we can also observe for melodic patterns, and identify motifs within the sections. The advantages of using this representation when analysing the melody is that we have now provided a graphical represention of the common melodic elements, listed in section 4.1, and therefore there is the ability to quantify to what degree they do conform to these proposed elements. Therefore from which we can provide a quantitative conclusion on it's 'musical turbulence' characteristics.
Furthermore we can use the code shown in figure 46 if we wanted to compare melodic lines between two sections. Uses of this representation could be to examine if sections re-used motifs presented previously, to determine if one section is described to be more 'busy' or has a more 'active' melodic line, or to suggest a section may exhibit a more turbulent 'feel' than the other.

2. Statistical moments against time difference
This is the plot which defines whether the characteristics of a turbulent flow is present in a piece of music as defined by Kolmogorov's theory. By plotting the statistical moments against the time difference, for varying values of n, we can determine the gradient of each, and evaluate the scaling function, with the use of the code presented in figure 51, with use of the function "moments(x,n)" defined in figure 50. We fit the line of least-square fit to a function of $y = bx^a$, where b is the y-intercept and a is the gradient of the line, and therefore the scaling function following the same method as stated by equation 2.1.2 to evaluate the scaling function. Similarly to that in the van Gogh paintings, if a piece of music were to exhibit characteristics found in turbulent flow, we'd expect the log-log function to be linear and therefore we would be able to fit the data to a straight line using the least-square method, as shown by figure 52. We note that the function "popt" returns the values of variable a and b and therefore can determine the scaling function through using the command "print(popt[0])".

Furthermore to confirm that the scaling function is of scalar order, as suggested by Kolmogorov's theory we use the code in figure 55, in which the n values are plotted against the scaling func-

tion. If a linear relation can be fitted again using the least-square fit then we can conclude that the scalar function is most likely to conform to that predicted by Kolmogorov. The line of least-square fit was fitted to a function of $y = px + q$ following the method of deriving the equation 2.1.3.
Due to time restraints, this method of coding is not as sophisticated as it can be, unfortunately a code that would run through range of n values and return all it's scaling function was not created, and therefore in order to plot this we had to crudely calculate each scaling function, by running the code in figure 52 for all n values considered and record this in a separate set.
We begin our study by first defining what we regard 'musical turbulence' to be.

4 Defining Musical 'Turbulence'

The following section discusses why chose to focus on the melodic line, and more specifically argue that investigating pitch differences would an appropriate variable that determines is a piece is described to be 'turbulent'. We begin by suggesting some common features of a melodic line which in turn then increases the expectation of the listeners, then we will explain the relation between expectation and prediction. From which we turn to demonstrating how a violation of the listener prediction can result in a negative reaction. We end this section with a category of musical pieces which are describe to be often unconventional and 'disobeys the musical rules' of the time, and so we will base our definition of 'turbulent music' to be pieces that exhibits characteristics of this group.

4.1 Melodic line and Expectation

We begin by stating some common features found melodic lines and therefore explain how the factor of 'expectancy' is affected by this. 'Musical expectancy' is explained as the involuntary reactive response music, whether this is a achievement or violation of the predicted outcome formulated on the stimulus presented. It is noted that examining the components of musical expectancy is not as simple as we might initially believed, and such components cannot be simply listed. This is because, firstly there are many different components making up what is known as musical expectancy, for example musical features such as the harmony, dynamics or melody, but also social concepts, for example the idea of 'Nature versus Nurture', whether these expectations are learnt or innate, therefore a question of cultural differences is opened and each influences the other in complex manners, often becoming difficult to disentangle a component from another. Secondly a piece of music can evoke multiple expectations at the same time, and the concept of expectation and prediction is personal to each person, based on aspects mentioned before.

While we have already discussed the difficulty in stating definitively aspects of music which can be predicted, we attempt to determine characteristics of the melodic line which are found commonly across mainly Western music. For example, Greenberg and Larkin hypothesised that listeners were able to detect a tone if the tone detected matched the one that was expected, from which it is suggested that there must be a degree of expectation being fulfilled when it comes to a positive reaction to music [17, p. 42]. We argue if melodic line does not follow these proposed characteristics, then there is a violation in musical expectation and a 'surprise' reaction is outputted and so the piece is described to be 'turbulent'.

4.1.1 Common Melodic features

Huron lists four common melodic features, in which he calls "Organizational Elements" and we give a brief explanation of each one:

1. Pitch proximity
   Huron argues that "successive pitch tend to be near on another"[17, p. 35]. A description

Scale degree	Name
1	Tonic
2	Supertonic
3	Mediant
4	subdominant
5	dominant
6	submediant
7	Leading tone

Table 6: Table listing the note's position in a scale against its scale degree name. We note the most harmonically stable degrees are the tonic, mediant and dominant, and triads are made up of these three degrees.

of this feature, is that if melodic lines were only built on pitch proximity, it is argued that the melodic line would look like a 'random walk' also known as *Brownian motion*[17, p. 35]. A study conducted by Aarden supports this claim as the results suggested that listeners were able to respond significantly quicker when asked if the successive pitch was higher, lower or remained the same, if the successive pitch was close to the previous [17, p. 75]. Furthermore the idea of small intervals commonly used in melodic lines can be generalised most cultures as Huron [17, p. 74] conducted a study of ten cultures in which the results were consistent to this pitch proximity element. However it noted that there were some exceptions for example "Swiss yodel-ing and Scandinavian "yoiks""[17, p. 75].

2. Central pitch tendency
   Following from pitch proximity, we remark that if melodic lines were dominated by pitch proximity it is highly likely that these melodies would often continually run until it became out of range, therefore there is the need for a central pitch tendency. The fundamental principle of this feature is that within a melodic line, there exists a pitch such that the overall line will gravitate back towards it. It is also noted, that if this is the predominant feature, the resulting melody is described as "Johnson noise" or "white noise"[17, p. 35].

3. Ascending leap tendency/ descending step tendency
   The melodic movement are commonly made up of "rapid upward movements (ascending leaps)"[17, p. 35] and "relatively leisurely downward movements (descending steps)"[17, p. 35].

4. Arch phrase tendency
   Finally, Huron notes that this last feature is commonly found in Western music, in which the melodic line would exhibit an arch-shape contour[17, p. 35].

Furthermore, if we focus on the theory side of music, we can argue for common pitches in related to the tonality. From table 6, we note that the most common scale degree is the dominant, followed by the mediant and then the tonic in a major key and in a minor key this is reversed, with the second most likely scale degree to be the tonic followed by the mediant. These three common pitches comprises what is known as the *triad*. Furthermore it is also noted that notes that are chromatic to consonant pitches i.e. pitches that are found in the key (therefore labelled to be dissonant) and notes that are not within the 'scale' of the key are expected to occur less frequently [17, p. 148].

4.1.2 I-R model

The I-R model, known as the *Implication-Realization Theory*, was proposed by Narmour and comprises of five subcategories as listed below. Narmour claims that the principles would "shape a listener's expectation for melodic continuation"[17, p. 95]. The goal of the I-R method, is to remove cultural or stylistic influence on analysis of music, but instead focus solely on the note-to-note relationship and determine what features would influence a listener's implications and realizations [3, p. 931]. Additionally Narmour believed there are two types of melodic continuations, the first is *implicative* in which there is a strong sense of knowing what will happen next, or *non-implicative* [17, p. 94] where there is a high difficulty in accurately predicting the next note for example. A statistical definition can be used to explain a non-implicative melody, for example we can suggest that there is an equal probability for the next to be struck to be any of the ones one a piano or to be a rest. Narmour claims that for a melodic line to be implicative, it follows the five dispositions laid out by the I-R model. Such violations of implications listed by Narmour can be regarded to be a deliberate decision in order to evoke a specific reaction [27, p. 77].

1. Registral Direction
   The first principle can be suggested to be a combination on Huron's pitch proximity and central pitch tendency elements, which is regarded to describe pattern of increasing and decreasing melodic lines, more specifically small intervals would continue in the same direction, whereas larger intervals would often result in a change of direction, as if to 'balance out' this interval [27, p. 77]. Additionally it supports the element of an arch phrase tendency, as Abbs and Gupta claims that it is this principle which dictates the "continuation of melodic contour"[3, p. 931].

2. Intervallic Difference
   This refers to the size of the difference between two notes sounded one after one another. It is suggested that a small interval differences would imply further intervals of similar sizes, whereas a large interval would result in intervals of smaller sizes [27, p. 78]. This support Huron's claim of pitch proximity and also there exists a central pitch tendency in melodic lines. Furthermore Narmour notes that the definition of 'similarly sized' intervals is dependent on whether the registral direction changes or remains the size. If it changes, it is defined to be a interval of the same size plus or minus three semitones, whereas for a melodic line to remain in the same small direction, a small change in the definition to a the same interval but plus or minus two semitones.

3. Registral Return
   The third principle refers to the change in direction in the melodic line, for example going from an upward direction to a downwards and the vice versa. Again this can be viewed as an extension of Huron's arch phrase tendency element, as Schellenberg refers to such patterns that exhibits a perfect registral return to be symmetrical (ABA) or near symmetrical (ABA') [3, p. 80] The closer the implicated note is to the initial note struck, increases the likelihood of being predicted.

4. Proximity
   Proximity claims that for a melody to be highly predictable, there is an inclination towards intervals that are small in size, the degree of expectation is highest at intervals of zero (being the same note) and falls linearly as the interval size increase [3, p. 932] (in which 'small' is defined to be intervals of no more than five semitones [27, p. 80]). It is claimed that this feature is consistent with other cultural music, and not just limited to that of Western music.

5. Closure
   The final category in the I-R model is a combination of pitch direction and interval sizes, and refers to the idea that a sense of closure is achieved if a implicative interval is both smaller in size and also changes direction in comparison to ones that preceded it.Again we

can argue that this is another variation on Huron's central pitch tendency, explaining that a common feature for melodies is that whilst it may not be direct route, there is still an underlying gravitation back towards the initial note.

Narmour claims the principles in the I-R Model are innate and a natural 'built-in' system, and not learnt or influenced by our experience as musicians or cultural differences. Furthermore the advantage of this model is that we are able to begin to distinguish characteristics that are universal, meaning they are consistent that in the I-R model, and melodies that do not are argued to be style specific [27, p. 76]. It is presumed that this model is applicable to listeners of different levels of experience or training, pieces composed of different periods and different culture[27, p. 81]. However experimental results do not support this claim completely, as shown by a study conducted by Schellenberg in which three experiments were carried out to determine the validity of the I-R model, the first of which on western tonal music (perhaps the most familiar style of music to most listeners), then an atonal piece (an unfamiliar style but still consistent with some Western music characteristic) of music and finally a non-western music, specifically a Chinese folk melody (arguably the most unfamiliar style of the three). The method used for all three experiments was that a melodic phrase would be played, followed by different 'test tones' in the middle of the phrase and participants were asked to rate these 'test tones' on its successful in continuing the phrase (1 for extremely bad continuation and 7 for an extremely good continuation) [27, p. 84].

The results of the first experiment, that there existed a "tonal hierarchy" within Western music, that is not considered by the I-R model. This factor meant that the degree of which the 'test tones' would be a good continuation did not only rely on the parameters the I-R model had predicted, for example did not only have to be a small interval, but there was an additional subconscious factor, in which if the 'test tone' was of tonal stability to the key then it could regarded as an implicative interval. Implying that regardless if there was a large pitch difference, it the 'test tone' is defined to be tonally stable, it had strong tendency to still be labelled as a 'good continuation'. Conclusions made from the results of this experiment were that the I-R model did seem to be an accurate model in order to predict the participants response to different 'test tones', and furthermore this did not depend on the participant's music training or experience as Narmour had claimed.
In order to determine the degree of which the "tonal hierarchy" would affect such judgements, Schellenberg repeated the experiment with atonal music, and therefore it is argued by the removal of a tonal structure participants are no longer able to use to tonic as a reference in stability of the 'test tone'[27, p. 94]. The results gained from this experiment were similar to that of the first, therefore arguably supports generalisability of the I-R model. However it was noted by Schellenberg that, whilst the I-R model did accurately predict responses, it did tend to exert a stronger influence on tonal music than it did on atonal. Suggesting that whilst there are similarities, it cannot be as generalised as initially claimed by Narmour.
The final experiment focuses on the idea of the successfulness of the I-R model depending on whether the material is familiar or not [27, p. 101]. This can be interpreted as if an interval is unfamiliar it is a violation of this implication and so a reaction, often negative, is evoked. The claim is that as the parameters defined in the I-R model are innate, then the effectiveness of the model will should not change depending on whether the participant is familiar with the melodic sequence or otherwise. Schellenberg focussed on Chinese folk melodies which are predominantly based upon pentatonic scale, an arguably unfamiliar concept in western music, as they do not follow the same pattern of intervals as major or minor scale. Therefore an interval that may be unusual in a western music, can be conventional in a Chinese folk melody, and the converse could be true. The results of this experiment were consistent with those found in the first and second experiment. Therefore supporting Narmour's claim that the I-R model's predictive ability were not affected by music of different cultures and that the parameters were not learnt. However it

is also argued that this successfulness could be because Chinese and Western music do share a number of systems, for example the duration of a note, it's position in the motif or whether it had been repeated, could also imply it's 'importance' and therefore could be used as a reference when determining the 'test tone' stability. This reiterates the difficulty in separating music into it's individual components to test a hypothesis as they do influence one another.

In summary, from experiments 1 and 2, Schellenberg proposes an added influence of tonality, however this could differ with participants based on their cultural training. Contradicting the model's claim to be universal for all listener's regardless of their cultural background or degree of training. Furthermore experiment 2 and 3, suggested that there is a significant influence if the participant has a high level of musical training especially when judging atonal and unfamiliar music [27, p. 109]. This could be because, arguably, participants who have more musical training are exposed to larger and more varied repertoire and therefore the collection of 'unfamiliar' motifs become smaller and so contradicts the claim that this is an innate ability.

We conclude this section be returning to the claim that there does exist a set of features that are commonly found in melodic lines, and therefore if these features are prominent, it is argued to be expected by the listener. However it also must be noted that this is not a rigid and comprehensive list and shown by the experiment conducted by Schellenberg. From which we add a further comment that the the model proposed by Narmour does provide a promising foundation in deciding if a melodic line is expected or otherwise, but there must also be room to consider exceptions.

4.2 Expectation and Surprise

The relation between expectation and surprise dates back to the evolutionary 'fight-or-flight' response, in which we use the process of expectations to generate predictions of future events [14, p. 1]. Therefore we can argue that subconsciously we make predictions[7] based on expectations that may be innate or based on experience i.e. are learnt. From which, if our predictions fail to predict the correct outcome, we exhibit a 'surprise' reaction. Huron argues from a biological perspective, a surprise response is never perceived as a positive reaction, regardless if the outcome is good [17, p. 21]. This is largely because having a positive response relies on being able to predict the future outcome accurately.

4.2.1 ITPRA model

The ITPRA model is used to explain how our brain process and evaluates a musical stimuli, in particular relation to the idea of musical expectation. The five principles within this model are (I) *imagination* response, (T) *Tension* response, (P) *Predictive* response, (R) *Reaction* response and (A) *Appraisal* response. For the purpose of this study we will focus on the Predictive Response. Huron emphasizes that when referring to 'good music', this is describing a piece of music which evokes a positive reaction as oppose to providing a judgement on the quality of the piece[17, p.140]. Furthermore he proposes that if a piece is 'highly predictive' then the overall prediction response are dominated by a positive reaction[17, p. 140]. On the other hand, if a piece is highly unpredictable it can often be perceived as unpleasant [17, p. 141].
Whilst we have now linked the concept of expectation and predictability with determining if an extract will evoke a positive or negative reaction, Huron does note that it is not as simple as we

[7] The concept of "making predictions" can be quantified using the *Information Theory* proposed by Shannon. A brief overview of this theory is that it quantify expectedness and uncertainty taking into account the context of the stimuli and calculates the entropy of probability densities of each outcome. More information on this theory and it's criticism can be found in [14]

have claimed, and that often the emotion evoked is a combination of different responses [17, p. 141].

4.3 Tempesta

The next section of discussion claims that there is a melodic style such that the pieces that exbhit this *Tempesta* characteristic will evoke a negative response. Huron claims that the two reactions of 'awe' and 'frisson' are the result of the 'surprise' reaction. The 'awe' reaction is related to the idea of 'freezing' in an event of surprise whereas 'frisson' is the reaction that is derived from the fight-or-flight response [22, p. 12]. McClelland claims that the Tempesta style will evoke the frisson response and is equated with a sense of terror[22, p. 12][8]. Some musical elements that are identified to evoke this response consists of "adjectives such as abrupt, rapid, sudden, new and unprepared suggest[ing] the precipitating musical events may be surprising or unexpected"[18, Musical Expectancy and Thrills, p. 594]. Another element is the theme of "high energy", for example increased loudness and/or addition of harmonic lines and the use of contrasts is also common when attempting to evoke a frisson response [18, Musical Expectancy and Thrills, p. 594].

McClelland defines the style of *Tempesta* music to be "disrupting the conventional musical language of the day, introducing in pitch, rhythm, timbre, and dynamics that would unsettle audiences"[22, p. viii].The style defined can be used as a device to express a wide variety of turmoil, both literal and metaphorical, for example it is often used to depict 'Storm music' and to imply a scene of rage, madness and conflict[22, p. ix]. Common melodic characteristics found in Tempesta music are listed as follows "Melody: disjunct motion, often fragmented, with very wide leaps, sometimes augmented or diminished leaps. Figurations: rapid scale passages, tremolo effects, repeated notes, *tirades*"[22, p. 219]. Furthermore it is noted that storm music in particular often exhibits conjunct motion and it is the rapid motion that creates the wave-like motion and an 'uneasy' response [22, p. 49]. However in contrast to this statement, McClelland does also explain that some composer would go against this tradition and introduce "angular [melodic] lines, including dissonant intervals or very wide leaps"[22, p. 55], conforming to that of 'disrupting the conventional language'. We could imply there is a constant oscillating of the definition of the conventional style as what was defined as disrupting the musical language will become accepted, and so has then become the conventional language of the time and the cycle repeats. Therefore perhaps the factor of the size of intervals between two notes impacting the degree of turbulence would arguably be dependent on the period that the piece was composed in, and cannot be used as a universal variable.
From McClelland definition of the Tempesta style we can note that there are already difficulties in defining 'musical turbulence'. Broadly speaking we have suggested that a piece is musically turbulent if it's melodic line is generally unexpected and therefore evokes a 'surprise' response. We began with identifying common melodic elements, and therefore proposed for a melody to be turbulent it will not exhibit many of these elements, conforming McClelland definition of "disrupting the conventional musical language"[22, p. viii]. However we note that the conventional language as denoted by the I-R model claims that these elements are universal regardless of culture, period or personal experience, whereas the Tempesta style is dependent on the period of composition. Therefore we argue that the proposing a representative definition of 'musical turbulence' is not as simply as initially assumed. We conclude that we have provided a foundational definition of 'musical turbulence' however again there needs to be room for exceptions.

[8] The 'awe' response is equated to a sense of horror and therefore coincide with an *Ombra* description

	Bars	Time	Note
0	5.0	0	-1
1	NaN	1	58
2	NaN	2	61
3	6.0	3	59
4	NaN	4	58
5	NaN	5	53
6	7.0	6	51
7	NaN	7	53
8	NaN	8	54
9	8.0	9	49
10	9.0	12	46
11	10.0	15	46
12	11.0	18	46
13	12.0	21	46
14	13.0	24	-1

Figure 12: The data file inputted into an excel spreadsheet listing the bars, time reference of each note in relation to the time unit chosen and the quantified note that was struck, following the method previously stated, in Satie's Gymnopédie No 1. . The very left column represents the row number of the spreadsheet.

5 Determining if musical pieces exhibit turbulent characteristics corresponding to that in Kolmogorov's theory of velocity fields

In order to determine whether the method proposed by Aragón can be applied to other forms of art, we propose to study four pieces, Satie's Gymnopédie No 1., Bach Minuet in G Major, Beethoven Für Elise and Mozart's Sonata in C Major and evaluate each piece individually using the method proposed in section 3.3.

5.1 Satie's Gymnopédie No 1.

The first piece we chose to analyse was Satie's Gymnopédie No 1., and decided to focus the analysis on the first melodic phrase (from bars 5-12), as shown by the highlighted section in figure 56. The data file created is shown in figure 12 . By convention, in classical music pieces, the melodic line is often the 'top line' and so we follow this when choosing the line to analyse, and therefore the 'lower' or bass notes are the accompaniment, and therefore ignored in this analysis. This is often the convention when analysing classical pieces, however we will see later this is not always true. A reliable method to determine the melodic line is by deciding which part of the piece 'one would sing along to'.

