Linear and Weakly Nonlinear Analysis of Rayleigh–Bénard Convection and Doubly-Diffusive Convection

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Abstract

In this report, we analyse the behaviour of Rayleigh–Bénard convection and doubly-diffusive convection in infinite horizontal layers of fluid.

We begin by formulating the equations governing Rayleigh–Bénard convection and doubly-diffusive convection from physical laws that describe the motion of fluids. We therefore saw that the dynamics of Rayleigh–Bénard convection is controlled by non-dimensional parameters Pr and Ra. Similarly, the dynamics of doubly-diffusive convection is controlled by the non-dimensional parameters Pr, Le and N.

We then perform linear stability calculations to find the critical values in terms of these parameters at which instability arises in the Rayleigh–Bénard and the doubly-diffusive problems. This involved subjecting the base state of the fluid to small infinitesimal perturbations, and then neglecting the nonlinear perturbations since these are even smaller than the linear perturbations.

We then perform a weakly nonlinear analysis by expanding the solutions in terms of a small parameter ϵ, and then successively finding solutions using the solvability condition for increasingly smaller terms. This then allows us to write Landau equations, which we then use to find the form of the primary bifurcations of each problem and for what parameter values they are subcritical or supercritical. We supplement this knowledge with bifurcation diagrams generated using the continuation and bifurcation package for MATLAB called pde2path.

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Chapter 1

Introduction

1.1 Motivation

In fluid dynamics, convection is broadly described as the motion of fluids. It is the sum of fluid motion by bulk, large-scale movement, called advection, and individual molecular movement, called diffusion. It may occur via many different mechanisms such as forced convection, which is a flow driven by an external source, or natural convection, which is a flow induced by density differences that generate buoyancy forces.

Many physical phenomena can be described by convective flows. Thermal convection, the transfer of heat due to convection, can be observed for example, when heating a glass container of fluid with a Bunsen burner. This will result in temperature changes in the container due to the circulation of the fluid from warmer to cooler areas. Similarly, solutal convection, the transfer of a solute in a fluid caused by convection, can be observed when mixing soluble ions in a container of water by stirring. More complex geophysical and astrophysical systems can also be explained by convective flows such as ocean circulations (Stommel and Arons, 1959a,b; Rahmstorf, 2003), plate tectonics (Wilson, 1963; Oxburgh and Turcotte, 1970; Weertman, 1978) and dynamo theory (Parker, 1955; Ovchinnikov and Enßlin, 2016).

In this report, we will be concerned with two natural convection problems: Rayleigh–Bénard convection and doubly-diffusive convection. Rayleigh–Bénard convection is the simplest convective problem we can study because of its straightforward geometry and thermodynamic properties. The mathematical analysis we perform will provide the groundwork for studying doubly-diffusive convection, a natural extension of the Rayleigh–Bénard problem, and potentially other more complicated flows.

1.2 Rayleigh–Bénard Convection

Consider a horizontal layer of fluid of depth d that is heated from below. We can then define the temperature at the bottom and the top of the fluid to be T_0 and T_1 respectively. The fluid particles at the bottom of the layer have more kinetic energy due to the applied heat. Therefore, they vibrate and move more, and maintain a greater average distance between themselves. Due to this thermal expansion, the fluid at the top of the layer will be denser than the fluid at the bottom. As this is a potentially unstable arrangement, the fluid will want to redistribute itself to a more stable arrangement. However, this is inhibited by the viscosity of the fluid. Therefore, in order for the instability to manifest, we expect the unfavourable temperature gradient that is maintained to exceed a certain critical value.

The first experiments that demonstrated the onset of this thermal instability in fluids

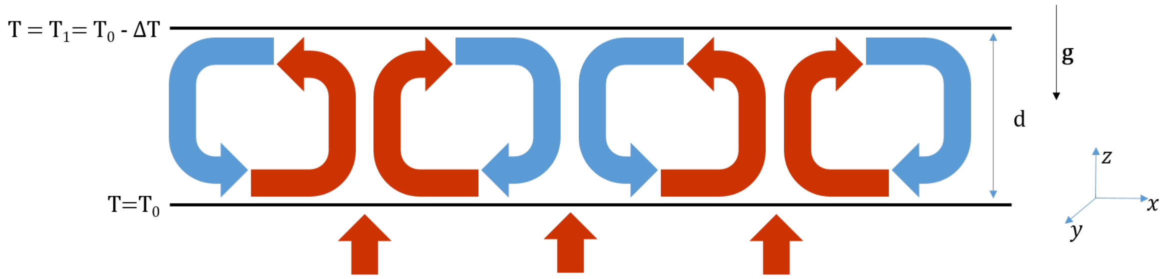

Figure 1.1: Schematic diagram of convection cells occuring in Rayleigh–Bénard convection. Red and blue represent higher and lower temperatures respectively.

were carried out by Bénard (1900). He found that when the unfavourable temperature gradient surpassed the critical value, the fluid settled into a regular hexagonal pattern of cells. In subsequent years, others (Schmidt and Milverton, 1935; Schmidt and Saunders, 1938; Bénard and Avsec, 1938; Saunders et al., 1935) have replicated the results using a variety of experimental methods, details of which are summarised in Chandrasekhar (1961, Chapter II, Section 18) and Koschmieder (1993, Part 1, Chapter 1).

Lord Rayleigh (1916) analysed the problem Bénard studied and showed that the stability of the fluid is determined by the value of what we now call the *Rayleigh number*, Ra. This is defined as

$$Ra = \frac{g\alpha\beta}{\kappa\nu} d^3, \tag{1.2.1}$$

where g is the acceleration due to gravity, d is the depth of the layer as defined above, $\beta = \frac{dT}{dz}$ is the uniform temperature gradient that is maintained, α is coefficient of volume expansion, κ is the thermometric conductivity coefficient and ν is the kinematic viscosity coefficient. How these coefficients arise will become apparent in Chapter 2. Instability occurs at the primary bifurcation, when Ra exceeds a critical value Ra_c. We will determine the value of Ra_c in Chapter 3.

1.3 Doubly-Diffusive Convection

Doubly-diffusive convection, as we have already said, is a natural extension of the classical Rayleigh–Bénard thermal convection problem. Motivated by the fact that many processes in engineering, biology, meteorology and other branches of science involve the transfer of mass through a concentration gradient, Veronis (1965, 1968) and Sani (1965) studied the convective instability that arises in a fluid layer that undergoes thermal and solutal convection. If we consider a fluid layer much like the one in the Rayleigh–Bénard problem, that is subjected to a higher temperature and concentration at the bottom of the layer, then we may observe oscillatory motions. The instability arises because if a fluid particle from the warmer and more concentrated region is raised in the layer, it finds itself in colder and less concentrated surroundings. As the rate of thermal diffusion is usually greater than the rate of solutal diffusion, heat will equilibrate faster than molecular concentration. This now leaves the particle heavier than its surroudings, which causes it to sink. However, the particle returns to its original position heavier than it began because the temperature field of the particle causes lag in the displacement field. Therefore it sinks further, leading to greater oscillations as time progresses. A schematic diagram of this is depicted in Figure 1.2.

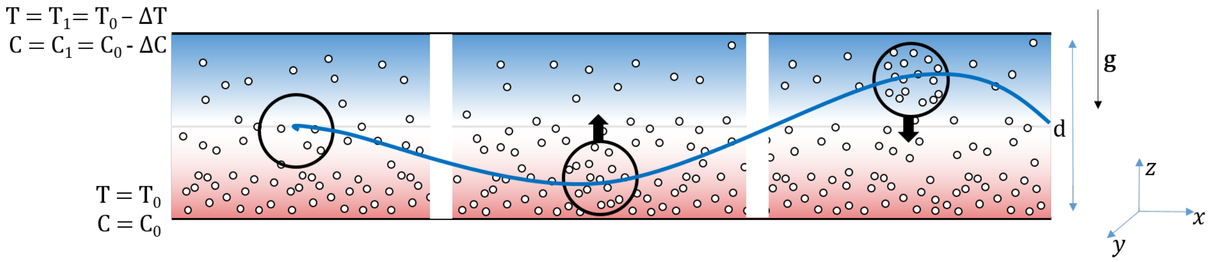

Figure 1.2: Schematic diagram adapted from Garaud (2018) of convection arising through an oscillatory instability that arises in doubly-diffusive convection. Red and blue represent a higher or lower temperature respectively. Fluid particles are depicted by white circles.

Motivated by an oceanographical phenomenon observed by Stommel et al. (1956), Melvin Stern (Stern, 1960) discovered the phenomena that is now known as the salt finger instability. This type of convection occurs in a fluid layer that is subjected to a higher temperature and concentration at the top of the layer. A schematic diagram of this phenomenon can be seen in Figure 1.3.

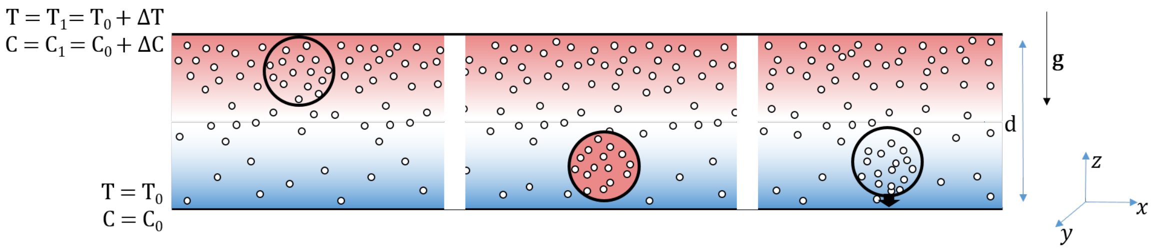

Figure 1.3: Schematic diagram adapted from Garaud (2018) of convection arising through the salt finger instability that arises in doubly-diffusive convection. Red and blue represent a higher or lower temperature respectively. Fluid particles are depicted by white circles.

This instability occurs because if a fluid particle from the warmer and more concentrated area is displaced down, its temperature equilibrates very quickly with its surrounds through diffusion, but its concentration does not. This decrease in temperature causes the density within the parcel to increase, which in turn causes it to sink further.

We will determine the critical values at which these instabilities occur in Chapter 3 and then use these to determine the types of primary bifurcations that occur in Chapter 4.

1.4 Report Outline

In order to study the behaviours of fluids in Rayleigh–Bénard convection and doubly-diffusive convection, we will first need to derive the governing equations. We do this from physical concepts and laws in Chapter 2.