We have chosen to begin with a short extract as a means to initially test the code that create the matrix is functioning as expected, as this can be checked by hand. This became our 'base case' and we used this to test all our other codes and revise where needed. Following the method outlined in section 3.1 and 3.2, we quantified each individual note and determine the time reference to be a crotchet, as shown in table 7.

Initially our code as shown in figure 43 would calculate the time difference between two notes at the initial moment they are struck. However it was concluded that this did not give an accurate representation when calculating the number of occurrences of an interval with a time difference. For example, if we look bar 8-9, as shown in figure 13, we can see how by only considering the time difference at which the note is struck, part (a), we would be missing some data compared to if we broke down longer note values into the set time reference. To elaborate on this, if we

Note Value	Time duration
Crotchet ♩	1
Minim 𝅗𝅥	2
Dotted minim 𝅗𝅥.	3

Table 7: Table stating the time values of each note and it's duration in reference to the time unit, chosen to be a crotchet Satie's Gymnopédie No. 1

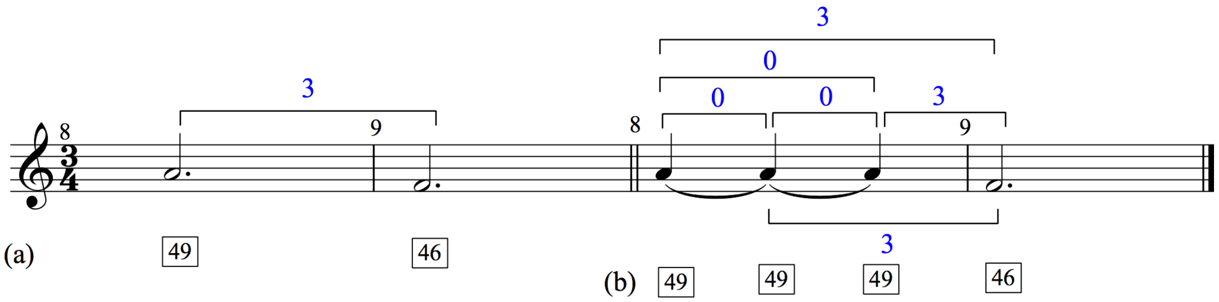


Figure 13: Bars 8-9 taken from Satie's Gymnopédie No 1.. From (a) shows the time values calculated by considering the moment the note was struck, where as (b) we can observe the missing data calculated when considering time to be discretized into it's time unit.

observe part (b) of figure 13, we being by breaking down the minim into three 'tied' crotchets and treating each crotchet as if it was separate notes being struck. A full matrix of bars 5-12 are shown in 14a and 14b, in which the second matrix is created with the code 'discretizing time' as shown in figure 44. From the two figures we can compare and examine the significant amount of data we would be missing if we only focused on the initial time at which the two notes are struck.

Furthermore another encountered problem was the presence of rests, as previously mentioned we considered rests to be a silent note and labelled it a pitch number of -1. However when it came to calculating the matrix size and the number of intervals that would be present, the code would take the note value of -1 into account and therefore not only output intervals that did not exist in our piece, but also a larger matrix was produced than needed. To rectify this, a separate list was created called 'actual_ notes' which ignored all 'notes' of -1. This ensured the the number of cells in each row did accurately reflect the possible intervals and also the intervals outputted did match that of the piece.

Additionally as previously discussed we are able to take the absolute value of intervals or consider positive and negative intervals. This is coded with the option "taking_abs = True (or False)" respectively. Figures 15a and 15b, demonstrates the difference. Reiterating an earlier explanation of how taking the absolute value will affect the matrix layout, we can observe that by taking the first row (using the command "matrix[0][:]", and noting Python begins counting elements at zero) i.e. intervals of a time difference of a crotchet $R = 1$, as shown in figures 16. We can see that in the figure 16a within bars 5-12, there were thirteen occurrences of an interval of zero, meaning that within a crotchet beat the note remained the same thirteen times - this evident from bars 9-12. Furthermore there were two occurrences of an interval of 1, also known as a semitone or minor 2nd interval, within two crotchet beats, or a minim, and so on. However when comparing to figure 16b, firstly the column representing an interval of zero is now the middle column, coded as "matrix[0][15]'. Subsequently we can also see that the two occurrences of a semitone interval is in fact it is one ascending semitone and one descending semitone. We can interpret this result to mean that by including positive and negative intervals we gain a

```
array([[0., 2., 3., 1., 0., 2., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [1., 1., 0., 2., 1., 0., 1., 1., 0., 0., 0., 0., 0., 0., 0., 0.],
       [4., 1., 1., 1., 0., 1., 0., 0., 2., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 2., 1., 1., 0., 1., 0., 1., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 1., 0., 2., 1., 1., 0., 0., 0., 0., 0., 0.],
       [2., 0., 0., 1., 0., 2., 0., 1., 0., 0., 1., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 1., 0., 0., 1., 1., 0., 0., 0., 1., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 1., 0., 1., 0., 0., 1., 0., 0., 0.],
       [1., 0., 0., 1., 0., 1., 0., 0., 0., 0., 0., 0., 0., 1., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 1., 1., 0., 0., 0., 0., 0., 0., 1.],
       [0., 0., 0., 0., 0., 0., 0., 1., 0., 0., 0., 0., 2., 0., 0., 0.],
       [0., 0., 0., 1., 0., 1., 0., 0., 0., 0., 0., 0., 0., 1., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 1., 1., 0., 0., 0., 0., 0., 0., 1.],
       [0., 0., 0., 0., 0., 0., 0., 1., 0., 0., 0., 0., 2., 0., 0., 0.],
       [0., 0., 0., 0., 0., 1., 0., 0., 0., 0., 0., 0., 0., 1., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 1., 0., 0., 0., 0., 0., 0., 0., 1.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 2., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.]])
```

(a)

```
array([[ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  2.,  0.,  1.,
         2.,  1., 13.,  1.,  1.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  1.,  1.,  1.,  1.,  3.,
         0.,  0., 12.,  1.,  0.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  2.,  0.,  0.,  2.,  1.,  3.,
         1.,  0., 10.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  1.,  0.,  1.,  0.,  1.,  1.,  3.,  3.,
         1.,  0.,  8.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  1.,  2.,  2.,  0.,  1.,  1.,  3.,
         1.,  0.,  7.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  1.,  1.,  1.,  2.,  0.,  2.,  1.,  3.,
         0.,  0.,  6.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  1.,  0.,  1.,  1.,  1.,  2.,  0.,  1.,  1.,  3.,
         0.,  0.,  5.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  2.,  0.,  1.,  1.,  1.,  2.,  0.,  1.,  0.,  3.,
         0.,  0.,  4.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  1.,  2.,  0.,  0.,  1.,  1.,  2.,  0.,  1.,  0.,  3.,
         0.,  0.,  3.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  1.,  0.,  0.,  1.,  1.,  2.,  0.,  1.,  0.,  3.,
         0.,  0.,  2.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  1.,  2.,  0.,  1.,  0.,  3.,
         0.,  0.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  1.,  2.,  0.,  1.,  0.,  3.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  1.,  2.,  0.,  1.,  0.,  2.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  1.,  2.,  0.,  1.,  0.,  1.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  1.,  2.,  0.,  1.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  0.,  2.,  0.,  1.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  0.,  1.,  0.,  1.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  0.,  1.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  0.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.]])
```

(b)

Figure 14: Comparison of two matrices, (a) for time values of note being struck, (b) considering of time values being discretized in respect to the chosen time unit.

more in-depth analysis that is arguably a more accurate reflection of the music. However by only taking absolute values we produce the 'bare-boned' structure, only giving an overview of the melodic line.

5.1.1 Predicting the melodic line

We can begin by giving an overview of the general melodic line, by plotting a line graph as shown by figure 17, with blue line as the data points taken from the piece and the black to be the 'overall' melodic line by looking at the line graph only. Looking at the line graph alone, we can evaluate a general descending melody with two peak ending with a static line, as shown by the black line in figure17a. By looking at the score with no musical knowledge we can see that this observation is supported, with the two arches present in bars 5-6 and 7-8, and the peak of each arch is at note number 61, (A) in bar 5 and note number 54 (D) in bar 7. However when analysing the motif from a musical perspective, as shown by figure 17b, the melodic line is different to that initially thought of in figure 17a, comparing the red line in figure 17b with that of the black line in figure 17a. This is because of the musical hierarchy of beats within a bar. For example, looking at the time signature of $\binom{3}{4}$, the notes on the 'strong' beats, beat one and then three of the bar, or the first and then third crotchet beats in this example, are hierarchically more important to the melody than those on 'weaker' beats, beat two, or second crotchet beat. Therefore, even though the melodic line begins on pitch 58, musically it is not deemed to be an important note of the melody due to it's position in the bar. Instead it can be seen as a 'upbeat', or a note to launch the melodic progression, and therefore from a musical perspective, the melodic line begins at pitch 61.

From this example, we establish the importance in analysing the melodic line in relation to it's position in the bar. Additionally we can view the notes between time five to eight to be decorative notes, embellishing the overall descending melody. From this we can conclude that

```
array([[13.,  2.,  3.,  2.,  0.,  2.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.],
       [12.,  1.,  0.,  4.,  1.,  1.,  1.,  1.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.],
       [10.,  1.,  1.,  3.,  1.,  2.,  0.,  0.,  2.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.],
       [ 8.,  0.,  1.,  3.,  3.,  1.,  1.,  0.,  1.,  0.,  1.,  0.,  0.,
         0.,  0.,  0.],
       [ 7.,  0.,  1.,  3.,  1.,  1.,  0.,  2.,  2.,  1.,  0.,  0.,  0.,
         0.,  0.,  0.],
       [ 6.,  0.,  0.,  3.,  1.,  2.,  0.,  2.,  1.,  1.,  1.,  0.,  0.,
         0.,  0.,  0.],
       [ 5.,  0.,  0.,  3.,  1.,  1.,  0.,  2.,  1.,  1.,  1.,  0.,  1.,
         0.,  0.,  0.],
       [ 4.,  0.,  0.,  3.,  0.,  1.,  0.,  2.,  1.,  1.,  1.,  0.,  2.,
         0.,  0.,  0.],
       [ 3.,  0.,  0.,  3.,  0.,  1.,  0.,  2.,  1.,  1.,  0.,  0.,  2.,
         1.,  0.,  0.],
       [ 2.,  0.,  0.,  3.,  0.,  1.,  0.,  2.,  1.,  1.,  0.,  0.,  1.,
         1.,  0.,  1.],
       [ 1.,  0.,  0.,  3.,  0.,  1.,  0.,  2.,  1.,  0.,  0.,  0.,  2.,
         1.,  0.,  1.],
       [ 0.,  0.,  0.,  3.,  0.,  1.,  0.,  2.,  1.,  0.,  0.,  0.,  2.,
         1.,  0.,  1.],
       [ 0.,  0.,  0.,  2.,  0.,  1.,  0.,  2.,  1.,  0.,  0.,  0.,  2.,
         1.,  0.,  1.],
       [ 0.,  0.,  0.,  1.,  0.,  1.,  0.,  2.,  1.,  0.,  0.,  0.,  2.,
         1.,  0.,  1.],
       [ 0.,  0.,  0.,  0.,  0.,  1.,  0.,  2.,  1.,  0.,  0.,  0.,  2.,
         1.,  0.,  1.],
       [ 0.,  0.,  0.,  0.,  0.,  1.,  0.,  2.,  0.,  0.,  0.,  0.,  2.,
         1.,  0.,  1.],
       [ 0.,  0.,  0.,  0.,  0.,  1.,  0.,  1.,  0.,  0.,  0.,  0.,  2.,
         1.,  0.,  1.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  1.,  0.,  0.,  0.,  0.,  2.,
         1.,  0.,  1.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  2.,
         1.,  0.,  1.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  1.,
         1.,  0.,  1.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  1.,
         0.,  0.,  1.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  1.,
         0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.]])
```

(a)

```
array([[ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  2.,  0.,  1.,
         2.,  1., 13.,  1.,  1.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  1.,  1.,  1.,  1.,  3.,
         0.,  0., 12.,  1.,  0.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  2.,  0.,  0.,  2.,  1.,  3.,
         1.,  0., 10.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  1.,  0.,  1.,  0.,  1.,  1.,  3.,  3.,
         1.,  0.,  8.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  1.,  2.,  2.,  0.,  1.,  1.,  3.,
         1.,  0.,  7.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  1.,  1.,  1.,  2.,  0.,  2.,  1.,  3.,
         0.,  0.,  6.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  1.,  0.,  1.,  1.,  1.,  2.,  0.,  1.,  1.,  3.,
         0.,  0.,  5.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  2.,  0.,  1.,  1.,  1.,  2.,  0.,  1.,  0.,  3.,
         0.,  0.,  4.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  1.,  2.,  0.,  0.,  1.,  1.,  2.,  0.,  1.,  0.,  3.,
         0.,  0.,  3.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  1.,  0.,  0.,  1.,  1.,  2.,  0.,  1.,  0.,  3.,
         0.,  0.,  2.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  1.,  2.,  0.,  1.,  0.,  3.,
         0.,  0.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  1.,  2.,  0.,  1.,  0.,  3.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  1.,  2.,  0.,  1.,  0.,  2.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  1.,  2.,  0.,  1.,  0.,  1.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  1.,  2.,  0.,  1.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  0.,  2.,  0.,  1.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  0.,  1.,  0.,  1.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  0.,  1.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  2.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  1.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 1.,  0.,  0.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
         0.,  0.,  0.,  0.,  0.]])
```

(b)

Figure 15: Two matrices comparing the difference between (a) absolute values of pitch differences or (b) considering if they're ascending or descending. For (a) the count for an ascending or descending interval of the same size is combined. In (b) the columns begins with the maximum descending interval (a negative pitch difference), the pitch difference of zero is now in the middle column and the most right column is the maximum ascending interval (positive pitch difference).

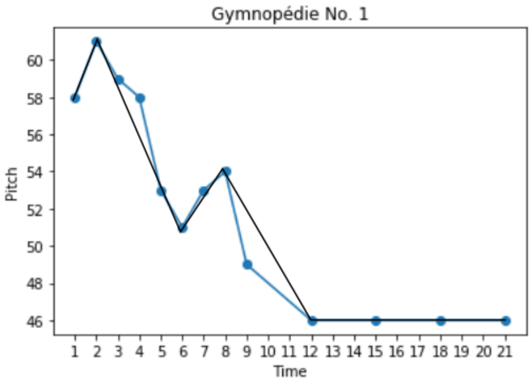
```
array([13.,  2.,  3.,  2.,  0.,  2.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
        0.,  0.,  0.])
```

(a)

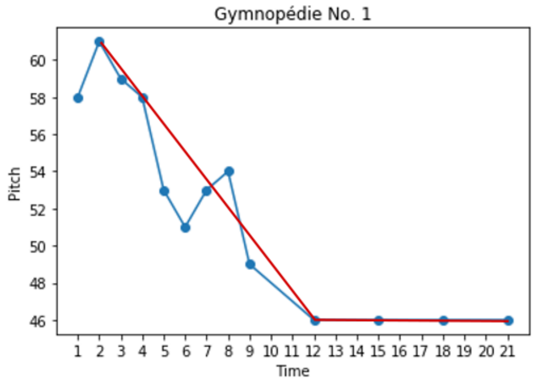
```
array([ 0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  2.,  0.,  1.,
        2.,  1., 13.,  1.,  1.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
        0.,  0.,  0.,  0.,  0.])
```

(b)

Figure 16: Comparing the first row, meaning a time difference of 1 time unit, of two matrices, (a) taking absolute values of intervalsand (b) considering ascending and descending intervals.



(a) An interpretation of the melodic line based on the data plotted alone.



(b) An overview of the melodic line with reference to the score.

Figure 17: Comparison of possible interpretation of the melodic line in relation to the data points alone or with reference to the score. The line is plotted with the time at which the note was struck, in relation to the time unit, and the quantified note number of the pitch.

using relying entirely on mathematical representation of data points to analyse the melodic line would not be representative of the musical understanding. Therefore it is vital to ensure a continually application of musical knowledge in the background when analysing the melodic lines using mathematical methods, or else the results may be interpreted differently to that expected by the composer.
As previously discussed, we can use the line graph to observe for patterns or motifs within the melodic line. Not only can we see a 'repeated arch' motif (time 1-4) , but also we can see that this is almost inverted in time 5-8, therefore suggesting a 'zig-zag' pattern in the melody, which can be founded in the score.

Furthermore we can comment on the rate of the melodic line through this representation of data, to which we conclude that the melodic line is arguably slow moving. For example it takes twelve crotchet beats, counting from time=1 to time= 12, (four bars), to descend from 58 to 46, and interval of an octave An alternative interpretation of this descent is to view it as a semitone fall per crotchet beat.

5.1.2 Predicting the tonality

By plotting a bar chart with the pitch against the occurrences we proposed a method to find the most 'popular' note, and from there an ability to predict the tonality of the piece. However because of the decisions to chose a short extract, we limit the validity of the analysis as this phrase alone does not provide a just representation of the tonality. As evident we can see from figure 18[9], the note with highest occurrence, almost half of the notes, is 46 (F♯), with just under an eighth of the notes struck being 49 (A), followed by an equal number of occurrence of notes 53 (C♯) and 58(F♯), and finally the 51 (B), 54 (D), 59 (G), 61 (A), each only occurring once throughout the phrase. The note number 46 having the highest occurrences can be explained is not due to a relation to music theory but is in fact because of the tied dotted minims in bars 9-12., similarly with note 49 with bar 8. The two notes 53 and 58 are not what we might expect to have the second highest occurrences in a tonality of D major, especially the C♯ as this is the leading tone. However an explanation to this is due to the melodic line again, and we argue that

[9]There is some error with this code as the last bar should be over 61 with an occurrence of 1, 60 should have zero occurrences. Therefore for all graphs created using this code, the last plot is checked with the score and adjusted if needed.

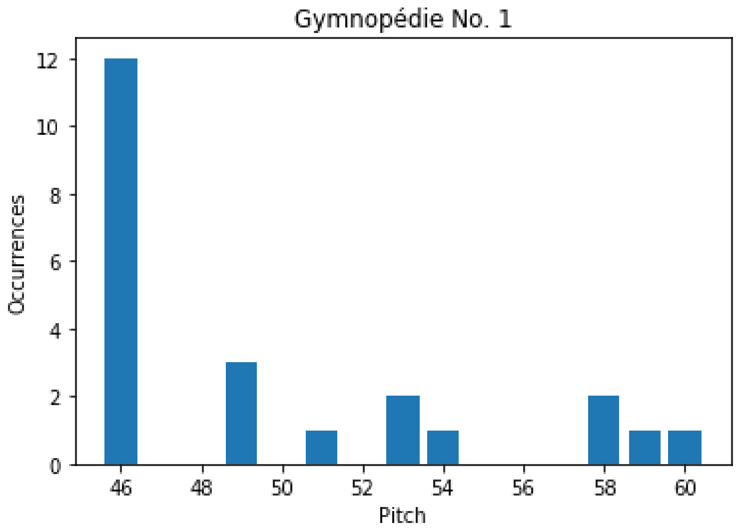


Figure 18: A bar chart plotting the quantified note number against the number of times it is struck in the extract.

the two notes are the 'mean' of the two arches, with 58 to be mean of the melodic arch in bar 5-6 and 53 in bar 6-7. One comment we can observe from this bar chart is the absence of pitch numbers 47,48, 50, 52, 55, 56, 57 and 60. We argue 48, 50, 52, 55, 57 and 60 are not included due to off-chance but instead because of the tonality of the piece. The corresponding notes are G♯/A♭, A♯/B♭, C♮, D♯/E♭ and F♮ and G♯/A♭, all of which would be unconventional in a piece where the only accidentals are C♯. and F♯.

5.1.3 Interval Analysis

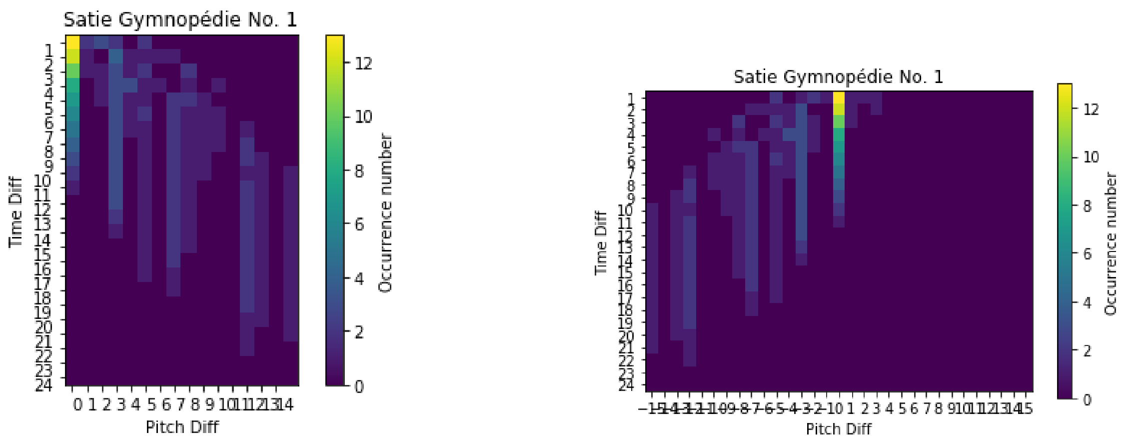
The following analysis will now focus on the interval size as a relation to time difference.