In deriving the governing equations, we will subject the fluid to perturbations before we begin with our linear and weakly nonlinear analyses in Chapter 3 and Chapter 4 respectively. The idea behind these perturbation equations is to obtain approximate analytical solutions to a problem so that we may understand it better. To illustrate this concept we will consider the following example adapted from Francis (2011). Suppose we have a rigid rod pendulum. This has two equilibrium points (or positions where the pendulum will remain stationary): one where the pendulum points directly downwards and one where the pendulum points upwards (similar to balancing a pencil on its end).

Now the question is how do we know whether the equilibria are stable or unstable? If we nudge, or perturb, the pendulum slightly from the downward-facing equilibrium point, we find that it will oscillate but then settle back towards the same equilibrium point. This indicates that trajectories that start near this equilibrium will tend towards it, implying a stable equilibrium. However, if we perturb the pendulum slightly from the upward-facing equilibrium, it will evetually settle at the downward-facing equilibrium. This indicates that trajectories that start near this equilibrium will tend away from it, implying an unstable equilibrium. Thus, by introducing a perturbation, we can determine local dynamics of the pendulum and therefore understand the system better.

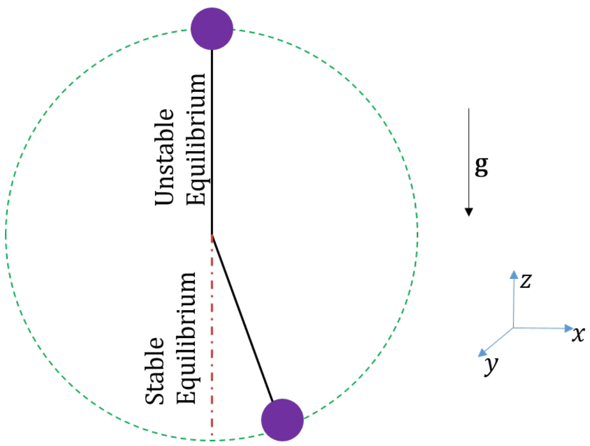

Figure 1.4: Diagram of a rigid rod pendulum and its equilibria.

Once we have subjected our equations to (small) perturbations, we will only consider terms up to a certain order. For example, in linear stability analysis, we subject the equations to infinitesimally small perturbations and then disregard the nonlinear terms. To illustrate this concept consider the following scenario adapted from Plait (2009). Suppose we are standing on a ship deck and are looking into the distance. How far away is the horizon?

If we assume that the Earth is a uniform sphere with a radius r, of $6,400$km and our eyes are at height $h = 10$m above sea level, then by Pythagoras' theorem, we find that the distance to the horizon d, is given by

$$d^2 = 2rh + h^2. \tag{1.4.1}$$

Suppose now we define the non-dimensional parameter δ to be $\delta \equiv \frac{h}{r}$ and rewrite the above equation as

$$\left(\frac{d}{r}\right)^2 = 2\delta + \delta^2. \tag{1.4.2}$$

For our particular values of h and r, we find that $\delta \approx 1.56 \times 10^{-6}$, so it is reasonable to neglect the δ^2 term since it is much smaller than the already small δ term.

Consequently, we may prefer to say the horizon can be approximated by

$$d^2 \approx 2rh, \tag{1.4.3}$$

which gives us $d \approx 11.3$km for the values we have assumed.

Why might we prefer this approximation to the exact answer? Suppose we got off the boat and now the height of our eyes above sea level has changed. In this example, calculating

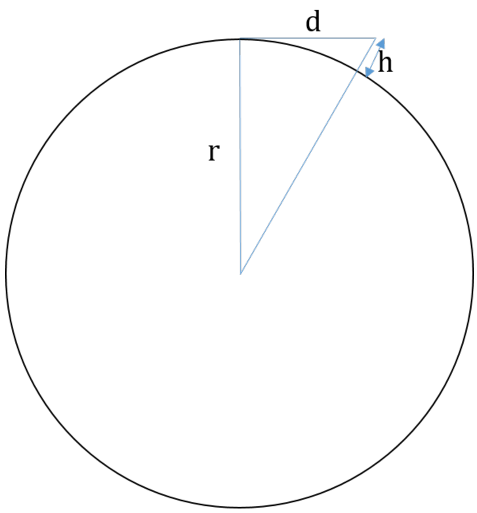

Figure 1.5: Schematic diagram showing the geometrical distance to the horizon.

the exact answer is not difficult, but for more complicated problems, it might much be easier to only have to deal with the leading order terms. We should also note that the Earth is not actually a perfect sphere, so the radius of the earth is not actually going to be uniform like we have assumed. The ocean is also not flat and may be perturbed by currents and gravity. Additionally, our vision may be obstructed by other things in the horizon such as clouds and haze. Considering all of these, the small δ^2 that we have neglected is likely not as important as the other complications of the formulation of the problem.

Using these concepts, we will now begin our study of Rayleigh–Bénard convection and doubly-diffusive convection.

Chapter 2

The Governing Equations

2.1 The Fundamental Hydrodynamic Equations

2.1.1 The Continuity Equation

The *principle of mass conservation* states that for a closed system, the mass of the system remains constant over time. Now, consider a fixed volume V with surface S and outward normal $\hat{\mathbf{n}}$. The total mass in V is given by

$$\int_V \rho dV, \tag{2.1.1}$$

where $\rho \equiv \rho(\mathbf{x}, t)$ is the density in terms of the position $\mathbf{x} = (x, y, z)$ and time t and $dV = dxdydz$ is a volume element. Due to the conservation of mass, expression (2.1.1) can only change if fluid is transferred in or out of V. The rate of this change of mass in time is then given by

$$\frac{d}{dt}\int_V \rho\, dV = -\int_S \rho\mathbf{u}\cdot\hat{\mathbf{n}}\, dS, \tag{2.1.2}$$

where $\mathbf{u} \equiv \mathbf{u}(\mathbf{x}, t)$ is the fluid velocity, dS is a surface element of S with outward normal $\hat{\mathbf{n}}$. Since V is fixed in space for all time t, we may write,

$$\int_V \frac{\partial\rho}{\partial t}\, dV = -\int_S \rho\mathbf{u}\cdot\hat{\mathbf{n}}\, dS. \tag{2.1.3}$$

Using the divergence theorem (see equation A.1.1), and rearranging we obtain

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right] dV = 0. \tag{2.1.4}$$

Since V is arbitrary, we have, for any volume V

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{2.1.5}$$

which is the general expression for the *continuity equation.*

2.1.2 The Momentum Equation

Consider again a fixed volume V of fluid, with surface S and outward normal $\hat{\mathbf{n}}$. *Newton's second law of motion* states that the rate at which the momentum of a body changes is proportional to the force applied. This change occurs in the direction of the applied force. If we let $\rho \equiv \rho(\mathbf{x}, t)$ be the density and $\mathbf{u} \equiv \mathbf{u}(\mathbf{x}, t)$ be the fluid velocity as defined in Section 2.1.1, then the rate of change of momentum in V is given by

$$\frac{d}{dt}\left(\int_V \rho\mathbf{u}\, dV\right) = \int_V \frac{\partial}{\partial t}(\rho\mathbf{u})\, dV, \tag{2.1.6}$$

since V is fixed in space for all time t.

By Newton's second law, this is equal to the net flow of momentum through the boundary S added to the net force acting on V. So we can write,

$$\int_V \frac{\partial}{\partial t}(\rho\mathbf{u})\, dV = -\int_S \rho\mathbf{u}\mathbf{u}\cdot\hat{\mathbf{n}}\, dS + \int_V \mathbf{F}\, dV + \int_S \mathbf{f}\, dS, \tag{2.1.7}$$

where the first integral on the right hand side is the net flow of momentum in V (since we have taken $\hat{\mathbf{n}}$ to be the outward normal), the second integral represents the macroscopic body forces acting on the fluid, with force density $\mathbf{F}$, and the third integral represents the microscopic stress interactions between fluid molecules either side of S, with stress density $\mathbf{f}$.

Using the divergence theorem (A.1.1), we can rewrite the previous equation as

$$\int_V \left[\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \mathbf{F} - \nabla\cdot\tau\right] dV = 0, \tag{2.1.8}$$

where we have chosen to define the stress tensor τ such that $\mathbf{f} \equiv \tau\cdot\hat{\mathbf{n}}$. Since V is arbitrary, we can write

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \mathbf{F} - \nabla\cdot\tau = 0. \tag{2.1.9}$$

This can also be written as

$$\mathbf{u}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}(\nabla\cdot(\rho\mathbf{u})) + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mathbf{F} - \nabla\cdot\tau = 0. \tag{2.1.10}$$

But since we have

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0,$$

from equation (2.1.5), we can reduce equation (2.1.10) to

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{F} + \nabla\cdot\tau, \tag{2.1.11}$$

the general expression of the *momentum equation.*

For a *Newtonian fluid*, the viscous stress is equal to rate of strain of the fluid. In tensor notation, this is written as

$$\tau_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial u_k}\delta_{ij}\right), \tag{2.1.12}$$

where P is the isotropic pressure at position $\mathbf{x}$ when there is no strain and μ is the dynamic viscosity.

In the absence of any macroscopic forces other than gravity, we can write $\mathbf{F}$ as

$$\mathbf{F} = \rho\mathbf{g}, \tag{2.1.13}$$

where $\mathbf{g}$ is the acceleration due to gravity. Substituting equations (2.1.12) and (2.1.13) into (2.1.11), we have the momentum equation in tensor notation,

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} = \rho g_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right], \tag{2.1.14}$$

or the momentum equation in vector notation,

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla P + \rho\mathbf{g} + \mu\left(\nabla^2\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\right). \tag{2.1.15}$$

2.1.3 The Heat Transfer Equation

The *principle of energy conservation* states that for a closed system, the energy of the system remains constant over time. If we consider a fixed volume V with surface S and outward normal $\hat{\mathbf{n}}$, then the rate of change in energy of the fluid in V per unit time t is

$$\frac{d}{dt}\left(\int_V \rho E \, dV\right) = \int_V \frac{\partial}{\partial t}(\rho E) \, dV, \tag{2.1.16}$$

where ρ is the density as defined before and $E \equiv \frac{1}{2}\|\mathbf{u}\|^2 + c_V T$ is the energy per unit mass of the fluid, with temperature T, specific heat capacity at constant volume c_V and the velocity of the fluid $\mathbf{u}$, as defined before.

Since energy must be conserved, equation (2.1.16) is equal to the sum of the net flux of energy in the form of heat across S and the rate of work done on the fluid by body and surface forces. Therefore, we can write,

$$\int_V \frac{\partial}{\partial t}(\rho E) \, dV = \underbrace{-\int_S \rho E \mathbf{u} \cdot \hat{\mathbf{n}} \, dS}_{\mathbf{I}} + \underbrace{\int_S k \nabla T \, dS}_{\mathbf{II}} + \underbrace{\int_V \mathbf{u} \cdot \mathbf{F} \, dV}_{\mathbf{III}} + \underbrace{\int_S \mathbf{u} \cdot \mathbf{f} \, dS}_{\mathbf{IV}}, \tag{2.1.17}$$

where **I** is the rate of energy convection across S by the mass motions in the form of heat, **II** is the rate of energy conduction across S, with heat conduction coefficient k, **III** represents the rate of work done on the fluid by macroscopic body forces, with force density $\mathbf{F}$, and **IV** represents the rate of work done on the boundary S by microscopic stress interactions, with stress density $\mathbf{f}$.

We can simplify equation (2.1.17) by rewriting **I** and **II** using the divergence theorem (A.1.1). **I** can be written as

$$\begin{aligned} -\int_S \rho E \mathbf{u} \cdot \hat{\mathbf{n}} \, dS &= -\int_S \rho \left(\frac{1}{2}\|\mathbf{u}\|^2 + c_V T\right) \mathbf{u} \cdot \hat{\mathbf{n}} \, dS \\ &= -\frac{1}{2}\int_S \rho \|\mathbf{u}\|^2 \mathbf{u} \cdot \hat{\mathbf{n}} \, dS - \int_V \nabla \cdot (\rho c_V T \mathbf{u}) \, dV, \end{aligned} \tag{2.1.18}$$

while **II** can be written as

$$\int_S k \nabla T \, dS = \int_V \nabla \cdot (k \nabla T) \, dV. \tag{2.1.19}$$

As in Section 2.1.2, if the only macroscopic force we consider is gravity, then we write $\mathbf{F} = \rho \mathbf{g}$ where $\mathbf{g}$ is the acceleration due to gravity. Then **III** becomes

$$\int_V \mathbf{u} \cdot \mathbf{F} \, dV = \int_V \mathbf{u} \cdot \rho \mathbf{g} \, dV. \tag{2.1.20}$$

We can rewrite **IV** by manipulating the general form of the momentum equation (2.1.11). Multiplying this equation by $\mathbf{u}$ and integrating by volume V, we have

$$\frac{1}{2}\int_V \rho \frac{\partial}{\partial t}\|\mathbf{u}\|^2 \, dV + \frac{1}{2}\int_V \rho(\mathbf{u} \cdot \nabla)\|\mathbf{u}\|^2 \, dV = \int_V \mathbf{u} \cdot \mathbf{F} \, dV + \int_V \mathbf{u} \cdot (\nabla \cdot \tau) \, dV. \tag{2.1.21}$$

Integrating the second integral on both the left and right hand side of equation (2.1.21) by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\int_V \left[\rho \frac{\partial}{\partial t}\|\mathbf{u}\|^2 - \|\mathbf{u}\|^2 (\nabla \cdot (\rho \mathbf{u}))\right] dV + \frac{1}{2}\int_S \rho \|\mathbf{u}\|^2 \mathbf{u} \cdot \hat{\mathbf{n}} \, dS \\ &\qquad = \int_V \mathbf{u} \cdot \mathbf{F} \, dV + \int_S \mathbf{u} \cdot \tau \cdot \hat{\mathbf{n}} \, dS - \int_V \tau \cdot (\nabla \cdot \mathbf{u}) \, dV. \end{aligned} \tag{2.1.22}$$

By using equation (2.1.5), the general expression of the continuity equation, we can write the first integral on the left hand side of the previous equation as

$$\begin{aligned}\frac{1}{2}\int_V \left[\rho\frac{\partial}{\partial t}\left\|\mathbf{u}\right\|^2 + \left\|\mathbf{u}\right\|^2\frac{\partial}{\partial t}\rho\right] dV &= \frac{1}{2}\int_V \frac{\partial}{\partial t}\left(\rho\left\|\mathbf{u}\right\|^2\right) dV \\ &= \frac{1}{2}\frac{d}{dt}\int_V \rho\left\|\mathbf{u}\right\|^2 dV,\end{aligned} \tag{2.1.23}$$

so that we have,

$$\frac{1}{2}\frac{d}{dt}\int_V \rho\left\|\mathbf{u}\right\|^2 dV + \frac{1}{2}\int_S \rho\left\|\mathbf{u}\right\|^2 \mathbf{u}\cdot\hat{\mathbf{n}}\, dS = \int_V \mathbf{u}\cdot\mathbf{F}\, dV + \int_S \mathbf{u}\cdot\tau\cdot\hat{\mathbf{n}}\, dS - \int_V \tau\cdot(\nabla\cdot\mathbf{u})\, dV. \tag{2.1.24}$$

Therefore, we can now rewrite **IV** as

$$\begin{aligned}\int_S \mathbf{u}\cdot\mathbf{f}\, dS &= \int_S \mathbf{u}\cdot\tau\cdot\hat{\mathbf{n}}\, dS \\ &= \frac{1}{2}\frac{d}{dt}\int_V \rho\left\|\mathbf{u}\right\|^2 dV + \frac{1}{2}\int_S \rho\left\|\mathbf{u}\right\|^2 \mathbf{u}\cdot\hat{\mathbf{n}}\, dS \\ &\quad - \int_V \mathbf{u}\cdot\mathbf{F}\, dV + \int_V \tau\cdot(\nabla\cdot\mathbf{u})\, dV,\end{aligned} \tag{2.1.25}$$

where we have defined $\mathbf{f} = \tau\cdot\hat{\mathbf{n}}$ as in Section 2.1.2.

For a Newtonian fluid, τ is as defined in equation (2.1.12). Using this, **IV** becomes

$$\begin{aligned}\int_S \mathbf{u}\cdot\mathbf{f}\, dS &= \frac{1}{2}\frac{d}{dt}\int_V \rho\left\|\mathbf{u}\right\|^2 dV + \frac{1}{2}\int_S \rho\left\|\mathbf{u}\right\|^2 \mathbf{u}\cdot\hat{\mathbf{n}}\, dS \\ &\quad - \int_V \mathbf{u}\cdot\mathbf{F}\, dV - \int_V P(\nabla\cdot\mathbf{u})\, dV + \int_V \Phi\, dV,\end{aligned} \tag{2.1.26}$$

where P is the isotropic pressure at position $\mathbf{x}$ when there is no strain and Φ is defined as

$$\Phi = \frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 - \frac{2}{3}\mu\left(\frac{\partial u_k}{\partial x_k}\right)^2, \tag{2.1.27}$$

where μ is dynamic viscosity.

Combining equations (2.1.18),(2.1.19),(2.1.20) and (2.1.26), and making use of the fact that E is defined as $E = \frac{1}{2}\left\|\mathbf{u}\right\|^2 + c_V T$, we obtain the following integral.

$$\int_V \left[\frac{\partial}{\partial t}\rho c_V T + \nabla\cdot(\rho c_V T\mathbf{u}) - \nabla\cdot(k\nabla T) + P(\nabla\cdot\mathbf{u}) - \Phi\right] dV = 0. \tag{2.1.28}$$

Since V is arbitrary, we can write

$$\frac{\partial}{\partial t}(\rho c_V T) + \nabla\cdot(\rho c_V T\mathbf{u}) = \nabla\cdot(k\nabla T) - P(\nabla\cdot\mathbf{u}) + \Phi. \tag{2.1.29}$$

Using the chain rule to expand the terms on the left hand side and using equation (2.1.5), we finally obtain the *heat transfer equation*

$$\rho\frac{\partial}{\partial t}(c_V T) + \rho\mathbf{u}\cdot\nabla(c_V T) = \nabla\cdot(k\nabla T) - P(\nabla\cdot\mathbf{u}) + \Phi. \tag{2.1.30}$$

2.1.4 The Mass Transfer Equation

If a fluid is composed of more than one component whose concentrations vary at different points, then there is a natural tendency for mass to be transferred in order to minimise the concentration difference throughout the system. This mass transfer is governed by *Fick's first law*, which states that the diffusive flux from higher to lower concentrations is proportional to the concentration gradient of the substance in question. This is represented mathematically as

$$\mathbf{j} = -D\nabla C, \tag{2.1.31}$$

where $C \equiv C(\mathbf{x}, t)$ is the concentration of a substance in the fluid, D is the diffusivity coefficient, and $\mathbf{j}$ is the diffusive flux that represents the amount of the substance that will flow through a unit area per unit time t.

Let us now consider a non-homogeneous fluid with a fixed volume V with surface S and outward normal $\hat{\mathbf{n}}$. The rate of change in concentration of a substance in the fluid in V per unit time t is then

$$\frac{d}{dt}\left(\int_V C\,dV\right) = -\int_S \mathbf{j}\cdot\hat{\mathbf{n}}\,dS. \tag{2.1.32}$$

By Reynold's transport theorem (Reynolds et al., 1903), we can rewrite equation (2.1.32) as

$$\frac{d}{dt}\int_V C\,dV = \int_V \frac{\partial C}{\partial t}\,dV + \int_S C\mathbf{u}\cdot\hat{\mathbf{n}}\,dS = -\int_S \mathbf{j}\cdot\hat{\mathbf{n}}\,dS, \tag{2.1.33}$$

where $\mathbf{u}$ is the fluid velocity as defined before.

Using the divergence theorem (A.1.1) we obtain

$$\int_V \left[\frac{\partial C}{\partial t} + \nabla\cdot(C\mathbf{u})\right] dV = -\int_V \nabla\cdot\mathbf{j}\,dV. \tag{2.1.34}$$

Since V was arbitrary, we may write equation (2.1.34) as

$$\frac{\partial C}{\partial t} + \nabla\cdot(C\mathbf{u}) = -\nabla\cdot\mathbf{j}, \tag{2.1.35}$$

which when expanded using the chain rule and noting Fick's first law (2.1.31), becomes

$$\frac{\partial C}{\partial t} + C(\nabla\cdot\mathbf{u}) + (\mathbf{u}\cdot\nabla)C = D\nabla^2 C. \tag{2.1.36}$$

Expanding the continuity equation (2.1.5) by the chain rule and rearranging, we may rewrite equation (2.1.36) as

$$\frac{\partial C}{\partial t} - \frac{C}{\rho}\left(\frac{\partial \rho}{\partial t} + (\mathbf{u}\cdot\nabla)\rho\right) + (\mathbf{u}\cdot\nabla)C = D\nabla^2 C, \tag{2.1.37}$$

where ρ is the density as defined before. This is known as the *mass transfer equation*.