Beginning with a heat map analysis of bars 5-12 of Satie's Gymnopédie No. 1, as shown in figure 19 and taking absolute values of the interval, we can observe from figure 19a, as expected, the highest occurrence are ones with lowest time difference and smallest interval (lowest pitch difference). From this we can interpret the phrase to melodically be compact, the notes are predominantly clustered together. Remarkably,there appears to be a linear (diagonal) decrease, showing that as the time difference increases, it opens more possibilities for larger intervals. From looking at figure 19b, we gain a better understanding of the phrase and can determine that almost all the intervals are descending ones.

5.1.4 Recreating the melody using statistical method

We hypothesized that we can re-create the melodic line through predicting the next note for a set time difference, through the use of a frequency function on the interval. We test this hypotheses by beginning on the pitch number 58, the same as the beginning pitch in the Satie's Gymnopédie No.1., with a time reference of a crotchet,R=1. Figure 20 calculates the weighted percentage of each intervallic difference for R=1. Table 8 is calculated from figure 20, to three significant figures, and states the percentage of the melody'landing' on that note in a given time difference, a crotchet, t=1, two crotchets, t=2, and three crotchets or a bar, t=3. The highest percentage is highlighted in red, and the second in blue.

We can observe from table 8 that the statistical model predicts the melodic line to be predominantly static for all time differences stated, notably an approximately 50% chance of remaining



(a) (b)

Figure 19: A comparison of heat maps, (a) taking absolute values of intervals (b) considering both ascending and descending intervals. The heat map, plotting the pitch difference between two notes and the time difference at which the notes are played, demonstrates the number of occurrences in a visual way, with the highest occurrence represented by the colour yellow and decreases to dark blue.

on the same pitch a crotchet beat later and almost 25% chance of remaining on the same pitch a bar later. The second highest possibility for a crotchet beat later are equally a major second descent or a perfect fourth descent. Without referring to the score, this does seem plausible as both a major second and perfect fourth are conventional intervals used in melodic lines and will not disturb the harmonic structure of the piece. However as the time difference increases to two and three crotchets, we can see that the statistical probabilities favours an pitch difference of a major second descent over the perfect fourth. A potential interpretation of this observation is that it suggests that the melodic line on a small scale is more disjoint, in comparison to the overall structure of the melodic line. Furthermore, it can also be interpreted that for every disjoint motion, there is some movement in the opposite direction to balance this, in order to satisfy the prediction of a small interval in a longer time difference. In other words, focussing on a time difference of three, it can viewed as within a bar there is an excursion in the melodic line, however there must be a point at which the movement begins to gravitate back towards the initial pitch a bar before.

It must be noted that such interpretation are arguably dubious for a number of reasons. Firstly, this results are based on a small amount of data points, half of which are dominated by a tied note (bars 9-12). Therefore the static motion is not a true representation of the whole extract, but a consequence of this subset of bars. In fact, when comparing to the score, we can see for bars 9-12 this statistical model has an 100% accuracy in predicting the melodic lines in these bar, however when analysing the first half of the motif (bars 5-7), we do not reach the same conclusion when evaluating the validity of this model for all time differences. Comparing with Satie's score, a crotchet beat away from the initial pitch of 48 is 61, then 59 in two crotchet beats and finally 58 a time value of a bar later, while we observe the model does accurately predict pitch a bar later, the model predicts a melodic descent first to a note value of 56 or 53, before either ascending or remaining on 56. Comparing this 'real' result to the one predicted by the model we can see that it is significantly inaccurate. Additionally, when choosing a different starting point, for example the first beat of bar 6 or the first beat of bar 7, we can see that the claimed 100% in accuracy of predicting the note a bar later, is in fact not true.
Secondly, while we do see this 'balancing of disjoint movement' especially in bars 5-6, in which

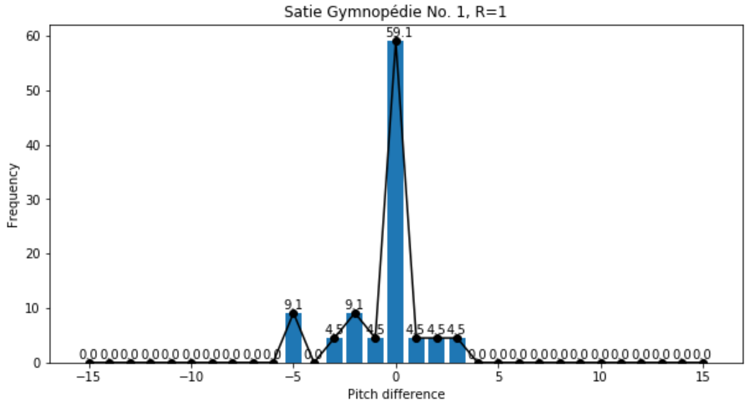


Figure 20: A bar chart plot of the pitch difference against it's frequency, for a time difference of R=1, or a time difference of a time unit

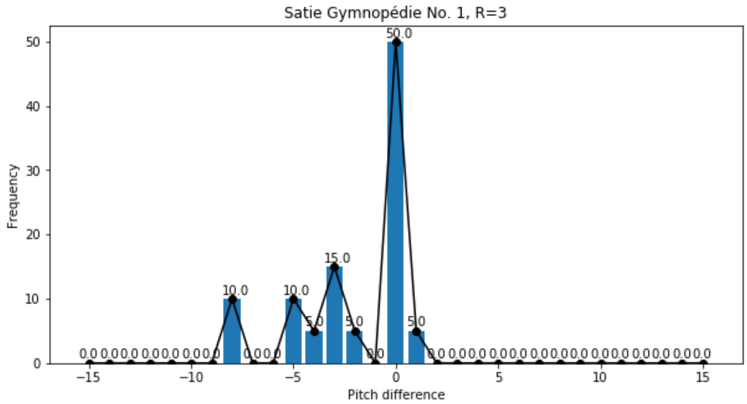


Figure 21: A bar chart plot of the pitch difference against it's frequency, for a time difference of R=3, or a time difference of a bar

an initial leap of three semitones is in fact balanced by a descending fall which does gravitate back to the initial note of 58 a bar later, this is not a generalised theme across the whole melodic line. For example this is not the case in bar 7-8 where the larger pitch difference of a perfect fourth (between 54 and 49) is not balanced by any movement in the opposite direction, as predicted by the statistical method. Instead there is a further descent to note 46, therefore we find an even larger interval a bar later as oppose to a gravitation back toward the initial note.

We test this model again, however this time we choose our time reference to be three crotchets, in other words a bar, R=2. We follow the same method as we did when using a crotchet as our time reference, and produce figure 21. Similarly to the previous example by using figure 21 we create table 9.

We hypothesize that table 9 may produce a more representative set of results, especially for a time reference of a bar, in comparison to 8, this could be arguably due to the statistical method being able to predict larger scale systems as oppose to smaller scale. We propose to compare this hypothesis with that of turbulent flow and if our proposal is valid, we can interpret the statistical method used to predict melodic lines is consistent to that predicting the velocity in a turbulent flow. We reiterate that the use of statistical method in predicting the velocity is due to irregularity of the flow, and that statistical methods would be deemed impractical in predicting small scale changes, for example being unable to describe the exact 'journey' of the molecule in the fluid accurately, however could give a reasonably accurate prediction to the

	t=0	t=1	t=2	t=3
46	0	0	0	0.00904
47	0	0	0	0.0167
48	0	0	0.0828	0.0607
49	0	0	0	0.00794
50	0	0	0.00820	0.0286
51	0	0	0.0166	0.0370
52	0	0	0.0102	0.0204
53	0	0.0910	0.116	0.0111
54	0	0	0.0205	0.0405
55	0	0.0450	0.0696	0.063 1
56	0	0.0910	0.122	0.121
57	0	0.0450	0.0649	0.0764
58	1	0.591	0.365	0.240
59	0	0.0450	0.0824	0.0841
60	0	0.0450	0.0526	0.0636
61	0	0.0450	0.0699	0.0702
62	0	0	0.0609	0.0489
63	0	0	0.0406	0.0414
64	0	0	0.0203	0.0197

Table 8: Table calculating the probability of landing on set pitch with respect to a time reference, t=0, 1, 2, and 3, with respect to the time unit being a crotchet in Satie's Gymnopédie No. 1.

		(time difference of 1 bar)	(time difference of 2 bars)
	t=0	t=1	t=2
42	0	0	0.0100
43	0	0	0
44	0	0	0
45	0	0	0.0200
46	0	0	0.0100
47	0	0	0.0300
48	0	0	0.0200
49	0	0	0.0100
50	0	0.100	0.133
51	0	0	0.0350
52	0	0	0.0275
53	0	0.100	0.115
54	0	0.0500	0.0625
55	0	0.150	0.155
56	0	0.0500	0.0650
57	0	0	0.00500
58	1	0.500	0.250
59	0	0.0500	0.0500
60	0	0	0.00250

Table 9: Table calculating the probability of landing on set pitch with respect to a time reference, t=0, 1, and 2, with respect to the time unit being a three crotchets, or a bar in Satie's Gymnopédie No. 1.

overall flow. In the same argument, if our hypothesis is proved true, we can argue that the melodic line, similar to that of turbulent flow is irregular, and therefore it is unreasonable to expect a statistical method to be able to predict the note progression per time reference, and instead can accurately give a description to the overall melodic progression.

Again from table 9 we can see that an interval difference of zero dominates, and can again suggest that it is due to the tied minims in bars 9-12 as it was in table 8. Therefore from this table alone we would interpret the large-scale melodic line to either be made up of static movements of an overall descending movement made up of a fall of three semitones. However, similarly to the case when we took our time reference to be the time unit, when referring to the score we can see this interpretation is not representative of the melodic line, especially in bar 5-8. Although there is a overall descending line present in those 4 bars, the descending interval of the minor third, per time duration of a bar, is not as dominating as the statistical model predicted. Therefore as both models do not predict the melodic lines to a high enough accuracy rating, it is difficult to conclude whether the statistical model would be more appropriate in predicting large or small scaled systems.
Furthermore, assuming the calculations are carried out correctly and to a high degree of accuracy, we would expect that the frequencies for table 8 at t=3 to match that of t=1 in table 9,as they both representing a time difference of a bar. However we can see that this is not true in the slightest. We can comment, while the actual frequency value is significantly different, the note with the highest probability is consistent with both tables. We conclude that this method proposed does provide a foundation to recreate melodic lines based on statistical methods, however perhaps due to human errors or a lack of accuracy will need to be reviewed and corrections applied where necessary.
We summarise that the statistical model produced by figure 20 and 21 does not reliably give an accurate prediction of the melodic line, but can if the right bars are chosen, therefore is not representative of the whole extract. However we should not dispute the use of statistical method to predict a melodic line, a parameter that may not have an obvious pattern, entirely. We can see from analysing a short extract that there does seem to be some validity of this hypothesis, however perhaps it is due to the lack of quantity of data that is comprises the method, as oppose to the method itself. It may be the case that if more data is inputted, the model becomes more accurate in predicting an overall structure, as oppose to highly accurate in some sections and significantly not in others. Furthermore if we were to recreate this statistical model with more data and the same observations are concluded, the argument linking it towards the theory of turbulent flow could be strengthened significantly, or ultimately disputed. The difficulty of executing such a claim is that the mathematical calculations can become very complex, and therefore constructing a code to carry this out would be advisable.

5.2 Bach Minuet in G Major

The next piece chosen to analyse was Minuet in G Major by J.S. Bach. Again the melodic line is easy to locate, being the top line (or the right hand part for this piano piece). We begin by following the same method outlined to quantify the notes and determined the time reference using a quaver as our time unit now, as shown in table 10, and input these results into a datafile. We note that for the purpose of simplicity, we ignore ornamentations and only quantify the principal note. For example in bar 8, we ignore the appogiatura, and quantify note 49, and in bars 11 and 13 we ignore the lower mordent, and consider only the notes 52 in both bars.

Note Value	Time duration
Quaver ♪	1
Crotchet ♩	2
Dotted minim 𝅗𝅥.	6

Table 10: Table denoting the note values with its time duration in reference to the time unit in Bach's Minuet in G Major.

Section A	**Section B**
Tonic	Dominant→Tonic
I	V→I

Table 11: Table representing the expected tonal structure in a Minuet of the Baroque period. The Minuet conventionally begins with section A, in the tonic key, I, followed by section B beginning in the dominant V and then modulating back to the home key, the tonic.

5.2.1 Minuet

Before we begin analysing the piece using mathematical methods, we comment on the musical background of a minuet. We note that, whilst later minuets were often of ternary form, this binary structure was considered a conventional of minuets composed in J.S. Bach's period. There are two main sections, Section A being from bars 1-32, and Section B from bars 33-64 (including repeats the end of each section), additionally we can see the sections clearly divided with repeats. Furthermore the minuet itself was originally composed to accompany an "aristocratic social dance [that is] dignified, graceful [and] relaxed"[19, Minuet] of time signature $\binom{3}{4}$. The conventional tonal structure is as shown in table 11.We can see that Bach follows this conventional structure, beginning with the tonic of G major through section A, and beginning with the dominant key D major in section B, before modulating back to the home key at the end. Furthermore reinforcing these tonalities are the use of perfect cadences, as shown in bar 15-16 for section A, and 45-48 in section B.

5.2.2 Predicting the tonality

We can see that the Minuet in G does follow the conventions of it's time both structurally and tonally, therefore we argue that listeners would not describe this piece to be 'unpredictable' or have any 'unexpected' characteristics. From which when determining if we can find turbulent characteristics within music, this may be argued to not be an appropriate piece to test the validity of the hypothesis fairly. However, the advantage of analysing a piece that is built upon a consonant melodic line in which everything 'behaves as expected' is that firstly, we now have a larger set of data in comparison to the Gymnopédie No 1., now with 383 data points in comparison to the 23 data points initially analysed in the Gymnopédie, and so arguably any conclusions achieved from analysing this larger set of data is more generalized and a more accurate representation of the piece. Secondly, we can now focus on interpreting the data without worrying about any accidentals that may corrupt or influence our interpretations, especially looking at a histogram plot of the pitch number against the occurrences, and the line graph depicting the melody in a graphical representation.

We begin with a histogram plot and a large amount of our analysis will be focussed on section B, as this is the section with two established tonalities. We have already explained what we might expect the plot to look like one tonality, so it would be interesting to determine if

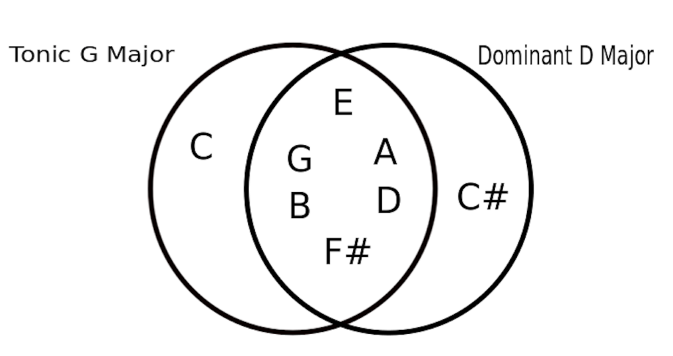


Figure 22: Taking the tonic and dominant keys as set of notes found within each tonality, we denote the set of notes that are consonant in both keys by the intersection of the two sets.

plotting a section of two tonalities will results in two 'expected graph'-like shape next to one another, or if the added tonality will disorientate the data until there is no obvious relation.

As shown by figure 22, because of the close relation between the tonic and dominant key, as expected they will share the majority of notes, differing in only one note, C♮ in G major and C♯ in D major. Therefore when analysing section B, especially, we already prepare for the appearance of both, and can disregard these as abnormalities from the tonality.

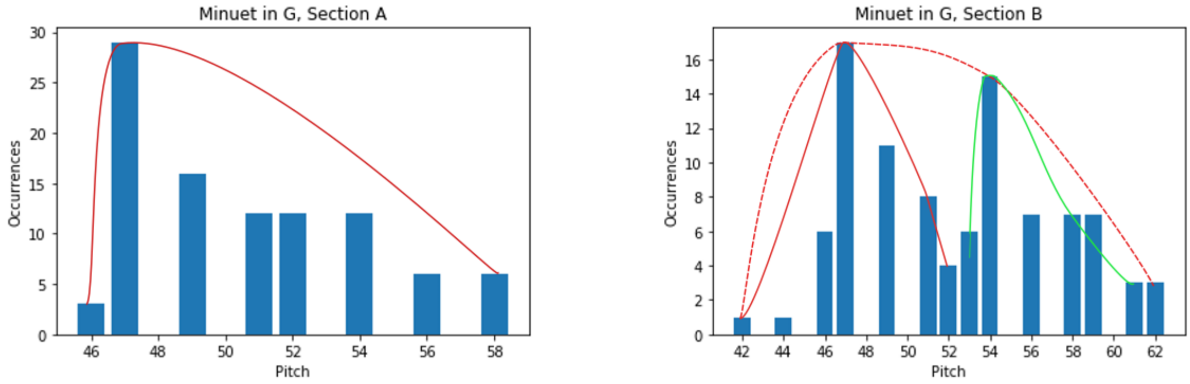
From figure 23, we can compare directly the difference between section A , which is in one key, as shown by figure 23a, and section B, that is in two, beginning in D major and then modulating back to G major, as shown by figure 23b.

We observe that the histogram plot in figure 23a, is consisted to that expected, as explained in section 3.3.2. Furthermore we can argue, through a simple observation that the first section is dominated by the note number 47, and that the melodic lines exhibits some ascending runs whilst always gravitating back to 47. However when comparing to the score this is not the case, in fact the note 47 is not as important as it may be interpreted from, but instead is a pedal note, with it main purpose to support the harmony. This reiterates the complexity of using mathematical methods to interpret a musical piece, as initially when predicting the usefulness of each method, we had assumed that all notes within a melody would have equal weighing of importance, such that they all would 'sound the same'. However this is not the general case in all melodic lines, as we see from figure 23b (and later in figures 24 and 26). We can conclude that by relying only on mathematical analysis, we can often misinterpret the composer's intention, and that the interpretation of a melodic line is not as simple as quantifying notes into numbers and executing rigorous mathematical methods. Instead it is made up of many complex parameters, some of which, for example the degree to which a note is important, is difficult to conclusively quantify.

Figure 23b shows the complexity of introducing multiple tonalities into a section. We note that within the three interpretations, the expected shape is still visible, further supporting that a universal shape for this plot may exist based upon the assumption we made in section 3.3.2. From harmonic assumption of the piece, we assume each note that is struck must belong to one of the two tonalities, and there does not exist an dissonant note for this section. Furthermore another factor taking into consideration when drawing the line of best fit, is that we already know note 52 (C) belongs to G major and note 53 (C♯) belongs to D major, and each are not used for the purpose of being dissonant in the other key. Our first observation is that two peaks, being at 49 (G) and 54 (D), and interpret these to have the highest occurrences because they are the tonic note of their respective tonalities. We then'force' the line of best fits for both the tonalities (G Major in red, and D Major (V) in green), through the use of our knowledge of triads and again the assumption that these will have higher occurrences in comparison to notes outside of the triad. We fit the red line to the following notes 42 (D, dominant), 47 (G, tonic), 51 (B, mediant) and 52 (C), and fit the green to 53 (C♯), 54 (D, tonic), 58 (F♯, mediant), 61 (A, dominant). Furthermore we can also argue there is an underlying G major character throughout section B, as shown by the dotted line in figure 23b, in which the notes fitted are 42 (D, dominant), 49 (G, tonic), 54 (D, dominant), and 63 (B, mediant). From this interpretation, even thought there a modulation to the dominant key of D major, the tonic is still preserved. This supports that claim that melodic line in this minuet is harmonically consistent, as expected of a piece composed in this time. Furthermore it can be argued that this suggests the melodic line revolves around the triad of G major, further supporting the harmonically consonance of the piece. Additionally we can suggest the the section in G major is in a lower register to that in D major, whihc is supported by the score.

However we cannot claim all these interpretations to all be universally true, as we have forced the lines of best fits through assumptions based on musical theory. For example, it may not be the case that the majority of notes within a triad is struck within the section of that tonality. In order to fit our line of best fit, we assumed that within G major, for example, there would be a higher occurrence of note 51 than in D major, as it is the mediant in G major and the submediant in D major. However this may not be the case, in fact there could be a melodic sequence that ascends and descends with note 51 as the average note, therefore it is not an 'important' note tonally, but will have a high number of occurrence.