2.2 The Boussinesq Approximation

The governing hydrodynamic equations that we derived in Section 2.1 can be simplified further by an important result called the *Boussinesq approximation*, first detailed by Boussinesq in 1903. (See also Chandrasekhar, 1961; Spiegel and Veronis, 1960 for further details.)

The basis of this approximation is to assume that we have a flow with small temperature variations. Then, the density ρ, that we defined in the previous section also varies by a small amount.

Let us consider a horizontal layer of fluid of depth d that is heated from below as in the Rayleigh-Bénard problem described in Section 1.2. We then let ρ_0 be the density at the bottom of the fluid at temperature T_0. For a small temperature difference between the top and the bottom of the fluid, we define the density to be

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{2.2.1}$$

where α is the coefficient of volume expansion, which is approximately $10^{-4} - 10^{-3}K^{-1}$ for a liquid. We call equation (2.2.1) the *equation of state*. For a temperature variation of a moderate amount, we can write

$$\frac{|\rho - \rho_0|}{\rho_0} = \alpha|T - T_0| \ll 1. \tag{2.2.2}$$

The variations of the coefficients k, μ and c_V must be of the same order, due to the small amounts of variation of density, and so can be ignored. However, we cannot neglect the variability of the density in the $\rho\mathbf{g}$ term in the momentum equation (2.1.15) since the acceleration from this term can be quite large.

We now take the fundamental hydrodynamic equations, substitute ρ with the expression in equation (2.2.1) and simplify them based on these remarks.

The continuity equation (2.1.5) is reduced to

$$\nabla \cdot \mathbf{u} = 0, \tag{2.2.3}$$

as for an incompressible fluid.

With this condition on $\mathbf{u}$, we consider the momentum and the heat transfer equations. For an incompressible fluid, equation (2.1.15) reduces to

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla P + \rho\mathbf{g} + \mu\nabla^2\mathbf{u} \tag{2.2.4}$$

which is also known as the *Navier–Stokes equation.* Treating μ as a constant with the above approximations, this becomes

$$\rho_0\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = -\nabla P + \rho_0(1 - \alpha(T - T_0))\mathbf{g} + \rho_0\nu\nabla^2\mathbf{u}, \tag{2.2.5}$$

where $\nu \equiv \frac{\mu}{\rho_0}$ is the kinematic viscosity.

We now consider the heat transfer equation (2.1.30). Due to equation (2.2.3), we can neglect the $-P(\nabla \cdot \mathbf{u})$ term. For an incompressible fluid, Φ is reduced to

$$\Phi = \frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2, \tag{2.2.6}$$

however, we can also neglect the term Φ since it is of the order 10^{-8}–10^{-7} (see Chandrasekhar, 1961, Chapter II, Section 8). Treating c_V and k as constants, the heat transfer equation then reduces to

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \kappa\nabla^2 T, \tag{2.2.7}$$

where $\kappa \equiv \frac{k}{\rho_0 c_V}$ is the coefficient of thermal diffusion.

Note that in the Rayleigh–Bénard problem, the fluid is assumed to be homogeneous, so we need not consider the mass transfer equation.

Let us now consider a binary fluid that occupies the space of length d between two planes as described in Section 1.3. We let ρ_0 be the density at the bottom (right) of the fluid in the horizontal (vertical) problem at temperature T_0 and concentration C_0. For small temperature and concentration differences between the two planes, we define the density to be

$$\rho = \rho_0[1 - \alpha_T(T - T_0) + \alpha_C(C - C_0)], \tag{2.2.8}$$

where α_T is coefficient of thermal expansion and α_C is the coefficient of solutal contraction. For temperature and concentration variations of a moderate amount, we can write

$$\frac{|\rho - \rho_0|}{\rho_0} = \alpha_T|T - T_0| + \alpha_C|C - C_0| \ll 1. \tag{2.2.9}$$

Following similar reasoning, the Boussinesq equations of motion for doubly-diffusive convection are given by

$$\nabla \cdot \mathbf{u} = 0, \tag{2.2.10}$$

$$\rho_0\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla P + \rho_0(1 - \alpha_T(T - T_0) + \alpha_C(C - C_0))\mathbf{g} + \rho_0\nu\nabla^2\mathbf{u}, \tag{2.2.11}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T = \kappa_T\nabla^2 T, \tag{2.2.12}$$

$$\frac{\partial C}{\partial t} + (\mathbf{u}\cdot\nabla)C = \kappa_C\nabla^2 C, \tag{2.2.13}$$

where κ_T is the coefficient of thermal diffusion and κ_C is the coefficient of solutal diffusion.

2.3 Mathematical Formulation of the Rayleigh–Bénard Problem

Now that we have derived the governing Boussinesq equations of motion we can formulate the systems we will study in the rest of the report, beginning with the Rayleigh–Bénard problem. Let us consider an infinite horizontal layer of fluid of depth d that is heated from below. The temperature gradient β is the driving force of the convection. We define the position to be $\mathbf{x} = (x, y, z)$ and let our variables to be functions of $\mathbf{x}$ and time, t. We also take gravity to be $\mathbf{g} = -g\mathbf{e_z}$, that is gravity acts in the negative z–direction.

2.3.1 Boundary Conditions

When we solve the governing hydrodynamic equations, we will seek solutions that satisfy certain boundary conditions, which we will determine in this section.

The fluid is found between two planes $z = 0$ and $z = d$. At these planes, the temperature satisfies the following boundary conditions

$$\begin{aligned} T &= T_0 && \text{at } z = 0, \\ T &= T_1 = T_0 - \beta d && \text{at } z = d, \end{aligned} \tag{2.3.1}$$

where $\beta = |\frac{dT}{dz}|$ is the temperature gradient.

We will consider two types of surfaces between which the fluid is confined: two *no-slip surfaces* and two *free-slip surfaces*. The boundary conditions imposed on the fluid velocity $\mathbf{u}(\mathbf{x}, t) \equiv (u, v, w)$ depend on which type of surface we are considering. However, in both cases, at the horizontal surfaces $z = 0$ and $z = d$, with outward normal $\hat{\mathbf{n}}$, the *impermeability* of the surfaces gives

$$\mathbf{u}\cdot\hat{\mathbf{n}} = w = 0 \quad \text{at } z = 0 = d. \tag{2.3.2}$$

No-Slip

The no-slip boundary condition requires the fluid to not be able to pass along the boundary. Therefore, all the components of the fluid velocity $\mathbf{u}$ must vanish, so we have

$$u = v = 0 \quad \text{at } z = 0 = d. \tag{2.3.3}$$

Since this must be satisfied for all x and y on the surface, it follows from the continuity equation (2.2.3) that

$$\frac{\partial w}{\partial z} = 0 \quad \text{at } z = 0 = d. \tag{2.3.4}$$

Free-Slip

The free-slip boundary condition, in contrast, allows the fluid to be able to pass freely along the boundary as no tangential stresses act along the surfaces. This implies that the tangential components of the stress tensor τ_{ij} defined in equation (2.1.12) are equal to zero. This gives us

$$\begin{aligned} \tau_{xz} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0, \\ \tau_{yz} &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0. \end{aligned} \tag{2.3.5}$$

Since we already have $w = 0$ for all x and y, from the impermeability condition, it follows that

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad \text{at } z = 0 = d. \tag{2.3.6}$$

Since this must be satisfied for all x and y, if we differentiate the continuity equation (2.2.3) with respect to z, we obtain

$$\frac{\partial^2 w}{\partial z^2} = 0. \tag{2.3.7}$$

Although the free-slip surfaces are physically unrealistic, we will consider these conditions in later calculations as it is the easiest to solve analytically. The same method can be applied in principle to solve the problem for two no slip surfaces. [1]

2.3.2 Base State

The *base state* is a solution that we construct to satisfy the governing equations we have derived. We will perturb this state in Section 2.3.3 and study the stability of the resulting equations in Chapter 3 and Chapter 4.

By convention, we assume that the base state is a steady state (that is, a state that is not time-dependent) with zero fluid velocity $\mathbf{u}$. We will further assume that the temperature T, only varies in the vertical z–direction, so T is a function of only z.

Since $\mathbf{u} = 0$, we write

$$\mathbf{u}_B = 0, \tag{2.3.8}$$

which trivially satisfies the continuity equation (2.2.3).

Following this, we find that the temperature distribution is given by a reduction of the heat transfer equation (2.2.7),

$$\frac{d^2 T}{dz^2} = 0, \tag{2.3.9}$$

[1] See Chandrasekhar, 1961 for an in-depth solution for two no slip surfaces, and for one free slip surface and one no slip surface.

which has solution

$$T_B = T_0 - \beta z. \tag{2.3.10}$$

The corresponding density ρ, which was defined in equation (2.2.1), is then given by

$$\rho_B = \rho_0(1 + \alpha\beta z). \tag{2.3.11}$$

Therefore, along with all the above assumptions, the momentum equation (2.2.5) can be written as

$$\frac{dP_B}{dz} = -g\rho_0(1 + \alpha\beta z), \tag{2.3.12}$$

which has solution

$$P_B = P_0 - \rho_0 z\left(1 + \frac{1}{2}\alpha\beta z\right) g\mathbf{e_z}, \tag{2.3.13}$$

where P_0 is the isotropic pressure at $z = 0$.

2.3.3 Perturbation Equations

We now subject the base state defined by equations (2.3.8), (2.3.10) and (2.3.13) to perturbations. (See **??** for a justification of why we do this.) This is given by the following equations,

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_B + \tilde{\mathbf{u}}, \\ T &= T_B + \tilde{T}, \\ P &= P_B + \tilde{P}, \end{aligned} \tag{2.3.14}$$

where is the perturbations are denoted by the variables with tildes.

Since we defined $\mathbf{u_B} = 0$, $\tilde{\mathbf{u}}$ automatically satisfies the boundary conditions outlined in Section 2.3.1. However, since T_B already satisfies the boundary conditions, we must have that

$$\tilde{T} = 0 \quad \text{at } z = 0, d. \tag{2.3.15}$$

Substituting equations (2.3.14) into the governing equations subject to Boussinesq's approximation (2.2.3), (2.2.5) and (2.2.7), we obtain the following perturbed equations

$$\begin{aligned} \nabla \cdot \tilde{\mathbf{u}} &= 0, \\ \frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} &= -\left(\frac{1}{\rho_0}\right)\nabla\tilde{P} + g\alpha\tilde{T}\mathbf{e_z} + \nu\nabla^2\tilde{\mathbf{u}}, \\ \frac{\partial \tilde{T}}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla\tilde{T} &= \beta\tilde{w} + \kappa\nabla^2\tilde{T}, \end{aligned} \tag{2.3.16}$$

where we have dropped the δ's and $\tilde{w}$ is the fluid velocity in the z–direction.

2.3.4 Non-dimensionality

It will be convenient to express the results we obtain with non-dimensional quantities that are combinations of the various parameters that we have introduced so far; by doing so we may easily compare our results with others even if different parameter values are used.

In order to non-dimensionalise the governing equations of the Rayleigh–Bénard problem, we rescale position by length scale d, the distance between the two boundaries of the fluid, time with unit time $\frac{d^2}{\kappa}$, the amount of time taken for heat to diffuse over distance d, and temperature with unit temperature β, the temperature gradient between the two boundaries. Thus our non-dimensional variables, denoted with hats, are defined as

$$\tilde{\mathbf{x}} = d\hat{\mathbf{x}}, \quad \tilde{t} = \frac{d^2}{\kappa}\hat{t}, \quad \tilde{\mathbf{u}} = \frac{\kappa}{d}\hat{\mathbf{u}}, \quad \tilde{T} = \beta\hat{t}, \quad \text{and} \quad \tilde{P} = \frac{\rho_0\kappa^2}{d^2}\hat{P}. \tag{2.3.17}$$

It should also be noted that $\nabla = \frac{1}{d}\hat{\nabla}$, and the boundaries are now at $\hat{z} = 0$ and $\hat{z} = 1$. Substituting these non-dimensional variables into equations (2.3.16), and dropping the hats, we have the non-dimensional perturbed governing equations

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla P + RaPrT\mathbf{e_z} + Pr\nabla^2\mathbf{u}, \\ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T &= w + \nabla^2 T. \end{aligned} \tag{2.3.18}$$

where $Ra = \frac{g\alpha\beta}{\kappa\nu}d^3$ is the *Rayleigh number*, and $Pr = \frac{\nu}{\kappa}$ is the *Prandtl number.*

Note that since the value of Pr is known for different fluids, the parameter Ra will be the bifurcation parameter in subsequent chapters.

2.3.5 Vorticity-Streamfunction Formulation

It will be convenient in Chapter 4 to express the non-dimensionalised hydrodynamic equations (2.3.18) in terms of the *streamfunction*, ψ and the *vorticity*, ω. Therefore, we derive this here.

Although so far, we have kept our equations general enough to be expressed in 3D, we can greatly simplify the Rayleigh–Bénard problem by adopting a 2D model. If we assume that the fluid does not move in the y–direction, we can now define our variables to be functions of x, z and time t.

The streamfunction ψ for a 2D flow, as in our simplified problem, is defined such that

$$\mathbf{u} = \nabla \times \psi \tag{2.3.19}$$

where $\psi = (0, \psi, 0)$ if $\mathbf{u} = (u, 0, w)$. This is equivalent to defining

$$u = -\frac{\partial \psi}{\partial z} \quad \text{and} \quad w = \frac{\partial \psi}{\partial x}. \tag{2.3.20}$$

The voriticity ω of a flow is defined as

$$\omega = \nabla \times \mathbf{u}. \tag{2.3.21}$$

Using these definitions, we now rewrite the governing equations in terms of the vorticity and the streamfunction. The continuity equation does not change, and is still

$$\nabla \cdot \mathbf{u} = 0, \tag{2.3.22}$$

as it is trivially satisfied by the way we have defined the streamfunction.

Using the definition for the vorticity, we can rewrite the momentum equation as

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\|\mathbf{u}\|^2\right) + \omega \times \mathbf{u} = -\nabla P + RaPrT\mathbf{e_z} + Pr\nabla^2\mathbf{u}, \tag{2.3.23}$$

where we have used the vector identity (A.2.16). Taking the curl of equation (2.3.23) gives us

$$\nabla \times \frac{\partial \mathbf{u}}{\partial t} + \nabla \times (\omega \times \mathbf{u}) = \nabla \times RaPrT\mathbf{e_z} + \nabla \times Pr\nabla^2\mathbf{u}, \tag{2.3.24}$$

since $\nabla \times \nabla f = 0$ for any twice differentiable scalar field f. This simplifies to

$$\frac{\partial \omega}{\partial t} + \nabla \times (\omega \times \mathbf{u}) = -RaPr\frac{\partial T}{\partial x}\mathbf{e_y} + Pr\nabla^2\omega. \tag{2.3.25}$$

Note that, by equation (A.2.23) we have

$$\begin{aligned}\nabla \times (\omega \times \mathbf{u}) &= \omega(\nabla \cdot \mathbf{u}) - \mathbf{u}(\nabla \cdot \omega) + (\mathbf{u} \cdot \nabla)\omega - (\omega \cdot \nabla)\mathbf{u} \\ &= (\mathbf{u} \cdot \nabla)\omega - (\omega \cdot \nabla)\mathbf{u},\end{aligned} \tag{2.3.26}$$

where we have used the fact that $\nabla \cdot \mathbf{u} = 0$ by the continuity equation, and that $\nabla \cdot \omega = \nabla \cdot (\nabla \times \mathbf{u}) = 0$ for any vector field $\mathbf{u}$.

Taking the y–component of equation (2.3.26) we obtain

$$\frac{\partial \omega_y}{\partial t} + (\mathbf{u} \cdot \nabla)\omega_y = -RaPr\frac{\partial T}{\partial x} + Pr\nabla^2\omega_y, \tag{2.3.27}$$

where ω_y is the y–component of the vorticity ω. Since we have defined our flow to only depend on x and z, then we can introduce a streamfunction ψ that satisfies equation (2.3.20). Therefore, equation (2.3.27) becomes

$$\begin{aligned}\frac{\partial \omega_y}{\partial t} + (\mathbf{u} \cdot \nabla)\omega_y &= \frac{\partial \omega_y}{\partial t} + \left(\left(-\frac{\partial \psi}{\partial z}, 0, \frac{\partial \psi}{\partial x}\right) \cdot \nabla\right)\omega_y \\ &= \frac{\partial \omega_y}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \omega_y}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \omega_y}{\partial x} \\ &= -RaPr\frac{\partial T}{\partial x} + Pr\nabla^2\omega_y.\end{aligned} \tag{2.3.28}$$

Letting J define the *Jacobian*, the momentum equation in terms of the vorticity and the streamfunction is

$$\frac{\partial \omega}{\partial t} + J(\psi, \omega) = -RaPr\frac{\partial T}{\partial x} + Pr\nabla^2\omega, \tag{2.3.29}$$

where we have relabelled ω_y as ω, and where $J(\psi, \omega) = \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \omega}{\partial x}$.

Similarly, the heat transfer equation in terms of the vorticity and the streamfunction is

$$\frac{\partial T}{\partial t} + J(\psi, T) = \frac{\partial \psi}{\partial x} + \nabla^2 T, \tag{2.3.30}$$

where $J(\psi, T) = \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial T}{\partial x}$.

We now rewrite the boundary conditions in terms of the vorticity and streamfunction. From the impermeability of the surfaces, we have

$$w = 0 \quad \text{at } z = 0 \text{ and } z = 1. \tag{2.3.31}$$

From the way we defined the streamfunction, we also have

$$w = \frac{\partial \psi}{\partial x} = 0, \tag{2.3.32}$$

which is equivalent to ψ being a constant. By the principle of mass conservation, we write

$$\int_0^1 u\, dz = -\int_0^1 \frac{\partial \psi}{\partial z} = 0, \tag{2.3.33}$$

which implies that the constant value of ψ must be the same at both $z = 0$ and $z = 1$. Without loss of generality, we may set

$$\psi = 0, \quad \text{at } z = 0 \text{ and } z = 1. \tag{2.3.34}$$

Recall that from subjecting the equations to perturbations we also have

$$T = 0 \quad \text{at } z = 0 \text{ and } z = 1. \tag{2.3.35}$$

For no slip surfaces, we require that $u = 0$ at both $z = 0$ and $z = 1$. Thus we also have the boundary condition,

$$\frac{\partial \psi}{\partial z} = 0 \quad \text{at } z = 0 \text{ and } z = 1. \tag{2.3.36}$$

For free slip surfaces, we require that $\frac{\partial u}{\partial z} = 0$ at both $z = 0$ and $z = 1$. Thus we also have the following boundary condition,

$$\omega = 0 \quad \text{at } z = 0 \text{ and } z = 1. \tag{2.3.37}$$

2.4 Mathematical Formulation of the Doubly-Diffusive Problem

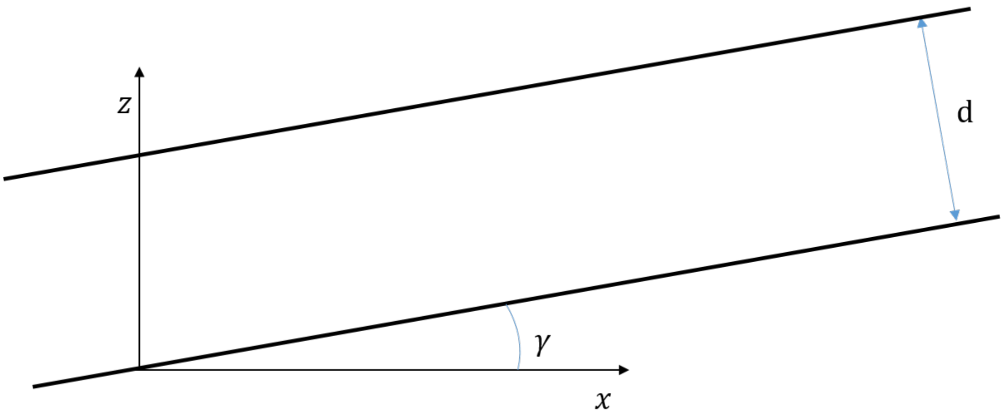

Figure 2.1: Schematic diagram of the different configurations of doubly-diffusive convection adapted from Thorpe et al. (1969).