To summarise, even when analysing a section with only two tonalities, we can already see the complexities into attempting to decipher the separate tonalities within a histogram combining both data. The main issue being that the majority of notes are shared by the two tonalities and therefore there is a complexity in deciding how to divide the 'height' of the bar into the percentage of the number of times the note was struck within the section of each tonality. For example, focussing on note 47, we do not know from figure 23b alone, what percentage of that note being struck was in the section G major in comparisons to the section of D major as it is highly likely to be in either. This problem is a direct consequence of the two tonalities being closely related. Perhaps, for future research into the usefulness of the method, a section should be chosen in which the tonalities do not relate to each other in any way, known as being extremely distant, for example a section that initially is in the tonic, for example C major and then modulates to the flatten submedient key, $G/flat$ major. However it must be noted, this may not be possible as it is unlikely to find a piece in which the two tonalities are harmonically stable enough to confidently be labelled as that tonality, but also be able to modulate to the extremely distant key successfully. Another comment is this method is most effective when analysing a piece in which the differing tonalities are all at different registers, although this is not a realistic expectation to be found in music.
From the evaluations stated, we can argue, that without some adjustments, this method to determine tonalities of a piece should not be extensively relied on as there are too many musical factors that can mislead the results interpreted from a mathematical perspective. This is not to render the method unworkable or completely inaccurate, as we have also commented on the useful of this method, but instead should be taken as precautionary comment, explaining the areas should be adjusted in order to increase the validity of any conclusion made using this method.



(a) (b)

Figure 23: Plots of two bar charts, (a) using the data from section A and (b) the data from section B. The red line denotes an approximated probability distribution function, with the reference of the tonic key, where as the green in (b) is in reference of the dominant key. The dotted red line in (b) represents an interpretation that whilst two tonalities are present, there exists an underlying presence of the tonic key throughout.

5.2.3 Predicting the melody line

We attempt to analyse the melodic line of each section more closely through the use of line graphs. Beginning with section A, we can observe from figure 24 that there can two interpretations of the plotted data. The first, figure 24a is what is argued to be an interpretation based on the data plotted alone, whereas figre 24b is the melodic line drawn using the score. From figure 24a alone, we might interpret the peaks of the line graph to be ‘turning points’ of the melody, and the main purpose of the melodic line is to drive it towards these extremities. However when referring to the score, we find that this is not the case. Again this reiterates the idea that the note at a set time value, not only within a bar, but also within a motif there is also a hierarchy of importance, and this is due to the phrasing of melodies. For example, looking at figure 24a, the climax of the interpreted melodic line is as follows, 59, 46, 59 and 47. However if we locate these alleged climaxes in reference to its position in a bar or in a phrase, we can quickly dismiss them as being notable notes of the melodic line.

If we analyse the score alone, we can see that section A is divided into two sections, the second being a repeat of the first. Each section comprises of four phrases, therefore the first note of each phrase conventionally is deemed as the most important[10] and a natural accent is placed upon this note. The first phrase is made up of the first motif shown in figure 25a, repeated, and the second phrases comprises of three appearances of the second motif shown in figure 25b before ending the phrase with a dotted minim. Then there is an almost exact repeat, only differing the last two bars in order to satisfy a cadential progression. We can observe the graphical representation of the melodic line in figure 24b.

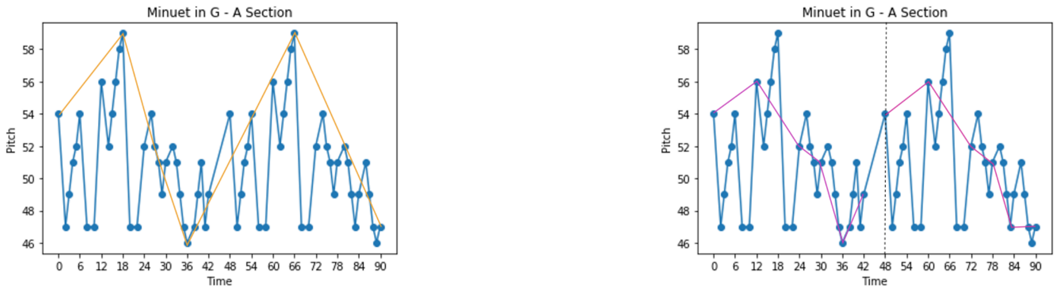
Returning to the analysis from figure 24a, we can now demonstrate why the climaxes interpreted form the data alone are not representative of the melodic line. Whilst the last alleged climax is a valid interpretation, the first three are note because of it’s position in the phrase. For example, if we observe the first ‘climax’ of 59 at a time reference of 18, whilst this is at the beginning of the bar, it is in the middle of the first phrase. Therefore musically it is perceived as a ‘decorative’ note within the phrase. A similar explanation can be given for the next two ‘climaxes’.

[10]This is demonstrated in the score

Furthermore, another aspect that we can ascertain from a line graph is if the melody is predominantly made up of conjunct or disjunct movement. We can do this through observing the gradients, for example a large interval would be represented by a steep gradients - a large change in pitch, in a short time difference. We can deduce from this that the melodic line does comprise of many interval falls, often balanced by conjunct movement in the opposite direction. Whilst the initial hypothesis claimed that a melodic line with many intervals could be the deciding factor onto whether a piece is described as turbulent, we can see from analysing this piece that this is not always the case. For example, in this section we can many steep gradients, which are in turn interpreted as a large interval, therefore from the graph alone we would predict that this piece may sound 'turbulent'. However this is not true as the majority of the large intervals are comprised of a fall of a perfect fifth, which is not unusual and therefore not 'unpredictable' or 'unexpected', providing an explanation to why this piece is often not described to be 'turbulent'. From which we can conclude that we cannot claim that large intervals will always equate a 'turbulent' piece. In fact, it would be more appropriate to adjust the hypothesis to specify that large intervals that are unconventional would be more appropriate in evoking a turbulent feel.

A further comment is that for both analysis, without and then with the score, there still seems to exhibit a few of the common melodic features that Huron has suggested. Firstly the melodic line does displays a small level of pitch proximity, with ascending pitches that are often no more than an interval of two pitches, in other words, ascending scalic movements made up of semitones or tones. Secondly, the melody does exert the element of central pitch tendency, as we can see that for every movement in one direction, it is followed by a return in the opposite, and therefore arguable is because the melodic line gravitates back to the 'centre'. Lastly, we can see quite clearly from the data points alone that the melody does exert some arch phrase tendency. Whilst, as explained the placement of the arches do differ depending on if the interpretation relied on musical knowledge or not, it cannot be denied that there these arch-like contours do exist within the melody. It could suggested that regardless of how we chose to interpret this set of data, the tendency of arch-like melodies could be an underlying phenomenon that naturally occurs in all melody, whether the composer had consciously intended it to. We note that this cannot be generalised to all pieces or melodic lines that Bach had composed or were composed in the same period, as it could equally be claimed that this was a coincidence just for this piece or even just for this section.
Overall this analysis support Huron's claim that there are some universal organizational elements that are used comprehensively in music. Furthermore as we have determined that these melody does exhibit common features, we can conclude again that we would not expect this to be described as 'turbulent'.
From analysing the relation of the pitch against the time reference that it is struck, we can reiterate further a previous statement in which commented on the representativeness of applying mathematical analysis on a set of data points to recreate the melody line of a piece. We can conclude that whilst this is an useful tool as a starting point, it must be used with some musical theory knowledge in order for it to be more accurate, for example understanding of the importance time reference, whether in a bar or a phrase, at which a note is struck.

We repeat this method with the the melody line is section B, as shown by figure 26. From this set of data points, it is arguably more difficult to determine a pattern, from the data points alone, in comparison to section A, and so two options are given. The first, the light blue line in 26a, suggests that section B comprises of two descending lines, The second, in the orange, suggests one singular descending line. However similarly to that in section A, in reference to the score, section B is divided into two sections, although in this section this is due to the change in



(a) (b)

Figure 24: Two line graphs denoting the graphical representation of the melodic lines in Section A. (a) is an interpretation based on the data alone, whereas (b) is the overall line with reference to the score. The comparisons is used to explain the importance in considering the importance of the note struck in relation to it's position in the bar.

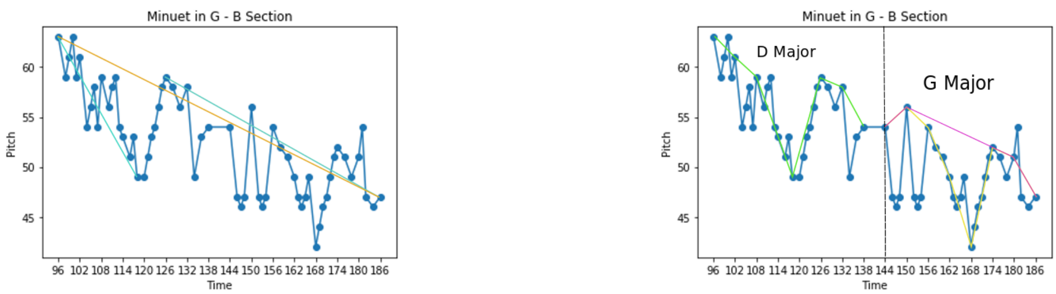
(a) Motif 1, taken from bars 1-2 (b) Motif 2, taken from bars 5

Figure 25: Two motifs that presented in section A of Bach's Minuet in G

tonality than a motivic reason, as demonstrated in figure 26b.This observation alone, discredits both interpretations of the data points in 26a. While we can state that this method is arguably ineffective to determine the overall melodic line especially in a set of data points in which the sections are divide by tonality changes, we still contend that it does have it's advantages in regard to detecting similar motivic patters. For example, if we refer to figure 24b again, in the second half, as shown by the pink line, we have a rhythmically compressed variation of the phrase in section A, which may not have been noticed with the score alone.

5.3 Beethoven Für Elise

The next piece we chose to analyse was Beethoven's Für Elise. This piece was chosen because as an attempt to 'move towards' pieces that could be described to be 'turbulent'. Similarly to van Gogh, Beethoven also suffered suffered from physical and mental adversities. It is noted that



(a) (b)

Figure 26: Two interpretations of the line plot of the melody in section B. (a) is based on the the data points alone and two possible interpretations are given in the blue and the orange. (b) shows the interpretation with the use of the score, and we note the return of motif 1 from section A, as denoted by the pink line.

Note Value	Time duration
Semiquaver	1
Quaver ♪	2
Crotchet ♩	4

Table 12: Table denoting the note values with it's time duration in relation to the time unit, a semiquaver in Beethoven's Für Elise.

in his last years he suffered "complete deafness and frequent illness [...].His chronic mistrust of individuals intensified and he viewed with increasing bitterness a world that had dealt him harsh personal failures and mixed profession success. Perhaps he also realized that his appearance and behaviour were perceived as ever more eccentric"[29, p. 18]. Therefore from this, we argue that Beethoven's pieces could be a musical reflection to that of choosing van Gogh paintings to investigate turbulent characteristics. We postulate whether Beethoven's state of mind would be translated in his composition in the similar way that was hypothesized for van Gogh's paintings.

Whilst arguably there are many compositions by Beethoven that are regarded more 'turbulent', for example the fourth movement of Beethoven's Symphony No. 6, literally titled *The Storm Movement*, however as it would not always as simple to establish the melodic line due to it being a orchestral piece. Furthermore, the added number of instruments would add additional complexities to the analysis, for example interplay between parts, harmony between parts and also the influence of dynamics[11]. Therefore for simplicity, we have chosen a piano piece, from which in theory, the melody should be easy to extract as with previous pieces analysed.

We postulate that based on looking at the score alone, the continuous chromatic movement, Für Elise could is arguably sway the piece towards being described as turbulent, as a result of continuous dissonant intervals movements. However in reality it is not regarded to be 'unpredictable' by the majority, and arguably one of the most famous melodic lines of which almost everyone can sing by heart. It opens the question to whether it is not 'turbulent' because it does exhibit conventional and common melodic features, as stated both Huron's organizational elements and follows the I-R model, or if it's because it is recognized by the majority that it becomes learnt and the 'turbulence musical characteristics' become familiar and therefore becoming part of our built in 'predictive system'.
We focus on analysing bars 1-30, the 'famous' melodic line.

Again the melodic line is in treble (the right hand), and we quantify the notes as per the method stated. In this piece our time unit is now a semiquaver, and table 12 illustrates the time values.

5.3.1 Analysing the melodic line and evaluating it's degree of predictability

We begin by analysing the melody with the use of a line graph plots the time reference at which the note is struck against the pitch number. We note that because of the two semiquaver upbeat at the beginning of the bar, the x-axis ticks are not aligned with each bar, but instead the note aligned with a 'ticked' time reference is actually the note two time units into the bar as oppose to the first note of that bar.

We can observe the division of this extract into two section, as shown in figure 27, as shown in the score by the repeat signs. Additionally we can see that the first section comprises of four repeats of what we have labelled motif 1, that is essentially made up of an initial descent

[11]This will further discussed in section 6

before two ascending lines in an almost 'climbing' movement. The second section is made up of three repeats of motif 2, followed by four repeats of motif, and concludes with an initial variation of motif 1 and then an almost exact copy of motif 1 with a slight deviation towards the end.

Analysing each motif in more detail, we can begin to evaluate the degree of 'turbulence' by comparing with parameters in the I-R model. For example if we analyse motif 1 as shown in figure 28a, we can observe a chromatic 'trill' motif followed by 'snake-ing' descent, and then followed by two arpeggiated ascent. The first part of motif 1, whilst chromatic movement does follow the proximity principle as outlined by Narmour's I-R model. Furthermore other parameters features include satisfying the closure, intervallic difference and registral return element, as for all large intervals are closed with a change in direction and a smaller interval, additionally all small intervals are followed by an interval of similar size. Therefore some from this analysis, as the small chromatic motif satisfies four out of the five principles outlines by the I-R model, it can argued that this would not exert a 'turbulent' mood. Motif 2, as shown by figure 28b, clearly exhibits the arch phrase tendency that Huron claims to be a common element found in melodies, of which all five principles of the I-R model are present within this short motif. Again arguing that there is a very conventional melodic technique to be used. Motif 3, on the other hand, comprises of large interval leaps,and therefore according to our hypothesis, exhibiting a turbulent characteristic. However as the leaps are made up of octave leaps semitone intervals, we conclude that this is not the case. If we were to analyse the octave leaps within this motif, based on the I-R model alone, as it fails all five principles, and therefore again we would claim this would be classified as highly 'turbulent'. Whilst there is a change in direction, the note followed after is not of an smaller interval, but instead retains the same interval, when analysing this motif musically, it is far from being 'turbulent' as firstly it comprises of octave leaps, around the dominant of the key, in which Schellenberg argues has a high stability value [27, p. 83].
From an analysis of the melodic line, we conclude that this extract would be unlikely to described as 'turbulent' because of it's melodic construction as oppose to the idea that is is 'learnt'. Additionally, any areas which arguably violated the I-R model can be easily justified with the use of musical theory knowledge.

5.3.2 Analysing the intervallic differences

We move onto interpreting the interval difference with the use of the heat map as shown by figure 29, from which we can argue supports the claim that the 'famous' melodic line, whilst being predominantly made up for chromatic movement is not 'turbulent' by our definition of 'musical turbulence'. We return to our main hypothesis of which we proposed that large intervals would results in a melodic line inclining towards becoming 'turbulent', however as previously argued that we must consider the relation between pitches as well. From analysing this short extract of Für Elise, we again must refine our hypothesis and state that it is not as simple to therefore suggest a melodic line is deemed 'turbulent' if there are a high occurrences of large intervals and that the intervals are dissonant, instead perhaps it is the combination of the two which is the decisive factor.

Another comment that we can evaluate about the melodic line from the figure 29 is there arguably is a horizontal reflection across the interval difference of zero. An interpretation of this is that the overall extract is 'intervall-y' balanced, meaning for every occurrence of an interval difference for a specific time difference, there is an equal number of occurrence of that same interval difference in the time difference, but in the opposite direction. This supports the argument that this extract exhibits the 'registral return' principle of the I-R model, which then supports that claim that this is not as turbulent as initially proposed. We can see the highest occurrence of of any interval difference against time difference is at an interval difference of zero (meaning the same note is struck again), at a time difference of approximately four bars. If we

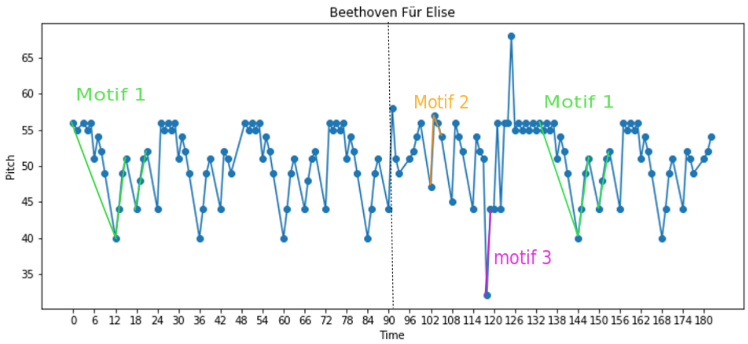


Figure 27: Line graph providing a graphical representation of the melodic line , plotting the quantified pitch against the time, in reference to the time unit, at which it was struck. We note there are two sections, the first section consists is of repeats of motif 1. The second begins with a sequence made up of motif 2, then sequence made up motif 3 and finally returning back to a two repeats of motif 1.

(a) Motif 1, taken from the upbeat and bars 1-4

(b) Motif 2, taken from bar 17

Figure 28: Two motifs present in Beethoven's Für Elise

translate this into the score, the representation from the heat map suggests that between any two notes, it is often exactly the same after four bars, in which a possible solution is that these four bars are essentially copied and pasted across, and therefore each note within those four bars are struck again four bars later. From which we can conclude that this high occurrences in the heat map is representative of the first half of the extract. In a similar sense, the second highest occurrence in the heat map, is also an interval difference of zero, however now the time difference is approximately eight bars, representing the repeats of the two sections. We summarise by explaining that the five highest occurrences with an interval difference of zero is a result the combination of repeated motifs and repeat signs.

5.3.3 Relating to the theory of turbulence

The next section of analysis will follow closely to that proposed by Aragón in which the natural logarithm of the nth order statistical moment , $\ln(\langle(\delta u_R)^n\rangle)$ and the natural logarithm of the time difference, ln(R) are plotted against one another and if linear relation is can be determined then we can conclude that it does exhibit similar turbulent characteristics, Kolmogorov's theory of turbulence. (We recap that δu_R is defined as the interval between two notes of a set time difference, R).
In previous analysis, we has already provided methods and assessed the degree to which a melody could be described as 'turbulent', however this was in relation to it's musical definition, as defined with the use of the I-R model and Tempesta characteristics. In order to provide an argument that a melody is turbulent consistent to that mathematically defined by Kolmogorov there must be a linear relation between natural logarithms of the statistical moment and R values.

Figure 30 presents the first order plot, as shown in blue. The three straight lines (in orange, green and red) are lines of least square fit depending on the range of input data. The orange is a fitted based on the first six points, therefore for a maximum time of a bar, the green line

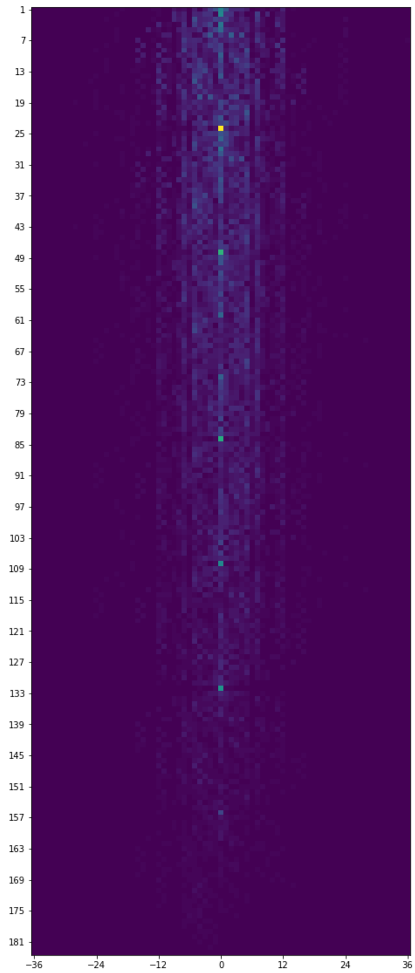


Figure 29: A heat map representation of the occurrences of an interval, taking both positive and negative intervals, for all time differences. The highest occurrence again is depicted by yellow, and as it decreases the luminance decreased.The data considered for this plot is taken from the entire extract analysed for Beethoven's Für Elise.

is based on all data after the first six, time differences of larger than a bar, and the red is for all data points, giving an overall result. By setting different fits, we suggest a method to differentiate small and large scale turbulence, taking the line fitted for $1 \leq R \leq 6$ as our small scale (orange line) and the green line, $R > 6$ to be larger scales. From which we postulate that there could exist different laws of turbulence depending if the system is small scale (comparing the pitch differences of two notes with maximum of six time units difference, or in terms of musical terminology, interval within a phrase, or a bar), or large scale (comparing pitch differences between two notes of larger than six time units apart, otherwise interpreted as the overall melodic progression within the piece or extract).