Let us consider an infinite layer of fluid of depth d, inclined at an angle γ to the horizontal (see figure 2.1). We will consider two configurations of this problem: the fluid in a horizontal layer with a higher temperature and concentration at the lower plane (such that $\gamma = 0$), and the fluid in a horizontal layer with a higher temperature and concentration at the upper plane ($\gamma = \pi$).The temperature gradient β_T and the concentration gradient β_C are the driving forces of the convection. We again define the position to be $\mathbf{x} = (x, y, z)$ and let our variables to be functions of $\mathbf{x}$ and time, t.

2.4.1 Boundary Conditions

The fluid is found between two parallel planes $z \cot(\gamma) = 0$ and $z \cot(\gamma) = \frac{d}{\sin(\gamma)}$. At these planes, we find that the temperature and the concentration satisfies the following boundary conditions

$$
\begin{aligned}
T &= T_0 && \text{at } z\cot(\gamma) = 0, \\
T &= T_1 = T_0 - \beta_T d && \text{at } z\cot(\gamma) = \frac{d}{\sin(\gamma)},
\end{aligned} \tag{2.4.1}
$$

$$
\begin{aligned}
C &= C_0 && \text{at } z\cot(\gamma) = 0, \\
C &= C_1 = C_0 - \beta_C d && \text{at } z\cot(\gamma) = \frac{d}{\sin(\gamma)},
\end{aligned} \tag{2.4.2}
$$

where $\beta_T = |\frac{dT}{dz}|$ is the temperature gradient and $\beta_C = |\frac{dC}{dz}|$ is the concentration gradient. We also take gravity to be $\mathbf{g} = -g\mathbf{e_z}$, that is gravity acts in the negative z–direction.

For the case when the fluid is inclined at angle $\gamma = 0$, the boundary conditions become

$$
\begin{aligned}
T &= T_0 && \text{at } z = 0, \\
T &= T_1 = T_0 - \beta_T d && \text{at } z = d,
\end{aligned} \tag{2.4.3}
$$

$$
\begin{aligned}
C &= C_0 && \text{at } z = 0, \\
C &= C_1 = C_0 - \beta_C d && \text{at } z = d.
\end{aligned} \tag{2.4.4}
$$

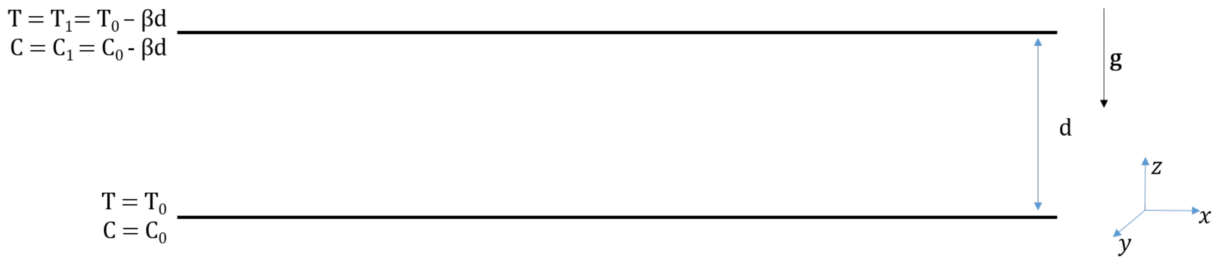

Figure 2.2: Diagram of doubly-diffusive convection in a horizontal layer of fluid subjected to a higher temperature and concentration at the bottom layer.

For the case when the fluid in inclined at $\gamma = \pi$, for the sake of consistency with the previous case, the boundary conditions become

$$
\begin{aligned}
T &= T_0 && \text{at } z = 0, \\
T &= T_1 = T_0 + \beta_T d && \text{at } z = d,
\end{aligned} \tag{2.4.5}
$$

$$
\begin{aligned}
C &= C_0 && \text{at } z = 0, \\
C &= C_1 = C_0 + \beta_C d && \text{at } z = d.
\end{aligned} \tag{2.4.6}
$$

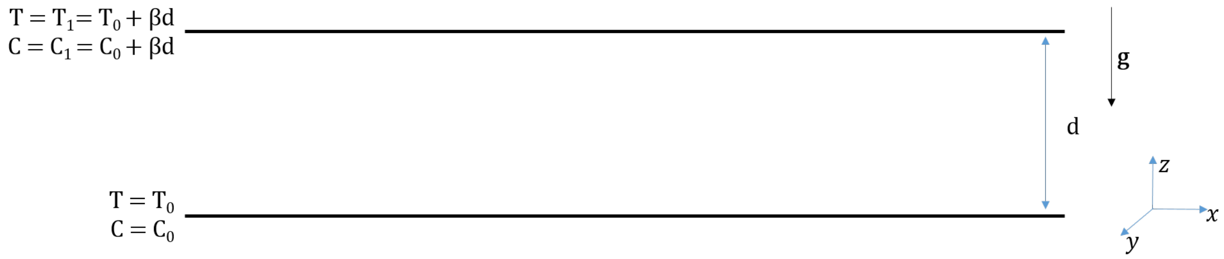

Figure 2.3: Diagram of doubly-diffusive convection in a horizontal layer of fluid subjected to a higher temperature and concentration at the top layer.

In both cases, we still assume that the planes bounding the fluid are impermeable surfaces. Therefore the following equation still holds

$$\mathbf{u} \cdot \hat{\mathbf{n}} = w = 0 \quad \text{at } z = 0 = d. \tag{2.4.7}$$

No-Slip

If the planes bounding the fluid are no-slip surfaces, then all the components of the fluid velocity $\mathbf{u}$ must vanish, so we have

$$u = v = 0 \quad \text{at } z = 0 = d. \tag{2.4.8}$$

Since this must be satisfied for all x and y on each surface, it follows from the continuity equation (2.2.10)

$$\frac{\partial w}{\partial z} = 0 \quad \text{at } z = 0 = d. \tag{2.4.9}$$

Free-Slip

If the planes bounding the fluid are free-slip surfaces, then the tangential components of the stress tensor τ_{ij} defined in equation (2.1.12) are equal to zero. Since $w = 0$ for all x and y by the impermeability of the surfaces, it follows that

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad \text{at } z = 0 = d. \tag{2.4.10}$$

Since this must be satisfied for all x and y, if we differentiate the continuity equation (2.2.10) with respect to z, we obtain

$$\frac{\partial^2 w}{\partial z^2} = 0. \tag{2.4.11}$$

It should be noted that whilst free-slip surfaces are physically unrealistic, we wish to consider these conditions since they are easier to solve analytically. [2]

2.4.2 Base State

We follow the same assumptions as in Section 2.3.2 and assume that the base state is a steady state with zero fluid velocity $\mathbf{u}$ and that the temperature T only varies in the z–direction (the direction perpendicular to the bounding surfaces). We will also assume that the concentration C also only varies in the z–direction.

Since $\mathbf{u} = 0$ we write

$$\mathbf{u}_B = 0, \tag{2.4.12}$$

which trivially satisfies the continuity equation (2.2.10).

The temperature distribution is then given by a reduction of the heat transfer equation (2.2.12),

$$\frac{d^2 T}{dz^2} = 0, \tag{2.4.13}$$

which has solution

$$T_B = T_0 \mp \beta_T z, \tag{2.4.14}$$

[2] The reader may wish to see the recent paper (Yang et al., 2016) that compares the effect of no-slip and free-slip boundary conditions in a horizontal layer of fluid.

if we are considering a fluid layer inclined at angles $\gamma = 0$ or if we are considering a fluid layer inclined at $\gamma = \pi$ respectively, recalling the way we have chosen to define the problem such that $\beta_T = |\frac{dT}{dz}|$.

Similarly, the concentration distribution is given by a reduction of the mass transfer equation (2.2.13)

$$\frac{d^2C}{dz^2} = 0, \tag{2.4.15}$$

which has solution

$$C_B = C_0 \mp \beta_C z, \tag{2.4.16}$$

if we are considering a fluid layer inclined at $\gamma = 0$ or if we are considering a fluid layer inclined at $\gamma = \pi$ respectively, recalling that $\beta_C = |\frac{dC}{dz}|$.

The corresponding density ρ, which was defined in equation (2.2.8), is then given by

$$\rho_B = \rho_0(1 \pm \alpha_T\beta_T z \mp \alpha_C\beta_C z), \tag{2.4.17}$$

if we are considering a fluid layer inclined at $\gamma = 0$ or if we are considering a fluid layer inclined at $\gamma = \pi$ respectively. Along with the all of the above assumptions, the momentum equation (2.2.11) can be written as

$$\nabla P = \rho_0(1 \pm \alpha_T\beta_T z \mp \alpha_C\beta_C z)\mathbf{g}. \tag{2.4.18}$$

This has solution

$$P_B = P_0 - \rho_0 z\left(1 + \frac{1}{2}\alpha_T\beta_T z - \frac{1}{2}\alpha_C\beta_C z\right) g\mathbf{e}_z, \tag{2.4.19}$$

if the fluid layer is inclined at $\gamma = 0$, or solution

$$P_B = P_0 - \rho_0 z\left(1 - \frac{1}{2}\alpha_T\beta_T z + \frac{1}{2}\alpha_C\beta_C z\right) g\mathbf{e}_z, \tag{2.4.20}$$

if the fluid layer is inclined at $\gamma = \pi$, where P_0 is the isotropic pressure at the plane $z = 0$ in both configurations.

2.4.3 Perturbation Equations

We now subject the base state defined by equations (2.4.12), (2.4.14), (2.4.16), (2.4.19), and (2.4.20) to perturbations. This is given by the following equations,

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_B + \tilde{\mathbf{u}}, \\ T &= T_B + \tilde{T}, \\ C &= C_B + \tilde{C}, \\ P &= P_B + \tilde{P}, \end{aligned} \tag{2.4.21}$$

where is the perturbations are denoted by the variables with tildes.

Note that since we defined $\mathbf{u_B} = 0$, $\tilde{\mathbf{u}}$ automatically satisfies the boundary conditions outlined in Section 2.3.1. However, since T_B and C_B already satisfy the boundary conditions, we must have that

$$\tilde{T} = 0 \quad \text{at } z = 0, d, \tag{2.4.22}$$

$$\tilde{C} = 0 \quad \text{at } z = 0, d. \tag{2.4.23}$$

Substituting equations (2.4.21) into the governing equations of motion subject to Boussinesq's approximation (2.2.10), (2.2.11), (2.2.12) and (2.2.13), we then obtain the following perturbed equations

$$
\begin{aligned}
&\nabla \cdot \tilde{\mathbf{u}} = 0, \\
&\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} = -\left(\frac{1}{\rho_0}\right)\nabla \tilde{P} + (\alpha_T \tilde{T} - \alpha_C \tilde{C}) g \mathbf{e_z} + \nu \nabla^2 \tilde{\mathbf{u}}, \\
&\frac{\partial \tilde{T}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{T}} = \beta_T \tilde{w} + \kappa_T \nabla^2 \tilde{T}, \\
&\frac{\partial \tilde{C}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{C}} = \beta_C \tilde{w} + \kappa_C \nabla^2 \tilde{C},
\end{aligned}
\tag{2.4.24}
$$

where the fluid is inclined at $\gamma = 0$,

$$
\begin{aligned}
&\nabla \cdot \tilde{\mathbf{u}} = 0, \\
&\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} = -\left(\frac{1}{\rho_0}\right)\nabla \tilde{P} + (\alpha_T \tilde{T} - \alpha_C \tilde{C}) g \mathbf{e_z} + \nu \nabla^2 \tilde{\mathbf{u}}, \\
&\frac{\partial \tilde{T}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{T}} = -\beta_T \tilde{w} + \kappa_T \nabla^2 \tilde{T}, \\
&\frac{\partial \tilde{C}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{C}} = -\beta_C \tilde{w} + \kappa_C \nabla^2 \tilde{C},
\end{aligned}
\tag{2.4.25}
$$

and where the fluid is inclined at $\gamma = \pi$.

2.4.4 Non-dimensionality

Following the same procedure as in Section 2.3.4, we rescale position by length scale d, the distance between the two boundaries of the fluid, time with unit time $\frac{d^2}{\kappa_T}$, the amount of time taken for heat to diffuse over distance d, temperature with unit temperature β_T, the temperature gradient between the two boundaries, and concentration with unit concentration β_C, the concentration gradient between the two boundaries. Thus our non-dimensional variables, denoted with hats, are defined as

$$
\tilde{\mathbf{x}} = d\hat{\mathbf{x}}, \quad \tilde{t} = \frac{d^2}{\kappa_T}\hat{t}, \quad \tilde{\mathbf{u}} = \frac{\kappa_T}{d}\hat{\mathbf{u}}, \quad \tilde{T} = \beta_T \hat{T}, \quad \tilde{C} = \beta_C \hat{C} \quad \text{and} \quad \tilde{P} = \frac{\rho_0 \kappa_T^2}{d^2}\hat{P}, \tag{2.4.26}
$$

with rescaled boundaries at $\hat{z} = 0$ and $\hat{z} = 1$.

Substituting these non-dimensional variables into equations (2.4.24), and (2.4.25), and dropping the hats we obtain the following non-dimensional perturbed equations,

$$
\begin{aligned}
&\nabla \cdot \mathbf{u} = 0, \\
&\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla P + Ra_T Pr (T - NC)\mathbf{e_z} + Pr \nabla^2 \mathbf{u}, \\
&\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{T} = w + \nabla^2 T, \\
&\frac{\partial C}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{C} = w + \frac{1}{Le}\nabla^2 C,
\end{aligned}
\tag{2.4.27}
$$

for a fluid inclined at $\gamma = 0$,

$$
\begin{aligned}
\nabla \cdot \mathbf{u} &= 0, \\
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla P + Ra_T Pr(T - NC)\mathbf{e_z} + Pr\nabla^2 \mathbf{u}, \\
\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{T} &= -w + \nabla^2 T, \\
\frac{\partial C}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{C} &= -w + \frac{1}{Le}\nabla^2 C,
\end{aligned}
\tag{2.4.28}
$$

for a fluid inclined at $\gamma = \pi$.

Here we have defined

$$Ra_T = \frac{g\alpha_T \beta_T}{\kappa_T \nu} d^3, \tag{2.4.29a}$$

$$Ra_C = \frac{g\alpha_C \beta_C d^3}{\kappa_T \nu}, \tag{2.4.29b}$$

$$N = \frac{\alpha_C \beta_C}{\alpha_T \beta_T}, \tag{2.4.29c}$$

$$Pr = \frac{\nu}{\kappa}, \tag{2.4.29d}$$

$$Le = \frac{\kappa_T}{\kappa_C}, \tag{2.4.29e}$$

where Ra_T is the *thermal Rayleigh number*, Ra_C is the *solutal Rayleigh number*, N is the *density ratio* such that $Ra_C = Ra_T N$, Pr is the Prandtl number and Le is the *Lewis number*.

Observe that the equations derived in this section suggest that the dynamics of doubly-diffusive systems are dependent on the parameters Pr, $\frac{1}{Le}$ and N. The values of Pr and Le are known for different fluids, but N is a measure of how the temperature and concentration gradients affect the density stratification of the fluid. For instance, when $N = 1$, the gradients are neutrally stratifying which result in a uniform density. Small N implies that the density stratification is controlled by the thermal component whilst large N implies that it is controlled by the solutal component.

We should also note that the governing equations that we have derived for both of these configurations are very similar. The only difference between equations (2.4.24) and (2.4.25) are the signs of the w terms.

2.4.5 Vorticity-Streamfunction Formulation

Let us assume again that the fluid does not move in the y–direction so that we may define our variables to be functions of only x, z and time t. Then, we can define the streamfunction ψ and the vorticity ω as in equations (2.3.20) and (2.3.21).

Following similar steps, we may rewrite the governing equations of doubly-diffusive convection in terms of the streamfunction and the vorticity.

For a fluid inclined at $\gamma = 0$ we write

$$
\begin{aligned}
\nabla \cdot \mathbf{u} &= 0, \\
\frac{\partial \omega}{\partial t} + J(\psi, \omega) &= -Ra_T Pr \frac{\partial}{\partial x}(T - NC) + Pr\nabla^2 \omega, \\
\frac{\partial T}{\partial t} + J(\psi, T) &= \frac{\partial \psi}{\partial x} + \nabla^2 T, \\
\frac{\partial C}{\partial t} + J(\psi, C) &= \frac{\partial \psi}{\partial x} + \frac{1}{Le}\nabla^2 C.
\end{aligned}
\tag{2.4.30}
$$

For a fluid inclined at $\gamma = \pi$ we write

$$\begin{aligned}
&\nabla \cdot \mathbf{u} = 0, \\
&\frac{\partial \omega}{\partial t} + J(\psi, \omega) = -Ra_T Pr \frac{\partial}{\partial x}(T - NC) + Pr\nabla^2 \omega, \\
&\frac{\partial T}{\partial t} + J(\psi, T) = -\frac{\partial \psi}{\partial x} + \nabla^2 T, \\
&\frac{\partial C}{\partial t} + J(\psi, C) = -\frac{\partial \psi}{\partial x} + \frac{1}{Le}\nabla^2 C.
\end{aligned} \tag{2.4.31}$$

We now rewrite the boundary conditions in terms of the vorticity and streamfunction. Recall that due to subjecting our equations to perturbations we have

$$T = C = 0 \quad \text{at } z = 0 \text{ and } z = 1. \tag{2.4.32}$$

By the same reasoning as in Section 2.3.5, we also have

$$w = 0 \quad \text{at } z = 0 \text{ and } z = 1. \tag{2.4.33}$$

$$\psi = 0 \quad \text{at } z = 0 \text{ and } z = 1. \tag{2.4.34}$$

For no slip surfaces, since we require that $u = 0$ at both $z = 0$ and $z = 1$ we also have the boundary condition,

$$\frac{\partial \psi}{\partial z} = 0 \quad \text{at } z = 0 \text{ and } z = 1. \tag{2.4.35}$$

For free slip surfaces, since we require that $\frac{\partial u}{\partial z} = 0$ at both $z = 0$ and $z = 1$, we also have the following boundary condition,

$$\omega = 0 \quad \text{at } z = 0 \text{ and } z = 1. \tag{2.4.36}$$

Chapter 3

Linear Analysis

Beginning with an initial flow that represented the base state of the system, we subjected the variables to perturbations and obtained the governing equations of the Rayleigh–Bénard problem and the doubly-diffusive problem that we will now work with. Note that we previously did not make any assumptions about the magnitude of the perturbations in the governing equations. In this chapter, we will now assume that these perturbations are infinitesimally small so that we may neglect the nonlinear terms. We will then perform a linear stability analysis in terms of normal modes on these equations in order to calculate the critical Rayleigh number and wavenumber at which convection occurs for the different configurations.

3.1 Linearisation

Recall the perturbed governing equations of the Rayleigh–Bénard problem (2.3.16), and of the three configurations of the doubly-diffusive problem (2.4.24), and (2.4.25), that we derived in Chapter 2. Let us now assume that the perturbations are infinitesimally small so that we may neglect the nonlinear terms. (See **??** for a justification of why we do this.)

The linearised equations for Rayleigh–Bénard convection are then given by

$$\nabla \cdot \mathbf{u} = 0, \tag{3.1.1a}$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla P + RaPrT\mathbf{e_z} + Pr\nabla^2 \mathbf{u}, \tag{3.1.1b}$$

$$\frac{\partial T}{\partial t} = w + \nabla^2 T. \tag{3.1.1c}$$

The linearised equations for a fluid in a horizontal layer that has a higher temperature and concentration at the bottom layer is given by

$$\nabla \cdot \mathbf{u} = 0, \tag{3.1.2a}$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla P + Ra_T Pr(T - NC)\mathbf{e_z} + Pr\nabla^2 \mathbf{u}, \tag{3.1.2b}$$

$$\frac{\partial T}{\partial t} = w + \nabla^2 T, \tag{3.1.2c}$$

$$\frac{\partial C}{\partial t} = w + \frac{1}{Le}\nabla^2 C. \tag{3.1.2d}$$

The linearised equations for a fluid in a horizontal layer that has a higher temperature and concentration at the top layer is given by

$$\begin{aligned}
&\nabla \cdot \mathbf{u} = 0, && (3.1.3a)\\
&\frac{\partial \mathbf{u}}{\partial t} = -\nabla P + Ra_T Pr(T - NC)\mathbf{e_z} + Pr\nabla^2 \mathbf{u}, && (3.1.3b)\\
&\frac{\partial T}{\partial t} = -w + \nabla^2 T, && (3.1.3c)\\
&\frac{\partial C}{\partial t} = -w + \frac{1}{Le}\nabla^2 C. && (3.1.3d)
\end{aligned}$$