We can observe that the fitted line for the small scale becomes highly inaccurate for large time intervals, and similarly the fitted line for large scales is inaccurate for small time intervals. We can interpret this observation by suggesting that the melodic progression within a phrase is not representative of the overall extract, and similarly the converse is true. Therefore we had added an extra plot in which takes all R values, resulting in a generalised plot.
Additionally we can conclude that for both large and small scales, the data is consistent with that the theory proposed by Kolmogorov, as it can fitted with a linear relation as derived by 1.5.6 derived. The scaling function, ξ_n is then obtained by determining the gradients of each line, as derived by equation 2.1.3, and shown from by first entry in each array in the key in figure 30. The results found from n=1 are also similarly to that found for n=2,3,4,5 as shown figure 59 in Appendix. The findings of this plot suggests that, contradicting the analysis based on the musical definition, for all three plots there does exert some turbulent characteristics as per the mathematical definition.

Figure 31 plots the n values against the scaling function, or the gradient of the nth order statistical moments. From which, again the data is fitted with a linear relation by a least-square fit. If the data can be accurately fitted to a linear relation, it supports the simple scaling exponent, similar to that of $\xi_n = \frac{n}{3}$ as was found in Kolmogorov's theory. Therefore by observing figure31, that the lines which considered high values of R, and the overall extract was consistent with the scaling function proposed by Kolmogorov. However, when considering smaller scales, R values of maximum six, we can see that this is not fitted accurately with a linear function, therefore suggesting that the scaling function for smaller scale is more complex than a scalar factor as proposed.

Furthermore we can also observe from figure 30, there is a descending scaling function for the the plot which focusses on large time differences, this suggests that as the time difference increases, the pitch difference decreases. We can interpret this to mean for large melodic structures, for example longer motifs or sections, there is a tendency to converge back towards similar notes. We can view this to suggest that the melodic progression within a phrase is faster than that in the overall melodic progression in an extract. Furthermore we can compare to that of eddies velocity within a turbulent flow, as small scales eddies would have a higher velocities and therefore higher velocity difference than those of smaller eddies.

Moreover we can use figure 30, we can comment on it's interpretation with relation to Huron's melodic organization elements. If we view the line of least-square fit based on the first six R values, we could argue that represent that of random walk, and therefore consistent with melodies within a phrase exhibits the element of pitch proximities. Furthermore the line least-square fit based on higher time differences, supports the claim of an existence of central pitch tendency and also an arch phrase tendency. These observation support the the original conclusion that the extract taken from Für Elise is not musically turbulent, as it satisfies many common melodic elements. However as we did also observe that the plots of statistical moments against time dif-

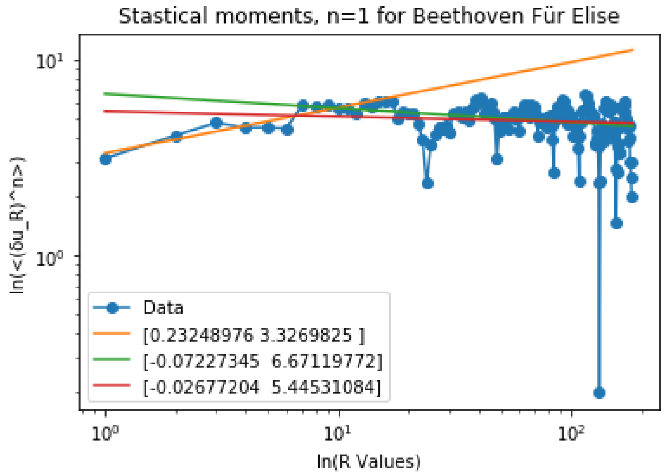


Figure 30: A plot of the natural logarithm of the statistical moment for n=1 against the R values, or time difference, with each R marked by circle as represented by the blue line. The data is then fitted by a least -square fit to approximate a linear function, considering the first six R values, denoted by the orange, all the data points after the first six, denoted by the green, and finally the red line takes all data points into consideration. The arrays in the key denotes the values of ξ_n and C_2 as labelled by equation 1.5.6

ferences were consistent to that in Kolmogorov's theory, it therefore must exhibit some turbulent characteristics. A possible explanation to this is reiterating that as we have chosen an extract is familiar to the majority and therefore a reason to why musically it is not often determined as turbulent is because of this 'learnt' aspect. Another explanation could be that the feature we chose to base our definition of musical 'turbulence', namely the melodic line and specifically the intervals, may not be the defining feature.

We end this analysis by concluding that whilst choosing a piece composed by a musician in a similar conditions to that of van Gogh, we did not obtain results that convincingly claimed that this extract chosen from Für Elise is turbulent. Although it did exhibit the characteristics, from taking the the nth order statistical moments, that were consistent with the structure functions, when analysing the melodic lines from a melodic definition of turbulence, we found it that, if anything, that melody could be argued as conventional. And so we conclude that this extract could be defined to be mathematically turbulent but not musically, therefore perhaps we should re-evaluate our definition of musical turbulence, and potentially choose another feature that is not melodic lines. Additionally it can also be argued that because we have chosen a small set of data, this could result in the conflicting results.

5.4 Mozart's Sonata in C Major First Movement

The next piece we chose is the first movement of Mozart's Sonata in C Major. Although similarly to Beethoven's Für Elise, this is arguably not turbulent, as it is a piece composed in the Classical period, the period that is famous for not only it's symmetry in overall structure and melodic construction but it's rigid tonal structure which can be described as 'well behaved' modulating to closely related and expected keys. Furthermore this sonata is in C major, which again is a tonality note regarded to be 'turbulent' in the slightest, and therefore naturally it is questionnable to why we have chosen to analyse piece with the aim of 'looking for turbulent characteristics' in a piece that can be considered near opposite to it's definition. We argue that the main advantage of analysing this piece is the large amount of data it holds, in total there are 2336 pitch values (taking the discretized time values of each note). Furthermore, similarly

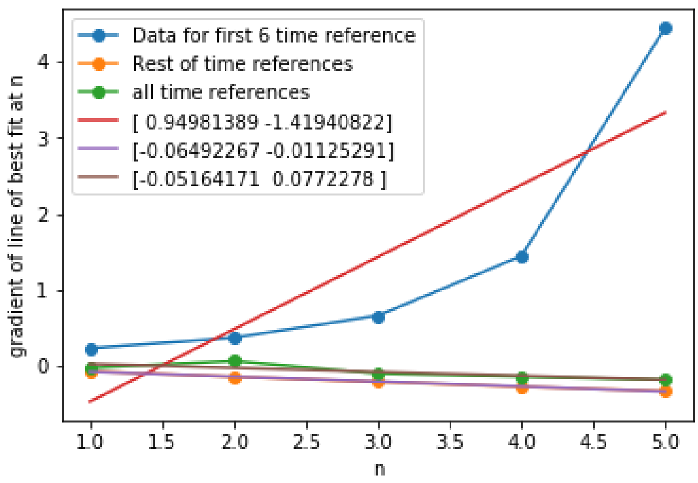


Figure 31: A plot of the scaling functions, or gradients in figure 30 and 59 , as shown in figure 53 against the n values. as shown by the blue line, orange and green. The data points are fitted by a least-square fit approximating a linear function, as shown by the red, purple and brown lines.The arrays in the key represents the values of p and q for the fitted linear relation as denoted by equation 2.1.2

to both van Gogh and Beethoven, whilst Mozart stylistic composing style was consistent to that of the classical period, he did exhibit signs of severe psychotic disorders, for example Tourette syndrome and displaying symptoms coherent to those found in ADHD and bipolar[28, p. 363]. Therefore we argue that in the similar way for van Gogh and his paintings, Mozart may have transcribed some of his state of mind into his compositions. We begin with a description of the sonata form before implementing any analysis.

5.4.1 Sonata Form

We give a brief definition on the sonata form, outlining key features commonly found in this structure. Hepokoski and Darcy states that the conventional sonata form is known as the Type 3 sonata, in which there are two main parts:(1) the exposition and (2) the development and recapitulation[16, p 16], with the two sections marked out by repeats. Furthermore it can be seen as an elaborated binary form, A||BA', where section A' is a variation of A[16, p. 16]. Figure 32 [16, p. 17] provides a graphical description of a typical sonata form, therefore suggesting there is some symmetry to the structure.

We now begin to dissect the overall structure and examine each section individually, again pointing out key characteristics. Section A, is made up of the Exposition, which is described to serve two purposes, the first is harmonic and the second is "thematic-texural" or "*rhetorical*" [16, p. 16]. The harmonic task is to ensure the tonic is outlined, and then through conventional harmonic sequences and cadences, a secondary key is established. In a major-mode sonatas, the secondary key is often the dominant (V), whereas in a minor-mode sonatas it is the major mediant (iii) (although occasionally there is a modulation is a minor dominant (v))[16, p. 16]. The second purpose of the exposition is described to be it's '*rhetorical* task', in which the themes are presented for the first time. Conventionally the exposition will begin with a *primary theme*, denoted as P, followed by a *secondary theme* denoted as S, in the secondary key. The exposition will then close with a perfect authentic cadence (PAC) in the new key, denoted as the following V:PAC, III:PAC or v:PAC dependent on if the sonata is in a major or minor mode[16, p. 18]. At the point of the PAC, this part of the exposition is known as the *essential expositional closure* (EEC). There are further subsections, for example the transition (TR) and medial caesira (MC)

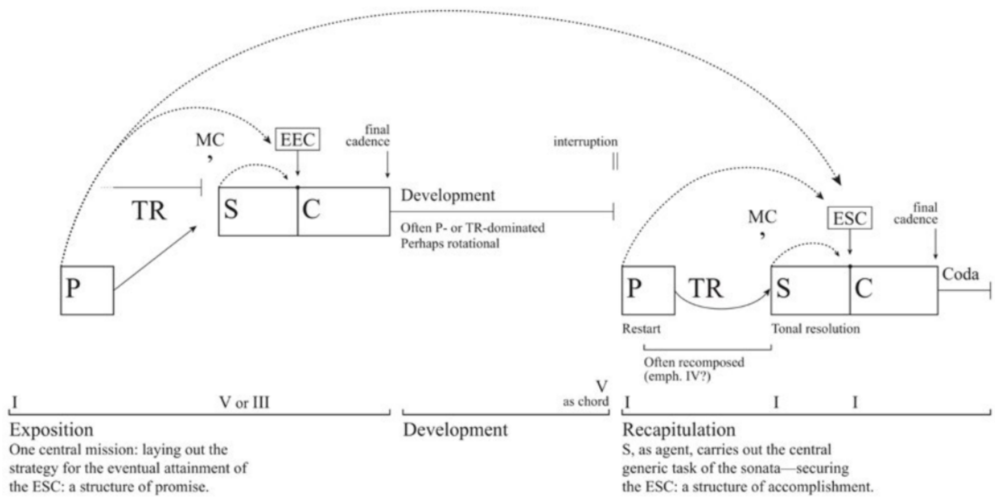


Figure 32: A layout of the sonata form, in which can see the first section, known as the exposition is essentially copied and this becomes the recapitulation. The two sections are separated by the Development. Furthermore the tonal structure can be observed from this figure. In the exposition begins with the tonic in the primary theme (P) then modulating to the mediant or dominant in the secondary theme (S). There is not state conventional key for the development to begin in but conventionally will end in the dominant. The recapitulation remains in the tonic throughout.

that have been purposely omitted from this introductory definition of an sonata form, however more information on these can be found in [16].

Section B of the 'binary form' is known as the development in the sonata structure. Hepokoski and Darcy describes the development to "initiate more active, restless, or frequent tonal shifts - a sense of comparative tonal instabilities [and h]ere one gets the impression of a series of changing, colo[u]ristic moods or tonal adventure often led (in major-mode works) through the submediant key, vi, or other minor-mode keys with shadowed, melancholy, or anxious connotations."[16, p. 18-19]. This contrasts with the exposition's 'strict' tonal structure. Another point to include is that the development will often use the themes presented in the expositions and develop variations on them.

The final section, A', is called the recapitulation and it's purpose is essentially what what it's called - it 'recaps' what was done in the exposition. Hepokoski and Darcy explain that the purpose of this section is to resolve the tonal tensions created in the development [16, p. 19]. Whilst the recapitulation retraces through the exposition's materials, the main difference is that the secondary theme is now presented in the tonic key and there is not modulation to the secondary key as there was in the exposition. Similarly the secondary theme ends with a PAC, for a major mode the cadence is denoted as I:PAC or i:PAC for a minor-mode , and this point is labelled as the *essential structural closure* (ESC).

5.4.2 Analysing the melodic line

We begin by quantifying the notes and determining the time reference following the method in section 3.3. In this piano sonata, again the melody is mostly distinctive, and therefore easy to decipher. However we note that this is not always the case, for example in bars 78-81 where the upper part is clearly the accompaniment and when the two parts combines and there is a 'call-and-response' between the two parts in bars 18-21, and so we take 'both' parts to be our

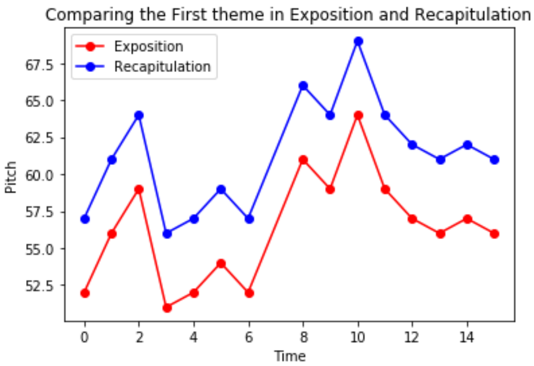
Note Value	Time duration
Semiquaver	1
Quaver ♪	2
Crotchet ♩	4
Mimim 𝅗𝅥	8
Semibreve 𝅝	16

Table 13: Table depicting the note values in reference to the time unit. The time unit in Mozart's Sonata in C is the semiquaver.

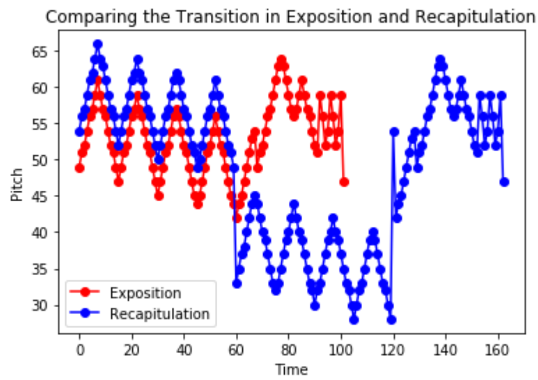
melody. Additionally, in bar 23, as shown in figure 60b, we ignored the four semidemiquavers, and instead took it as an 'upper turn' ornamentation (with the principle note being 61) of time value equating to a quaver. We take the semiquaver as our time unit, therefore we can list the time values of each note in relation to this, as shown by table 13.

We begin by determining to what degree does this sonata conform with that of a conventional sonata form as defined by Hepokoski and Darcy. The easiest to determine this, is to compare the exposition and recapitulation and observe firstly they're of the same size and then if the themes presented in the expositions can be found in the recapitulation in the order expected. To test this hypothesis we plot a line graph of the first theme, transition, second theme and codetta of both sections, as shown by figure 33. From which we can observe that generally the melodies in the exposition (shown in red) and recapitulation (shown in blue) do correspond with one another, the only differing factor is that it is often been transposed corresponding to the differing tonalities. The most obvious difference between the exposition and recapitulation is that in the transition period, shown in figure 33b, in which the recapitulation is extended by repeating the first motif an octave and a sixth lower. An explanation to this is the extension is used for a tonal purpose as in the recapitulation begins in F Major and therefore would take a 'longer route' to the dominant key of G Major to avoiding sounding 'jarring', in comparison to the tonic moving to the dominant as in the exposition. Additionally we can also see in figure 33c, whilst the melody in the second theme of the recapitulation, generally exhibits the same shape as in the exposition, we can see that the intervals between motifs are slightly different. For example approximately in the middle of second theme, we can see the recapitulation melody is no longer parallel to that of the exposition, meaning we cannot say that the second theme recapitulation is transposed copy of the exposition. We can also conclude that the translation of lines is due to the change of tonality, it can be interpreted as Mozart essentially 'copying and pasting' these themes and shifting it up or down so that the theme is consonant in the new key.

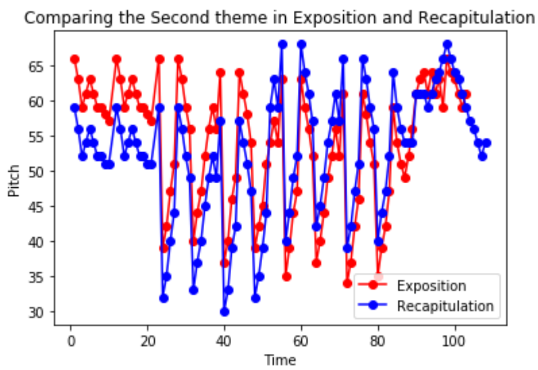
Furthermore if we analyse the score tonally, we can deduce that whilst the exposition behaves as expected beginning with the tonic (C Major) and then modulating to a secondary key, the dominant (G Major) in the second theme (Bar 13), the recapitulation does not. According to the conventional structure that Hepokoski and Darcy proposed, the recapitulation should begin on the tonic and remain in that key until the end. However in Mozart's sonata in C, the recapitulation begins on F Major, the submediant key before modulating back to the tonic in the second theme. Therefore we can postulate that, in contrast to initial assumptions, because it does not follow the standard tonal structure, we could expect some turbulent characteristics.
Although there are some differing areas between the melodies in the exposition and recapitulation, for simplicity we will assume that it does follow the convention of the classical sonata form and therefore assume that the recapitulation is a 'melodic copy' of the exposition. In regards to the recapitulation being tonally unconventional, we note that we will keep this in mind when interpreting results, however as our proposed definition for both musical and mathematical turbulence, we have chosen to focus interval difference in a melody and so we can assume that not



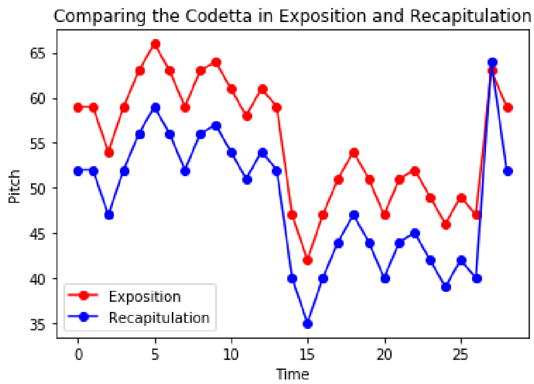
(a) Primary Theme (P)



(b) Transition



(c) Secondary Theme (S)



(d) Codetta

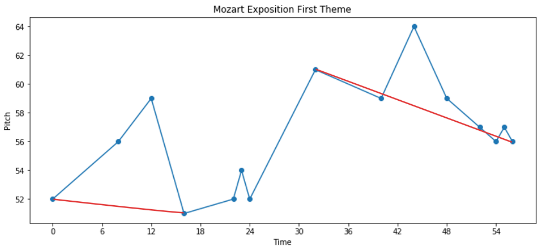
Figure 33: Line graphs comparing the melodic lines between the four sections of the exposition and recapitulation in Mozart's Sonata in C. The melodic line in the exposition is depicted in red and the recapitulation in blue.

changes to tonal structures will not affect the results significantly. Therefore our analysis will mainly focus comparing the exposition to the development as an attempt to compare a 'non-turbulent' melody with a 'turbulent' melody. McClelland notes that because of the conventional use of major keys in storm period, particularly in the late seventeenth century and the early eighteenth [22, p. 41], that composers would rely on melodic figurations, such as rapid motion, as oppose to focussing on dissonant harmonies in order to depict the turmoil that is associated with the Tempesta style. Whilst this sonata was composed in 1788, it can argued that Mozart was aware of these techniques and perhaps could be inferred that this feature is the driving element in pushing this sonata towards being 'musically turbulent'. The exposition is made up of three 'motifs' as shown by figure 34, the first is the primary theme (bars 1-4, shown in yellow highlight in figure 60a),graphically represented figure 34a from which we can interpret, with the use of the score, that it is made up of largely two descending intervals, each 'decorated' with an arch. The second motif is the secondary theme (bars 14-15, shown by the pink highlight in figure60a). From figure 34c, the'shape' of secondary theme is in highlighted by the green line, which is made up of a interval fall followed another downward step. Additionally we can observe the arppegiated 'call-and-response' motif, denoted by the pink line. Finally, the third motif is found in the codetta motif (bar 26, in the orange box in figure 60b). Within this motif, as demonstrated figure34d, it consists broadly of just one small pattern based around a triad, shown by the orange line, and two repeats of it's vertical and horizontal reflection, denoted y the green line. From a quick analysis of these motifs, it supports the claim that it would be regarded as non-turbulent because it satisfies many of the principles the I-R model, namely registral return and closure and we can see most obviously from these graphical representation that there is a high arch phrase tendency as predicted by Huron. Although arguably 'large' intervals are present within all the motifs, it is not regarded as 'unpredictable' as they are often an interval of a third, fifth or octave which extremely diatonic and conventional of pieces of this period.

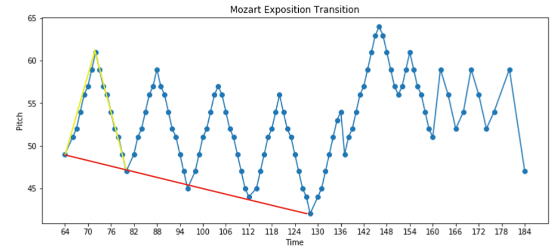
Furthermore, from looking at figure 33b, it further consolidates the prediction that the exposition, in particular, will be not described as 'musically turbulent'. Firstly the arch phrase tendency is predominant feature, in which the arch is made up of scalic movements - therefore small intervallic differences, but with reference to it's time of composition could be regarded as evoking some musical turbulent characteristics. However the overall melodic progression is quite slow, descending a tone or semitone at the end up each phrase, and therefore we lose the 'sweeping' motion that depicts the motion of of a "storm-tossed sea"[22, p. 51], and so we conclude that it unlikely to be considered as evoking a negative reaction, as expected by Tempesta characteristics.

We now proceed by analysing the graphical presentation of the development. As shown by figure Hepokoski and Darcy described this section to be "active [and] restless"[16, p. 18]we might expect to observe larger intervals and a quicker overall melodic progression than we did in the exposition, and so we can conclude as more 'musically turbulent'. However when analysing figure 35, we actually come to the conclusion that it is not likely to be turbulent. We can see that is predominately made up of the motif found in the codetta and a series of scalic patterns. McClelland does argue that rapid scalic passages are consistent with tempesta-like characteristics[22, p. 219], and therefore this could be the factor in driving the melodic line, however as the section does not exert many other elements, it cannot be strongly claimed to be turbulent.
We note four distinct sections of the development, the first, labelled (1) is the codetta motif. (2) is a scalic pattern consisting of a ascending scale in the bass followed by a return in the treble, exhibit a central pitch tendency predicted by Huron and proximity and registral return categories of the I-R model. We note that (3) is section (2) but with an interval reverse between the interplay. To expand, we observe a large interval leap after the scalic run in (2), and (3) retains this ascending scalic run however now there is a their transition to one another is flipped horizontally. Finally (4) is simply the second half of both (2) and (3) repeated in succession between the treble and bass. The use of previous material is not uncommon for the development, and Hepokoski and Darcy does state an expectation of returns of the primary and secondary theme presented in the exposition, with some variations on them (meaning some parts of each theme could be taken out separated and used in a sequence for example). Therefore we question how the development can referred to the section which contrasts with the exposition's strict structure and acclaimed to have more freedom when it is built on melodic motifs already presented. From a melodic perspective , we can argue that if anything the development is more restricted than the exposition as it relies on the themes presented there. Therefore we suggest that it is seen a more "active [and] restless" because of a different feature, for example tonality, but not because of it's melodic content.

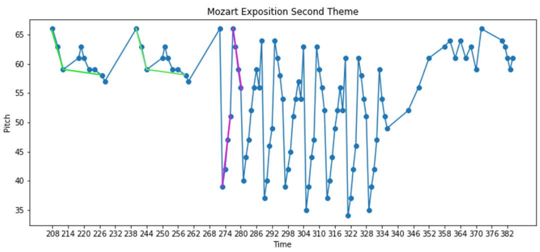
For the next part of the analysis, we compare bar chart plot of the frequencies against intervals of the exposition and development focussing on a time difference of the time unit, r=1, or a semiquaver, as shown by figures 36 and 37. The most distinct difference between the two figures is that the frequency graph for the exposition has an almost exact symmetry around pitch difference of zero, whereas in the development there is a slight tendency towards descending intervals. Additionally, another obvious difference between the two figures is that in the exposition, the predominant interval within this section is that of zero, meaning there is a highest probability (having a frequency value of 39.6) that within a semiquaver beat that the note



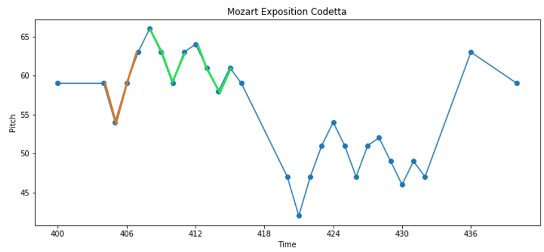
(a) Primary Theme (P)



(b) Transition



(c) Secondary Theme (S)



(d) Codetta

Figure 34: Line graphs depicting the melodic lines of (a) Primary Theme, (b) Transition, (c) Secondary Theme and (d) Codetta, the data is taken to plot these are taken from the Exposition, as shown in blue and each note struck is denoted by the marker. We point out the overall pattern of each motif within the sections.

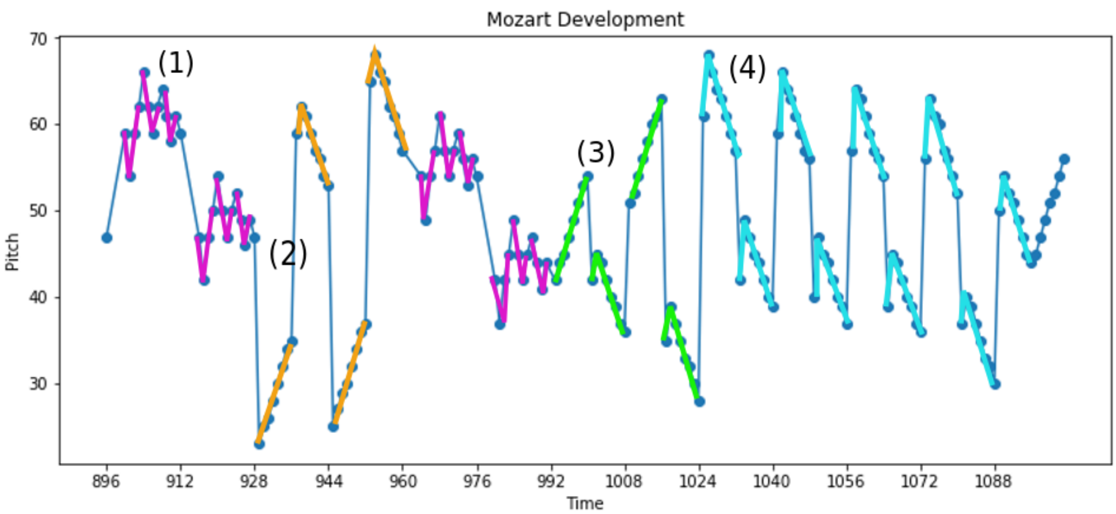


Figure 35: Line graph plotting the quantified notes within the Development with it's respective time reference. Motifs found in this section are as follows: (1) pink, Codetta (2) orange, scalic pattern (3) green, variation on motif (2) and (4) turquoise, built from motif (2).

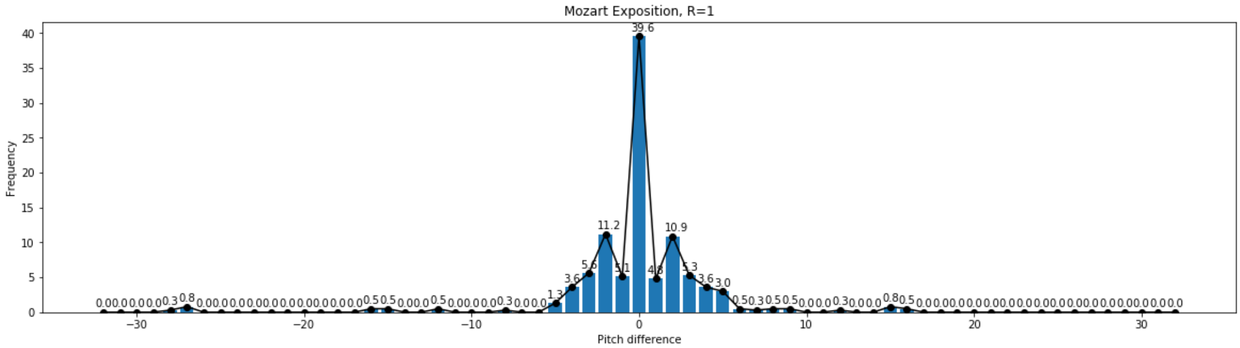


Figure 36: A bar graph representing the probability of an interval in the time difference of a time unit, a semiquaver for the exposition.

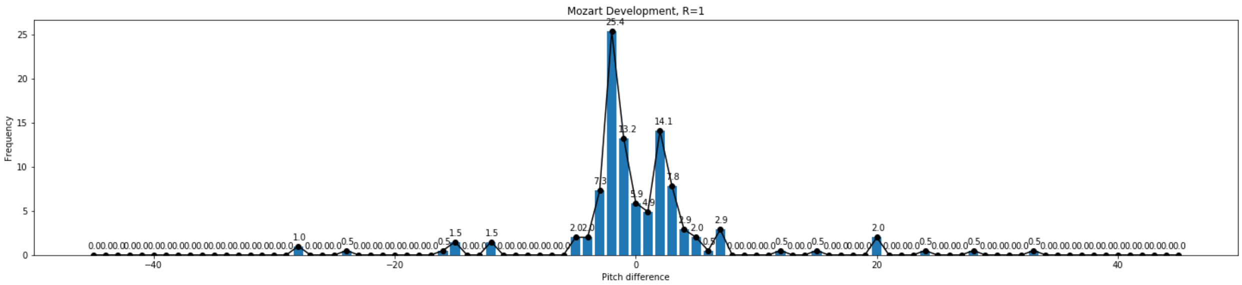


Figure 37: A bar graph representing the probability of an interval in the time difference of a time unit, a semiquaver for the development.

struck next will the same, however in the development we see that it is a pitch difference of two that has the highest probability, meaning that from figure 37 we'd expect the development to have a predominant descending scalic pattern made up of intervals of tones. Furthermore we can suggest that the exposition does conform to the 'symmetrical characteristic' that is often associated with this pieces of this period. Not only does the overall sonata structure exhibit some symmetrical relation as shown by the comparison of the melodies in the exposition and recapitulation, but also this characteristic is present in small scale structures, namely the melodic lines in the exposition. We conclude that in a similar manner to that of turbulent characteristics being exhibited in both large and small structures, the converse can be said of this piece.

An additional point of comparison from the last conclusion, is whilst we have already stated the development does not display as much 'musical turbulence' as we had initially assumed based on Hepokoski and Darcy's remarks about the section, we could argue that from figure 37, especially when comparing it to the exposition, does exhibited some tendency towards turbulence, as asymmetric shape arguably results in a higher degree of unpredictability. Furthermore, while the frequencies are small and therefore less likely to occur, the development does exhibit higher probabilities of larger intervals in comparison to that in the exposition.

We conclude that while the development is not necessarily 'obviously' turbulent when analyse based on it's melodic content, we observe that in reference to the 'well-behaved' and symmetrical exposition, perhaps it does conform to the description of being "restless [and] active"[16, p. 18], therefore tending towards'musically turbulent' in relation to the exposition. Perhaps it is combination of the dominating probability that the melodic line will descend a tone between each semitone and the occasional large interval jumps or falls that forces the melodic line to continually be "restless and active", despite it consisted of motifs we have already determined to not be 'musically turbulent'.

5.4.3 Relating to theory of turbulence

We attempt to determine if Mozart's Sonata in C, in particular the development section, exhibits turbulent characteristics as defined by Kolmogorov's theory. Again we take statistical moments for different orders, and if a linear relation can be fitted using the least-square fit, then we can comment that it consistent with being 'mathematically turbulent'.

Figure 38 plots the natural logarithms of the statistical moment for n=1 against the R values, i.e. time differences. We note that as the piece ends with a rest, there does not exist any pitch differences for a 'maximum' time difference, explaining the sudden drop in statistical moment in the plot at maximum R value. Furthermore we can observe that similarly to that of Beethoven's Für Elise, there seems to be two sections to this plot, and more specifically two separate linear relations. The first of which encompasses the first four data points, and the second the rest of the plots. Again this reiterates back to the idea of finding turbulent characteristics in small and large structures, however again we note that the scaling function associated with the small and large structures would be not be applicable to the other, indicated they are, in fact separate structures. As an linear relations can be fitted to a reasonable degree of accuracy, we can conclude that, especially for small scale structure, that the melodic lines within Mozart's Sonata in C does exhibit turbulent characteristics as defined by Kolmogorov's theory. Another comment to add is the small dips we observe in the plots. We propose that each 'dip' is approximately at R values such that the time difference is equal to a bar, two bars and so on. Therefore from this plot we can conclude whilst generally as the time difference increases, in particular for n=1, the average interval sizes between two pitches also increases, however when we reach a time difference equating to a bar or a multiple of a bar, as the average interval sizes decreases, we can interpret this to correspond to Huron's organizational element of 'central pitch tendency'. As a time difference of some multiple of a bar, there is a return back towards the initial note struck, from which we can propose a melodic line in which in time difference of less than a bar the melody essentially 'runs' from the initial note, and then when the time difference is equated to a bar or two bars, there is a interval jump or fall back towards the initial note. We find similar results for n=2,3,4 and 5 shown in figure 61.
Furthermore the interpretation proposed for this plot is strongly supported by the melodic motifs found in the score, in particular in the transition in the exposition as shown by figure 33b. From this we can observe the 'running' of the melody, however the larger scale is simply a descending motif around the beginning note of the 'smaller' motif. This is not to say that the interpretation from figure 38is limited to just the transition period, in fact all four sections of the expositon exhibits this 'central pitch tendency' however the transition period is the one that conforms to that of a central pitch tendency per bar, and so the section that most closely fits with the results as shown by figure 38.

We hypothesized that the development may exhibit a strong tendency towards turbulence than the exposition, therefore we propose to execute the same method, however this time instead of inputting the full score's data we limit to just the development section. If our hypothesis is valid, we would expect the linear relation to fit more closely to that of the overall data. From comparing figures 38 and 39 we can see that again there are two distinctly linear relations, the difference is that in the development the first linear relation extends to the first eight data plots (a time difference of half a bar), approximately double to that in the overall data. Therefore we interpret this to mean the small scale structures in the development are double the size of the ones in the exposition. If we assume that small scale structures corresponds to phrases and motifs, and large scale structures are the overall melodic progression, we can suggest that the phrases within the development sections are perhaps longer than the generalised one in the whole sonata. Again we note the 'dips' in the plot, and we can comment that the 'central note tendency' element is also present in this section. As the development is made up of motifs pre-

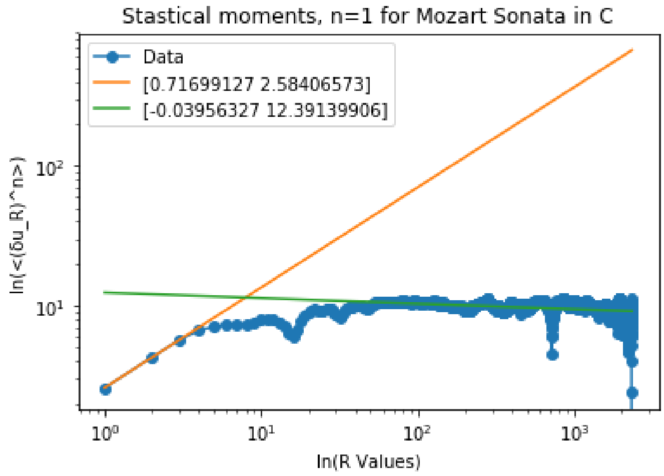


Figure 38: A plot of the natural logarithm of the statistical moment for n=1 against the R values, or time difference, with each R marked by circle as represented by the blue line. Similarly to the statistical moment plot for n=1 for Beethoven's Für Elise. Again the data is then fitted by a least -square fit to approximate a linear function,, considering the first four data points, denoted by the orange and all the data points after the first four semiquavers are denoted by the green. The arrays in the key denotes the values of ξ_n and C_2 as labelled by equation 1.5.6

sented in the exposition and then repeated in the recapitulation, this should not be surprising. From analysing the development alone, we reach the similar conclusion to that when we analysed the whole sonata. Therefore we suggest that the development does not necessarily exhibit significantly higher degree of turbulence than the overall structure.

In order to determine if the order of the scaling function found in Mozart's Sonata in C is consistent to that of Kolmogorov's theory in a velocity field, we plot the n values against the scaling function, ξ_n. If a linear relation can be fitted using the least-square fit to an reasonable degree of accuracy, we can conclude that the scaling function is equal to $\xi_n \propto \lambda n$, for λ is a scalar, following that of Kolmogorov's theory. As shown by figure 40, similar to figure 31 for Beethoven's Für Elise, the small scale structure (denoted in both figures by the blue line), is not accurately fitted by linear function, and therefore can be argued that it's scaling function does not correspond to that Kolmogorov had observed for velocity fields. However, for large scale structure, we observe this is almost corresponds with the scaling function as expected, the only outlier being the gradient at n=4, however this could be due to a humanistic error or a lack of accuracy.

We conclude that after analysing Mozart's Sonata in C, we reach a similar conclusion to that in Für Elise, in which both pieces are not regarded as musically turbulent as they exhibit many features of the I-R model, and not a significant amount from the Tempesta style, however we were able to observe some mathematical turbulence as both pieces did conform to a linear relation between it's statistical moments and R values. As a results of this contradiction in musical and mathematical definitions of turbulence, we propose that perhaps we had chosen the wrong musical element to analyse, and if we chosen another for example looking at the harmonic structure, rhythm or dynamics, or even adjust our definition looking into melodic lines, we may be able to conclude a continuity between definitions.

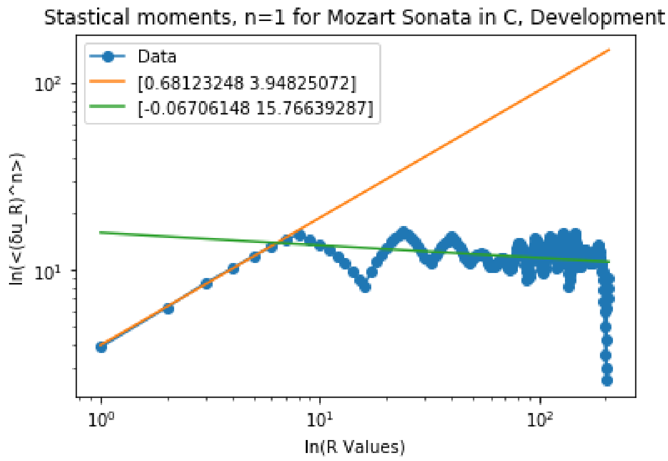


Figure 39: A plot of the natural logarithm of the statistical moment for n=1 against the R values for a data set limited to just the Development section. The method and variables are defined corresponding to the of figure 38

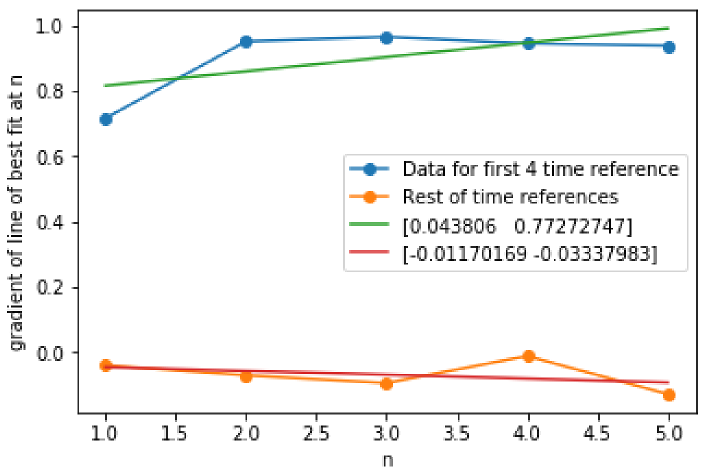


Figure 40: A plot of the scaling functions, or gradients in figures 38 and 61 against the n values, as shown in figure 54, as shown by the blue and orange line, with a marker denoting the gradient at each n value. The data points are fitted by a least-square fit approximating a linear function, as shown by the green for the first four data points, and the red for the rest of the data points.The arrays in the key represents the values of p and q for the fitted linear relation as denoted by equation 2.1.2

6 Conclusion and suggestion for further research

Some areas for further research is repeating this method except analysing only pieces that are inspired by van Gogh's *Starry Night*, for example looking at 'Timbres, espace, mouvement, La nuit etoilée (The Starry Night)' composed by Henri Dutilleux or Rautavaara's 'Symphony no. 6 1st Movement -Starry Night. From which we propose that by composing a piece that is influenced by the painting, potentially the turbulent characteristics found when analysing the painting could be transcribed. The two pieces mentioned are both orchestral pieces, the latter being an opera, and therefore we could observe the effect in adding more melodic lines into the analysis, and observing the effect of interplay between the parts. Furthermore the addition of instruments will also increase the energy projected, inferred as a increased in loudness.
A interesting choice of music to analyse would also be 'Vincent(Starry Starry Night)' by Don McLean, not only as it is a piece that will have some relation with van Gogh's painting and therefore could exhibit the turbulent characteristics similar to that found by Aragón et al. , but also as it is not a classical piece, it would be interesting to observe how turbulent characteristic could present itself in a different 'style' of music. An additional area of research is analysing pieces of difference genres and from different cultures as for this study we focussed on Western classical music. For example, if we still focus on the melodic line, what is accepted as conventional western classical music may not be the same for other generals, for example jazz or rock music or for musical of different cultures, which may not rely on any tonal structure.

Furthermore, pieces that are inspired by the cosmic could be a factor in relating music with that of van Gogh Starry Night, and perhaps could influence the degree of 'turbulence' exhibited by the melody, as Aragón et al. suggests that it is the eddies that transmit the turbulent flow [4, p. 276]. Therefore if we were to analyse a piece that are inspired by the cosmic and preferably with some mentions of eddies such as a gas cloud, we may be able to observe a greater correlation between the musical and mathematical presence of turbulence. A famous example to that could be deemed appropriated might be'The Planets' by Holst, and in particular 'Mars' and 'Mercury'.

As we have chosen to focus on the melodic line for this study, and by focussing this feature alone we may have limited our analysis or arguably not have provided a fair representation of the entire piece. Additionally from our analysis, we often concluded that a piece that arguably did not exhibit musical turbulence, but did show characteristics consistent to mathematical turbulence, implies a suggestion for future research into analysing different features, for example the rhythm or the harmonic structure of a piece. We note that analysing the rhythm within a piece would be achievable, however it is arguably may not be an as interesting feature to analyse, especially in piece that do not vary rhythmically. On the other hand if analyse the harmonic structure and the relation between harmonies, therefore now analysing a piece vertically as oppose to horizontally when taking time as a variable, could arguably be the defining feature in determine if a piece is 'musically turbulent' however it might be very complex to quantify and therefore not a practical element to consider. An example could be looking at Debussy's Claire du Lune as it is chordal, we propose some further research could be analysing both the melody in a horizontal direction but also analyse harmony between the notes in a vertical direction. Additionally another suggestion would be to a combination of a few factors McClellend [22, p. 13] states that "no single musical feature can [provoke this emotional response] in isolation] It is the combination of different aspects of pitch, rhythm, dynamics and colour that achieves that effect". Therefore by focussing on the melodic line alone, we have not provided a fair representation of the overall emotion evoked from the piece, which may be remedied by taking another feature or the combination of features. We defined 'musical turbulence' based on the idea of disrupting common melodic elements with goal to evoke an emotion, however the idea of evoking an emotion is complex as one stimuli can evoke many emotions, and additionally the converse in which multiple stimulis can evoke a single emotion. The concept of evoking an

emotion is arguable both innate and also based on person experience and therefore a stimuli can evoke multiple emotions such they contradict one another.
Another limitation we find by choosing the melodic line is when we began to analyse the data collected, we concluded that our initial hypothesis was overly simplified and too many assumptions were taken. We conducted this study with the hypothesis that large intervals would equate to 'musical turbulence'. However when we conducted the study we began noting we could not rely on the size of interval alone, but also the tonal stability of the interval. Furthermore we noted when labelling the pitch difference with it's musical interval name that we could not assume the pitch difference would be the conventional one as oppose to on that is diminished or augmented. Therefore we should take into account the two notes in question to determine if the interval is unstable being augmented or diminished. Another example of this limitation is that enharmonics were labelled with the same note number, as we assumed they were the 'same note' when in fact, for example in D Major, C♯ is a conventional note, however D♭ would not be, even though they would be labelled with the same number. As previously mentioned, for this study we did not consider the different weighting of notes within a tonality and assumed all notes were of equal importance, and so arguably lost the complexity of layers that melodic lines exerts. We argue that by quantifying the notes, we essentially lose the richness that comes with the complexity of music theory. We therefore suggest that the method to quantify note may need to be reconsidered in order to retain the richness that present in music.

The main aim of our study was to determine if we could replicated the study of paintings, conducted by Aragón et al. onto another art form, music. Whilst we did have some success in repeating the method, we note some limitations that we found by choosing to analyze music. We have already commented that arguably music can be deemed to be more complex with more abstract variables playing into effect than that of music, however there were some physical limitations as well. One example being that unlike in paintings where the range of 'R values' infinite and therefore the amount of data analysed can be adjusted simply by choosing a smaller or larger pixel size unit, in music we are limited by the number of notes that the composer has chosen to use in the piece, and so this cannot be adjusted without assuming what the composer has intended and doing some 'free-lance' composition. Another limitation we find by analysing music instead of paintings is that the sections within each differ in their ability to 'separate'. For in the Starry Night, the analysis could be conducted the top half and bottom half simply by splitting the paintings into, from which Aragón et al. could draw some conclusion on small and large scale turbulence. Whilst we had attempted this in our study, the conclusions we drew often were specify to that piece, as some pieces were naturally divided into sections by tonality changes and others by a change in motifs and phrase. The main difficulty with this analysis is that in music, there are layers of structures, in which each are embedded within one another, some of which may overlap therefore being difficult to extract.

In summary, from this study into determining if we were able to find turbulent characteristics in a piece of music in a similar method to the study conducted on a painting, we can conclude that in Beethoven's Für Elise and Mozart's Sonata in C the melodic line did exhibit characteristics that were consistent to that in a velocity field as proposed by Kolmogorov. Therefore we argue that as an initial study into this topic, there does seem to be some validity of our initial hypothesis. However we do note that we did encounter several problems and limitations and therefore there is definite room for improvement and suggestion for further research in order to provide a more conclusive assessment into this study.

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7 Appendix

Figure 41: Full reference of labelling of pitches in relation to it's key number of an 88-key keyboard.

```
#Pseudocode to calculate interval and difference in time and matrix
#import data/table from excel
#Let i= the number of row of that table  i=df.shape[0]
#for j,k less than the length of the table
#start with j=row 0, and k= row 1 df.iloc[2,2]- first 2 gives row number, next 2 gives column (if want all columns use :)
#do time(j)-time(k) and absolute(pitch(j)-pitch(k)) (absolute=abs)
#continue with k=k+1
#j=j+1
#output matrix
```

Figure 42: Pseudocode that creates a plan for the code to create the matrix calculating the number of occurrences of a pitch difference for a time difference.

```
#CODE FOR NOTE STRUCK
taking_abs=True #or False
piecenum = 0
for i in range (piecenum,piecenum+1):
#For specific pieces change range
  data=pd.read_excel (r'/Users/nicole/Documents/University/Master/Summer Dissertation/Base pieces Note Change.xlsx', sheet_name=i)
  df= pd.DataFrame(data, columns=['Bars','Time','Note'])
  #print (df)
  actual_notes=[r for r in df.Note[1:] if r !=-1.0]
  if taking_abs==False:
      matrix=np.zeros((max(df.Time)-min(df.Time),int(2*(max(actual_notes)-min(actual_notes))+1)))
  else:
      matrix=np.zeros((max(df.Time)-min(df.Time),int(max(actual_notes)-min(actual_notes)+1)))
  if np.isnan(df.iloc[0,2])==True:
      start=1
  else:
      start=0
#matrix starts with min(df.Note[1:]) in (0,0) and ends with max(df.Note[1:] in (0,matrix.shape[1])
  #print(i)
  #print(df.shape[0])
  for a in range(start,df.shape[0]):
      for b in range (a, df.shape[0]):
          if a!=b:
              if df.iloc[a,2]!=-1.0 and df.iloc[b,2]!=-1.0:
                  time_diff= df.iloc[b,1]-df.iloc[a,1]
                  if taking_abs==True:
                      note_diff = abs(df.iloc[b,2]-df.iloc[a,2])
                  else:
                      note_diff = (df.iloc[b,2]-df.iloc[a,2])+int((max(actual_notes)-min(actual_notes)))
                  matrix[time_diff-1, int(note_diff)]+=1
#Each row correpsonds to a time difference
#position of matrix (m,n)=(time difference-1, note difference) [can't have a time difference of 0]
#m=row number, n=column number
```

Figure 43: Code used to create a matrix in which calculates the pitch difference for time difference of times when the pitch is struck. This code takes the data from the datafile as commanded in "piecenum" and the labelling of pieces follows the table 5. The option of taking absolute values or considering positive and negative interval is denoted by changing the 'taking abs' function. A new set of notes, called 'actual notes', is created in which we remove all note values of -1. The code begins with creating the 'right' size matrix of zeros, depending if absolute value of intervals is taken or considering positive and negative intervals. The code begins with the first row and then runs through each row of comparing the time reference and the note number, appending a value of 1 into the relevant cell within the matrix. After the running through running all the rows has finished, the code begins again with the next row down and repeats.

```
#CODE FOR TIME DISCRETIZED
taking_abs=True #or False
piecenum = 3
for i in range (piecenum,piecenum+1):
#For specific pieces change range
  data=pd.read_excel (r'/Users/nicole/Documents/University/Master/Summer Dissertation/Base pieces Note Change.xlsx', sheet_name=i)
  df= pd.DataFrame(data, columns=['Bars','Time','Note'])
  Time_diff=df.iloc[:,1]
  Note=df.iloc[:,2]
  New_time=range(min(df.Time),max(df.Time)+1)
  New_note=[]
  for j in range(0,len(Note)-1):
      difference = int(Time_diff[j+1] - Time_diff[j])
      for repeat in range(0,difference):
          New_note.append(Note[j])
  New_note.append(Note[len(Note)-1])
  actual_notes=[r for r in New_note if r !=-1.0]
  if taking_abs==False:
      matrix=np.zeros((max(New_time)-min(New_time),int(2*(max(actual_notes)-min(actual_notes))+1)))
  else:
      matrix=np.zeros((max(New_time)-min(New_time),int(max(actual_notes)-min(actual_notes)+1)))
  if np.isnan(df.iloc[0,2])==True:
      start=1
  else:
      start=0
#matrix starts with min(df.Note[1:]) in (0,0) and ends with max(df.Note[1:] in (0,matrix.shape[1])
  for a in range(start,len(New_note)):
      for b in range (a, len(New_note)):
          if a!=b:
              if New_note[a]!=-1.0 and New_note[b]!=-1.0:
                  time_diff= New_time[b]-New_time[a]
                  if taking_abs==True:
                      note_diff = abs(New_note[b]-New_note[a])
                  else:
                      note_diff = (New_note[b]-New_note[a])+int((max(actual_notes)-min(actual_notes)))
                  matrix[time_diff-1, int(note_diff)]+=int(1)
```

Figure 44: The code follows in a similar way to that of figure 43. The difference is that we recreate two new lists named ‘new time’ and ‘new note’. New Time is a list which begins at the zero and ends at the maximum time reference. The New Note list is coded by running through the old time and for any ‘gaps’ between time references, the note number for the time before the gap is repeated the time difference time.

```
#PLotting a line graph - can see the melodic lines in terms of ascending and descending lines (and leaps)
#plot (x-axis, y-axis)
x=df['Note']
x_pos=x[x>0.0]
y=df['Time']
y_pos=[]
for i in range (0,len(x)):
    if x[i]>0:
        y_pos.append(y[i])
#scale, stretch(x, y)
plt.figure(figsize=(12,5))
plt.plot(y_pos,x_pos, marker='o')
plt.xticks(np.arange(min(y_pos), max(y_pos)+1, 16.0));
plt.xlabel('Time')
plt.ylabel('Pitch')
plt.title('Mozart Development')
```

Figure 45: Code used to create a line graph taking the data from the development in Mozart’s Sonata in C. The code begins by creating a new list x, which removes all note values that is less than zero. Then the pitch number is plotted against the time reference it is struck.

```
#Comparing pieces
all_note=[]
for i in range (3,16):
#For specific pieces change range
  data=pd.read_excel (r'/Users/nicole/Documents/University/Master/Summer Dissertation/Base pieces Note Change.xlsx', sheet_name=i)
  df= pd.DataFrame(data, columns=['Bars','Time','Note'])
  all_note.append(df.Note)
#print(all_note)
x_1=all_note[5];x_2=all_note[12]
x_pos_1=x_1[x_1>0.0];x_pos_2=x_2[x_2>0.0]
#plt.figure(figsize=(10,10))
plt.plot(x_pos_1, 'r', marker='o', label='Exposition')
plt.plot(x_pos_2, 'b', marker='o', label='Recapitulation')
plt.legend()
plt.title('Comparing the Codetta in Exposition and Recapitulation')
plt.xlabel('Time')
plt.ylabel('Pitch')
```

Figure 46: Code used to compare the melodic line of the exposition and recapitulation in Mozart’s Sonata in C. The pieces chosen to compare are denoted again by the numbering of figure 5

```
#Plotting a histogram graph - showing most common note
x=[]
for i in range (0,len(actual_notes)):
    if np.isnan(actual_notes[i])==False and actual_notes[i]>0:
        x.append(actual_notes[i])

plt.hist(x, bins=(max(x)-min(x)+1),rwidth=0.8, align='left');
plt.xticks(np.arange(min(x), max(x)+1, 2.0,));
plt.xlabel('Pitch')
plt.ylabel('Occurrences')
plt.title('Mozart Sonata in C Major- Development')
#Can set range of plots using range[....] inside plt.hist
```

Figure 47: Code used to create a histogram plot of the pitch against the number of occurrences. The code begins with creating a new set 'x' in which removes all note values that are 'NaN' and below zero and then a histogram is plotted using the python's inbuilt histogram function. We note that there is some error in this code as the graph plotted will always plot the last bar over the second last x value, this has been noted and before any analysis is conducted, we have checked the score and made adjustments where necessary.

```
##NON-ABS (PDF) Frequence vs note_diff
#plt.figure(figsize=(20,5))
R=0 #R=row number, starting at zero = time difference starting at 1
weighted=[]
x=range(-(max(actual_notes)-min(actual_notes)), max(actual_notes)-min(actual_notes)+1)
#adding up column
sum=0
for num in matrix[R]:
    sum+=num
print(sum)
for i in range(0,len(x)):
    p=(((matrix[R][i])/sum)*100)
    percentage=round(p,1)
    weighted.append(percentage)
#Gives each x as a percentage
for i in range (len(x)):
    plt.text(x=x[i]-0.5, y=weighted[i]+0.7, s=weighted[i], size=10)
plt.bar(x,weighted)
#plt.xticks(np.arange(-(max(actual_notes)-min(actual_notes)),max(actual_notes)-min(actual_notes)+1),2.0)
plt.plot(x,weighted, marker='o', color='k')
plt.xlabel('Pitch difference')
plt.ylabel('Frequency')
plt.title('Mozart Development, R=1')
#Note we label the R to be one larger than what what is imputted as
#We chose to beging coutning time differences, R from 1, whereas python
#begins counting at zero.
```

Figure 48: Code used to calculate the frequency of each interval, considering both positive and negative intervals, for a set time difference. In this example we take R=0, meaning a time difference of one time unit. The code begins with a empty set called "weighted", and adds all values in the row and this value is labelled as sum. Then it runs through each value of the row, divides by the total and multiplies by 100 to determine it's frequency.

```
#Code for Heat Map
taking_abs=False
plt.figure(figsize=(10,20))
if taking_abs==True:
    Heat_map=plt.imshow(matrix[:,:],extent=[-0.5,max(actual_notes)-min(actual_notes)+0.5,max(df.Time)-min(df.Time)+0.5,0.5]);
    ax = plt.gca()
    plt.xticks(np.arange(0,max(actual_notes)-min(actual_notes),1))
    plt.yticks(np.arange(1,max(df.Time)-min(df.Time)+1,1));
    plt.setp(ax.get_yticklabels(),va='top');
    plt.setp(ax.get_xticklabels(),ha='left');

else:
    Heat_map=plt.imshow(matrix[:,:],extent=[-(max(actual_notes)-min(actual_notes))-0.5,max(actual_notes)-min(actual_notes)+0.5,max(df.Time)-min(df.Time)+0.5,0.5]
    plt.xticks(np.arange(-(max(actual_notes)-min(actual_notes)),max(actual_notes)-min(actual_notes)+1,))
    plt.yticks(np.arange(1,max(df.Time)-min(df.Time)+1,1));
    plt.setp(ax.get_yticklabels(),va='top');
    plt.setp(ax.get_xticklabels(),ha='center');
plt.xlabel('Pitch Diff')
plt.ylabel('Time Diff')
#We take absolute if we want to focus on large vs small interval sizes
#Non-abs will tell us whehther the melody is generally ascending or descening
#large leaps should be balaned with step in opposite direction
cmap=plt.cm.rainbow
c_bar=plt.colorbar(Heat_map)
c_bar.set_label('Occurrence number')
```

Figure 49: Code to create heat map of the pitch difference against time differences. Again the option of taking absolute values are set by setting taking abs to be true or false. Therefore the size of the heat maps and the adjustment of interval values on the x values are adjusted respectively.

```python
###Defining moment equation###
def moment(x,n):
    taking_abs=True
    #x=sequence,column of matrix[R] for time_diff=R+1
    #n=nunmber, i.e., order of moment
    power=[]
    if taking_abs==False:
        note_diff=range(-(max(actual_notes)-min(actual_notes)),(max(actual_notes)-min(actual_notes))+1)
    else:
        note_diff=range(0, (max(actual_notes)-min(actual_notes))+1)
    for i in range (0, len(x)):
        #Want to sum for all i (x[i]^n)/len(x)
        power.append(x[i]*(note_diff[i]**n))
    #print(power)
    y=np.sum(power)/np.sum(x)
    return(y)
```

Figure 50: Code used to define the 'moment' equation. Again there is the option of taking the absolute value of intervals or otherwise, from which the note difference will be adjusted. Moments are calculated with the formula $x[i] * (\text{note diff}[i] * *n)$ where x is the set of notes and n is the order of which we want to calculate the moment to.

```python
##Code for putting moments
#plt.figure(figsize=(20,10))
n=1
moment_list=[]
for R in range(0,matrix.shape[0]):
    x=matrix[R]
    y=moment(x,n)
    #print(y)
    moment_list.append(y)
R_values=range(1,matrix.shape[0]+1)
plt.loglog(R_values,moment_list, marker='o')
```

Figure 51: Code used to plot the a log-log plot of the set of intervals for all values of R, time differences, against the computed the statistical moment.

```python
##Plotting least mean square####
actual_moment_list=[]
actual_R_values=[]
for i in range (0,len(moment_list)):
    if np.isnan(moment_list[i])==False and moment_list[i]>0:
        actual_moment_list.append(moment_list[i])
        actual_R_values.append(R_values[i])
#Line of least-square fit
def func(x, a, b):
    return b*x**a

#First section
popt_1, pcov_1 = curve_fit(func, actual_R_values[0:6],actual_moment_list[0:6])
print(popt_1)
#Second section
popt_2, pcov_2 = curve_fit(func, actual_R_values[6:],actual_moment_list[6:])
print(popt_2)
#All
popt_3, pcov_3 = curve_fit(func, actual_R_values,actual_moment_list)
print(popt_3)


ydata=[]
ydata2=[]
ydata3=[]
for i in range(len(R_values)):
    ydata.append(func(R_values[i],popt_1[0],popt_1[1]))
    ydata2.append(func(R_values[i],popt_2[0],popt_2[1]))
    ydata3.append(func(R_values[i],popt_3[0],popt_3[1]))

plt.loglog(R_values,moment_list, marker='o')
plt.loglog(R_values,ydata)
plt.loglog(R_values,ydata2)
plt.loglog(R_values,ydata3)
plt.gca().legend(('Data', popt_1,popt_2, popt_3))
#plt.show()
plt.xlabel('ln(R Values)')
plt.ylabel('ln(<(\u03B4u_R)^n>)')
plt.title('Stastical moments, n=1 for Beethoven Für Elise')
```

Figure 52: Code used to plot straight line by fitting the least-square fit to data of the log-log of statistical moments against the time difference. In this partiular example it plots the least-square fit for the data of Beethoven's Für Elise.

```python
#Gradient values for Beethoven
a_values_1=[0.23248976,0.37032546, 0.65842031, 1.44441965, 4.44451213]
a_values_2=[-0.07227345, -0.14229115, -2.10349352e-01, -2.74316540e-01, -3.30874115e-01]
a_values_3=[-0.02677204, 0.0628331, -1.00608551e-01, -1.40750274e-01, -1.83188900e-01]
```

Figure 53: The scaling functions for the small, and large scales and for all the data for Beethoven's Für Elise.

```python
#Gradient values for Mozart
a_values_1=[0.71699127, 0.95230444, 0.96630625, 9.45903855e-01, 9.39221571e-01]
a_values_2=[-0.03956327, -7.03731212e-02, -9.36265074e-02, -1.12067473e-02, -1.27654925e-01]
```

Figure 54: The scaling functions for the small, and large scales for Mozart Sonata in C.

```python
### Line of best for gradients of all n##
n_values=range(1,6)
def func_1(x,p,q):
    return ((p*x)+q)

popt_a, pcov_a = curve_fit(func_1, n_values, a_values_1)
print(popt_a)
popt_b, pcov_b = curve_fit(func_1, n_values, a_values_2)
print(popt_b)
#popt_c, pcov_c = curve_fit(func_1, n_values, a_values_3)
#print(popt_c)
scalingfunction_1=[]
scalingfunction_2=[]
#scalingfunction_3=[]
for i in range(len(n_values)):
    scalingfunction_1.append(func_1(n_values[i],popt_a[0],popt_a[1]))
    scalingfunction_2.append(func_1(n_values[i],popt_b[0],popt_b[1]))
    #scalingfunction_3.append(func_1(n_values[i],popt_c[0],popt_c[1]))
plt.plot(n_values, a_values_1, marker='o')
plt.plot(n_values, a_values_2, marker='o')
#plt.plot(n_values, a_values_3, marker='o')
plt.xlabel('n')
plt.ylabel('gradient of line of best fit at n')
plt.plot(n_values, scalingfunction_1)
plt.plot(n_values, scalingfunction_2)
#plt.plot(n_values, scalingfunction_3)
plt.gca().legend(('Data for first 4 time reference', 'Rest of time references', popt_a,popt_b,))
```

Figure 55: Code used to plot the scaling function against the n values, for small and large scale structures. The values for a values 1 and a values 2 are inputted by hand, as shown by figure 53 and 54

.

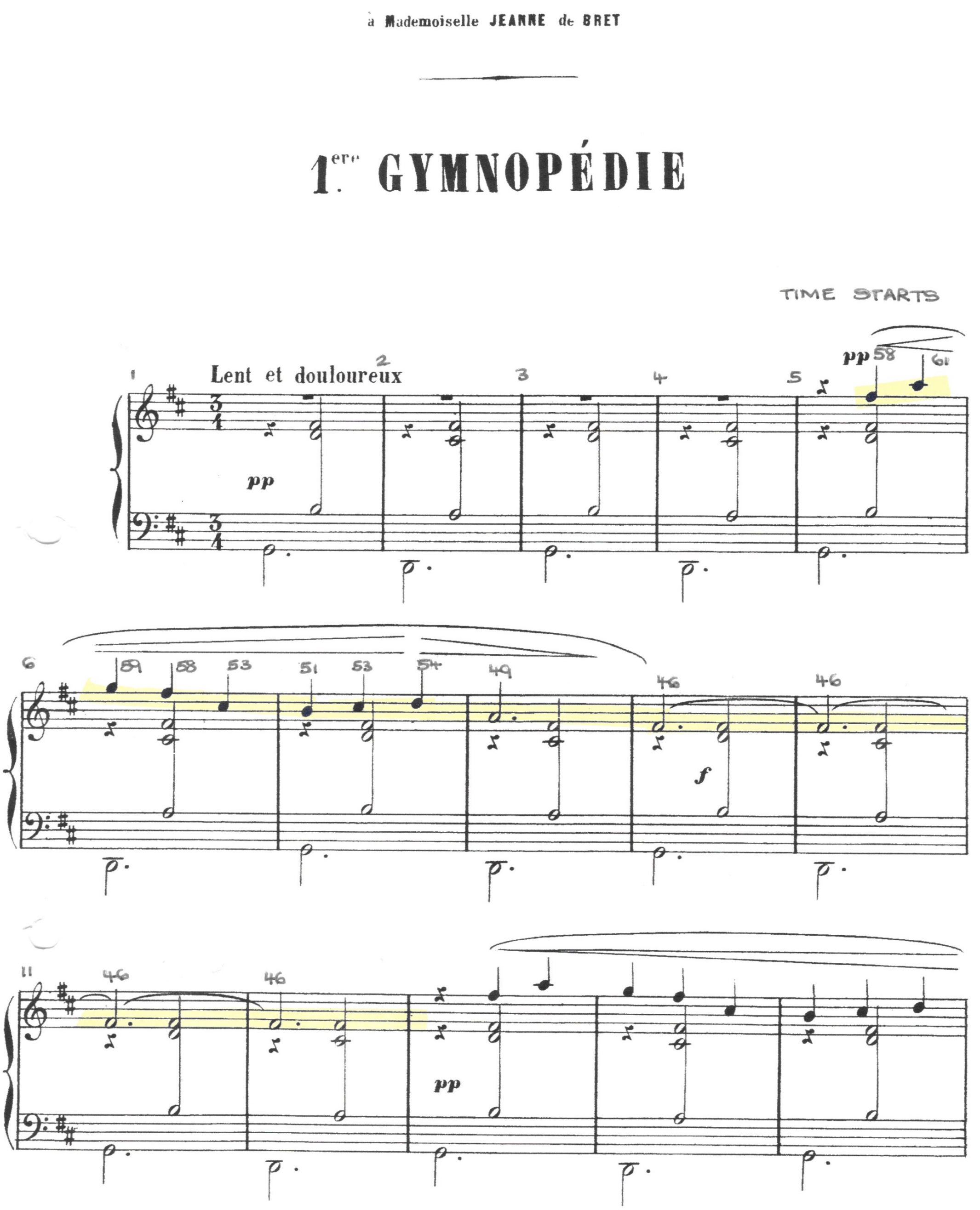


Figure 56: Full score of Satie's Gymnopédie No. 1 [25]. The bar numbers are denoted right of each bar and the note number above each note. The melodic line is highlighted in yellow.

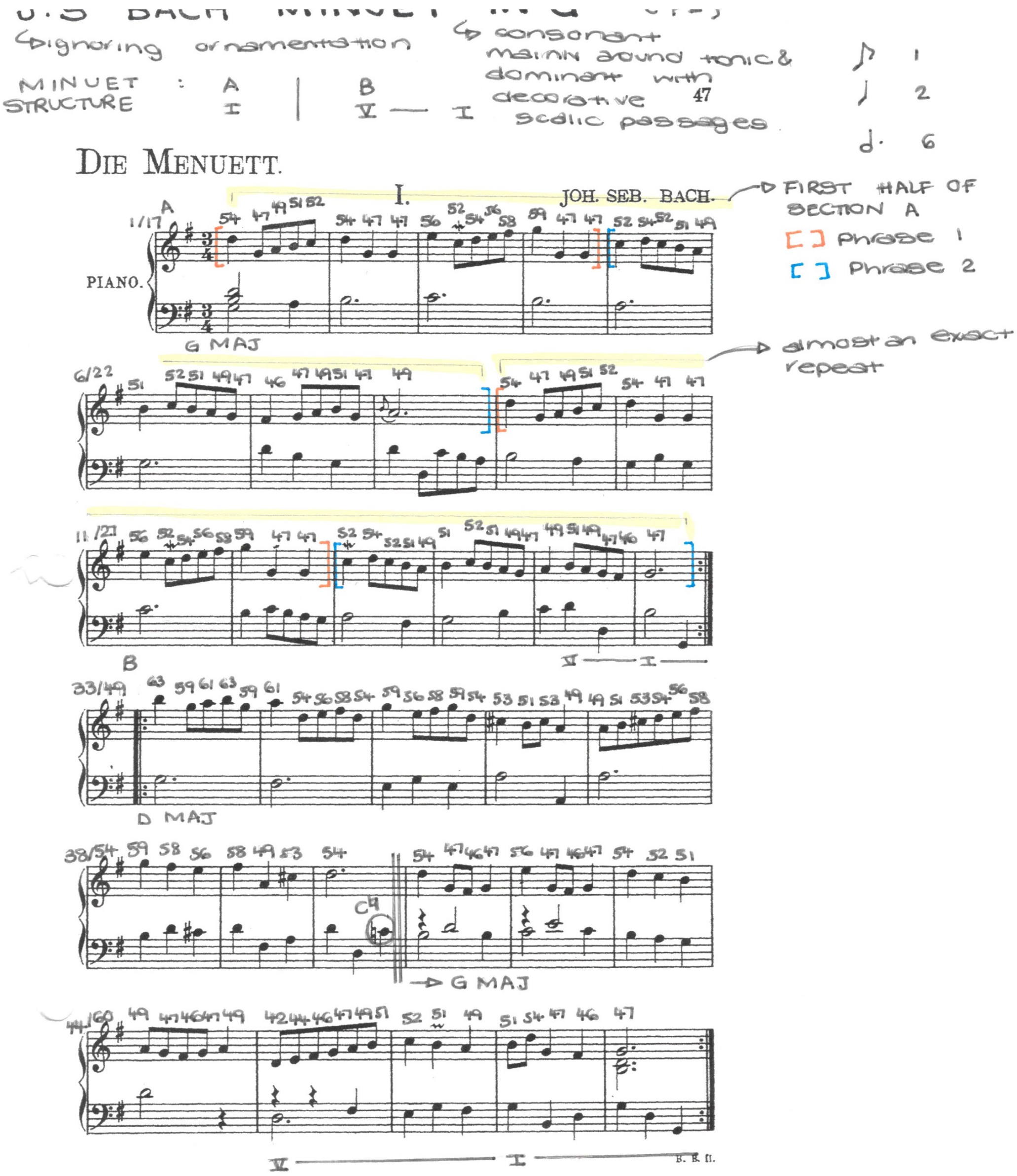


Figure 57: Full score of Bach's Minuet in G [5]. The two sections are labelled with A and B. In section A, phrase 1 is denoted by a orange square brackets and phrase 2 in blue brackets. The two phrases are denoted in the yellow.

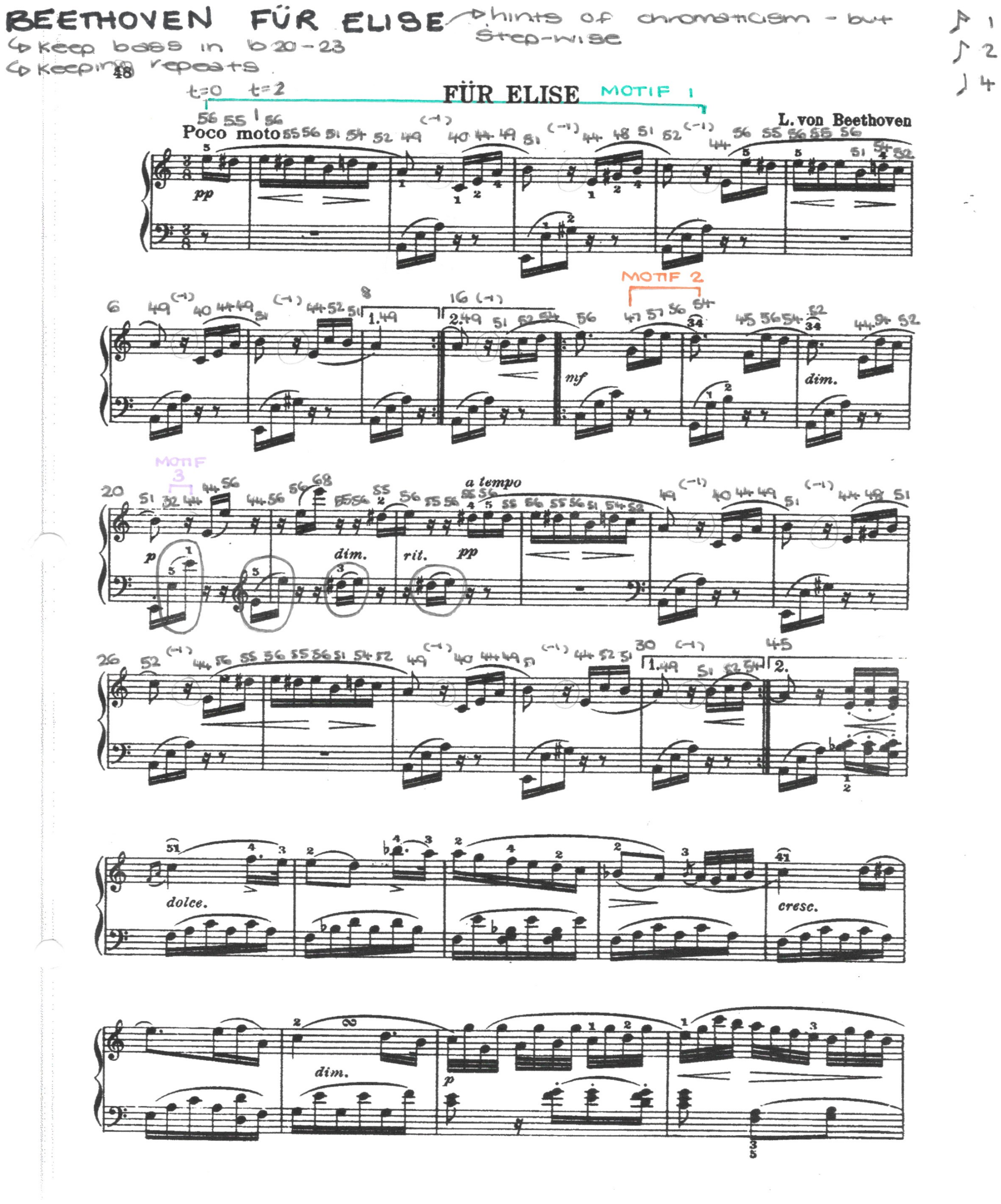
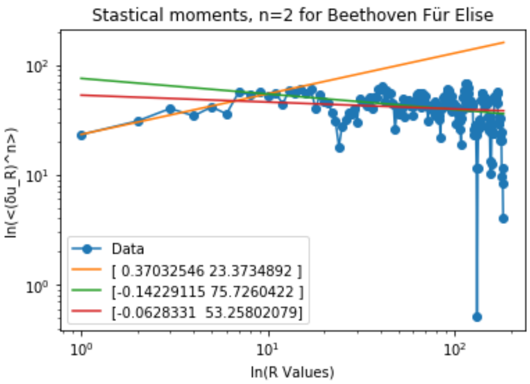
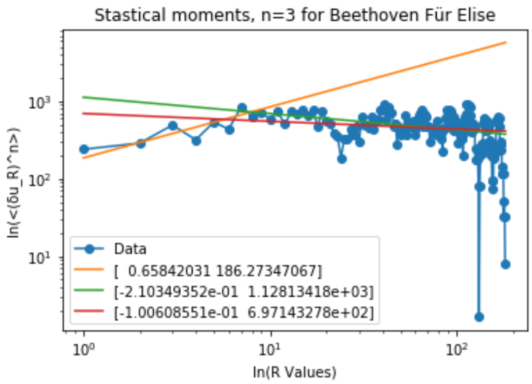


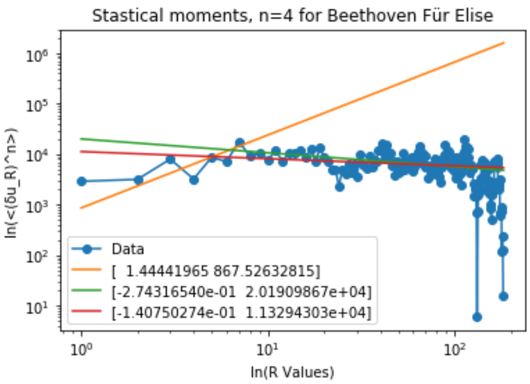
Figure 58: Full score of Beethoven's Für Elise [8]. The three motifs are labelled in the score, the first in turquoise, the second in oange and the third in pink.



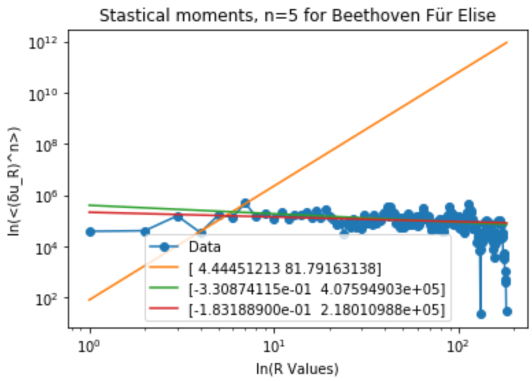
(a) n=2



(b) n=3



(c) n=4



(d) n=5

Figure 59: A plot of the natural logarithm of the statistical moment for n=2, 3, 4, and 5 against the R values, or time difference, with each R marked by circle as represented by the blue line, for the data in Beethoven's Für Elise. The data is then fitted by a least-square fit to approximate a linear function, considering the first six R values, denoted by the orange, all the data points after the first six, denoted by the green, and finally the red line takes all data points into consideration. The arrays in the key denotes the values of ξ_n and C_2 as labelled by equation 1.5.6

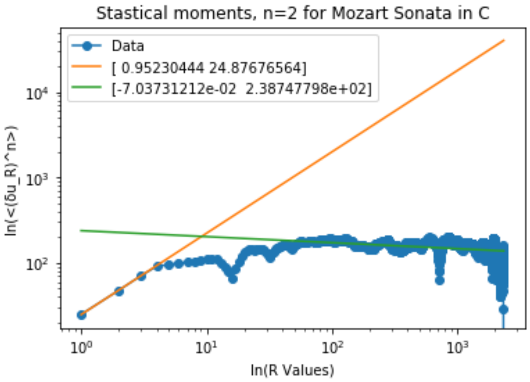
(a) p. 1

(b) p. 2

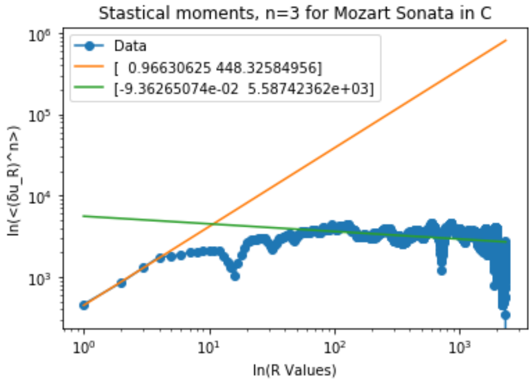
(c) p. 3

(d) p. 4

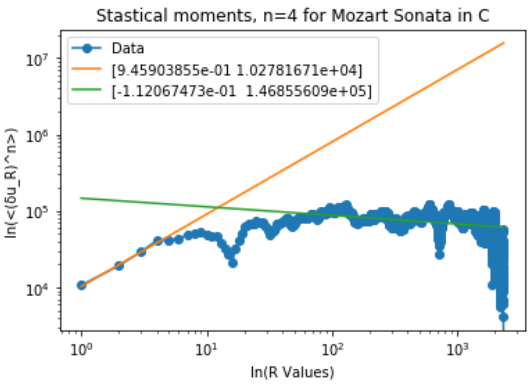
Figure 60: Full score of Mozart's Sonata in C [23]. The section for the exposition, development and recapitulation are labelled on the score as well as the tonalities of each sections. The primary theme is highlighted in yellow, the secondary theme in pink, and the motif in the codetta highlighted with an orange box.



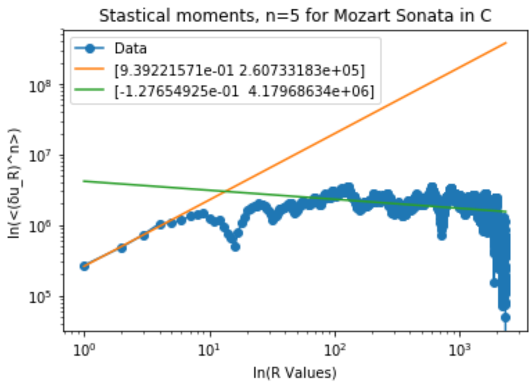
(a) n=2



(b) n=3



(c) n=4



(d) n=5

Figure 61: A plot of the natural logarithm of the statistical moment for n=2, 3, 4, and 5 against the R values, or time difference, with each R marked by circle as represented by the blue line, for the data in Mozart's Sonata in C. The data is then fitted by a least -square fit to approximate a linear function, considering the first six R values, denoted by the orange, all the data points after the first six, denoted by the green, and finally the red line takes all data points into consideration. The arrays in the key denotes the values of ξ_n and C_2 as labelled by equation 1.5.6