3.2 Rayleigh–Bénard Convection

3.2.1 Normal Mode Analysis

Before we begin with the normal mode analysis, we will simplify the equations (3.1.1a), (3.1.1b) and (3.1.1c) by eliminating the pressure term. Note that if we take the divergence of equation (3.1.1b), we have

$$\begin{aligned}
\nabla \cdot \frac{\partial \mathbf{u}}{\partial t} &= \frac{\partial}{\partial t}(\nabla \cdot \mathbf{u})\\
&= \nabla \cdot \left(-\nabla P + RaPrT\mathbf{e_z} + Pr\nabla^2 \mathbf{u}\right)\\
&= -\nabla^2 P + RaPr\frac{\partial T}{\partial z} + Pr\nabla^2(\nabla \cdot \mathbf{u}).
\end{aligned} \tag{3.2.1}$$

By equation (3.1.1a), this reduces to

$$0 = -\nabla^2 P + RaPr\frac{\partial T}{\partial z}. \tag{3.2.2}$$

Let us now consider the z–component of equation (3.1.1b) and multiply it by ∇^2. We have

$$\frac{\partial}{\partial t}\nabla^2 w = -\frac{\partial}{\partial z}\nabla^2 P + \nabla^2 RaPrT + Pr\nabla^2 w, \tag{3.2.3}$$

which by equation (3.2.2), we can write as

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^2 w &= -RaPr\frac{\partial^2 T}{\partial z^2} + RaPr\nabla^2 T + Pr\nabla^4 w\\
&= RaPr\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)T + Pr\nabla^4 w.
\end{aligned} \tag{3.2.4}$$

As a further simplification, we assume that we are dealing with a 2D flow so the fluid does not vary in the y–direction, however, we note that the following analysis could be easily modified to consider a 3D flow. Thus, we now have to solve the following equations.

$$\begin{aligned}
&\frac{\partial}{\partial t}\nabla^2 w = RaPr\frac{\partial^2}{\partial x^2}T + Pr\nabla^4 w, && (3.2.5a)\\
&\frac{\partial T}{\partial t} = w + \nabla^2 T && (3.2.5b)
\end{aligned}$$

We now postulate that these equations have separable normal mode solutions of the form

$$\begin{aligned}
&w(x,z,t) = f(x)\bar{w}(z)e^{\lambda t}, && (3.2.6a)\\
&T(x,z,t) = f(x)\bar{T}(z)e^{\lambda t}, && (3.2.6b)
\end{aligned}$$

where λ is the complex eigenvalue whose real part determines the system's stability.

Note that when $\mathrm{Re}(\lambda) < 0$, the solutions of this form exponentially decay, implying that the system is stable. Conversely, when $\mathrm{Re}(\lambda) > 0$, the solutions exponentially grow, implying that the system is unstable. $\mathrm{Re}(\lambda) = 0$ is the point at which the system is marginally stable, that is the point when the system is neither stable nor unstable. Thus we can consider λ to be a bifurcation parameter of the system, with $\lambda = 0$ the corresponding local bifurcation. We can distinguish between two types of marginal stability here: the instability that arises when a purely real eigenvalue crosses the imaginary axis in the complex plane, often called a stationary instability, and the instability that arises when a pair of complex conjugate eigenvalues cross the imaginary axis, called overstability or an oscillatory instability. Note that the latter type is also a Hopf bifurcation.

Substituting equations (3.2.6a) and (3.2.6b) into equation (3.2.5a), we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\nabla^2(f\bar{w}e^{\lambda t}) &= \lambda\left[\bar{w}\frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 \bar{w}}{\partial z^2}\right]e^{\lambda t} \\ &= RaPr\frac{\partial^2}{\partial x^2}(f\bar{T}e^{\lambda t}) + Pr\nabla^4(f\bar{w}e^{\lambda t}) \\ &= \left[RaPr\frac{\partial^2}{\partial x^2}(f\bar{T}) + Pr\nabla^4(f\bar{w})\right]e^{\lambda t}. \end{aligned} \tag{3.2.7}$$

Dividing the both sides of the equation by $e^{\lambda t}$ we have

$$\lambda\left[\bar{w}\frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 \bar{w}}{\partial z^2}\right] = RaPr\frac{\partial^2}{\partial x^2}(f\bar{T}) + Pr\nabla^4(f\bar{w}). \tag{3.2.8}$$

Similarly, equation (3.2.5b) becomes

$$\lambda f\bar{T} = f\bar{w} + \bar{T}\frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 \bar{T}}{\partial z^2}. \tag{3.2.9}$$

Dividing both sides of this equation now by f, we have

$$\lambda\bar{T} = \bar{w} + \bar{T}\frac{1}{f}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial z^2} \tag{3.2.10}$$

Thus, we can only seek separable solutions when

$$\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = constant \equiv -a^2, \tag{3.2.11}$$

which leads us to suppose that f is of the form

$$f(x) = e^{iax}, \tag{3.2.12}$$

where we can think of a as the wavenumber of a particular normal mode.

Multiplying equation (3.2.8) by $\frac{1}{f}f$, the left hand side becomes

$$\lambda\left[\frac{1}{f}\frac{\partial^2 f}{\partial x^2}\bar{w} + \frac{1}{f}f\frac{\partial^2 \bar{w}}{\partial z^2}\right]f = \lambda\left[\frac{d^2}{dz^2}\bar{w} - a^2\bar{w}\right]f. \tag{3.2.13}$$

The right hand side of (3.2.8) becomes

$$
\begin{aligned}
RaPr\frac{1}{f}\frac{\partial^2 f}{\partial x^2}\bar{T}f + \frac{1}{f}Pr\nabla^4 f\bar{w}f &= -a^2RaPrf\bar{T} \\
&\quad + Pr\left[\frac{1}{f}\left(\left(\frac{\partial^2}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial z^2}\right) + \left(\frac{\partial^2}{\partial z^2}\right)^2\right) f\bar{w}\right] f \\
&= -a^2RaPrf\bar{T} + Pr\left[(-a^2)^2 - 2a^2\left(\frac{\partial^2}{\partial z^2}\right) + \left(\frac{\partial^2}{\partial z^2}\right)^2\right]\bar{w}f \\
&= -a^2RaPrf\bar{T} + Pr\left[\left(\frac{d^2}{dz^2} - a^2\right)^2 \bar{w}f\right].
\end{aligned}
\tag{3.2.14}
$$

Dividing both sides by f and rearranging, this becomes

$$
\left(D^2 - a^2\right)\left(D^2 - a^2 - \frac{\lambda}{Pr}\right)\bar{w} = a^2Ra\bar{T}, \tag{3.2.15}
$$

where we have let $D = \frac{d}{dz}$ and $D^2 = \frac{d^2}{dz^2}$. Similarly, equation (3.2.10) becomes

$$
\left(D^2 - a^2 - \lambda\right)\bar{T} = -\bar{w}. \tag{3.2.16}
$$

Note that in this notation the free-slip boundary conditions become

$$
\bar{w} = D^2\bar{w} = \bar{T} = 0 \quad \text{at} \ \ z = 0, 1. \tag{3.2.17}
$$

If we evaluate equation (3.2.15) at the boundary conditions (3.2.17), we find that we also have that

$$
D^4\bar{w} = 0 \quad \text{at} \ \ z = 0, 1, \tag{3.2.18}
$$

which in fact further implies that any even-powered derivative of $\bar{w}$ is equal to zero at $z = 0$ and $z = 1$. Therefore, we expect solutions $\bar{w}$ to be of the form $\bar{w} = \bar{w}_0\sin(n\pi z)$ for $n \in \mathbb{N}$. Equation (3.2.16) suggests that solutions $\bar{T}$ have the same parity as $\bar{w}$ and so are of the form $\bar{T} \approx \sin(n\pi z)$ as well.

Now rearranging equation (3.2.16) we can obtain an expression for $\bar{T}$. Substituting this into equation (3.2.15) we can eliminate $\bar{T}$ and obtain one equation for $\bar{w}$ which is given by

$$
\left(D^2 - a^2 - \lambda\right)\left(D^2 - a^2\right)\left(D^2 - a^2 - \frac{\lambda}{Pr}\right)\bar{w} = -a^2Ra\bar{w}. \tag{3.2.19}
$$

We want to solve this sixth order characteristic value problem for the boundary conditions given in equations (3.2.17) and (3.2.18). Given these boundary conditions, there is a countably infinite number of eigenvalues λ_n with associated eigenfunctions $\bar{w}_n$ that we have already deduced are of the form $\bar{w}_n = \bar{w}_0\sin(n\pi z)$. For given values of a, Ra, and Pr, the complete set of solutions $\bar{w}_n$ are called the normal mode solutions.

The form of $\bar{w}_n$ implies that $D^2\bar{w}_n = -n^2\pi^2\bar{w}_0\sin(n\pi z)$. Therefore, substituting this expression for $\bar{w}$ into equation (3.2.19) we have

$$
(-n^2\pi^2 - a^2 - \lambda_n)(-n^2\pi^2 - a^2)\left(-n^2\pi^2 - a^2 - \frac{\lambda_n}{Pr}\right)\bar{w}_0\sin(n\pi z) = -a^2Ra\bar{w}_0\sin(n\pi z). \tag{3.2.20}
$$

Equation (3.2.19) is therefore only satisfied if the following equation is true:

$$
(-n^2\pi^2 - a^2 - \lambda_n)(-n^2\pi^2 - a^2)\left(-n^2\pi^2 - a^2 - \frac{\lambda_n}{Pr}\right) = -a^2Ra. \tag{3.2.21}
$$

Equation (3.2.21) is the characteristic equation or dispersion relation that we solve to determine the eigenvalues of the system.

3.2.2 The Principle of the Exchange of Stabilities

We prove here in a similar manner to Drazin and Reid (2004) and Chandrasekhar (1961) that the eigenvalues of the Rayleigh–Bénard problem are real and so the instability occurs at $\lambda = 0$. This is equivalent to saying that the principle of the exchange of stabilities is valid.

First we multiply equation (3.2.15) by the complex conjugate of $\bar{w}$, which we denote $\bar{w}^*$, and integrate over the layer of fluid, between $z = 0$ and $z = 1$. This gives us

$$\int_0^1 \bar{w}^*(D^2 - a^2)\left(D^2 - a^2 - \frac{\lambda}{Pr}\right)\bar{w}\, dz = a^2 Ra \int_0^1 \bar{w}^* \bar{T}\, dz. \tag{3.2.22}$$

Expanding the left-hand side of this equation, we can write this as

$$\int_0^1 \bar{w}^* \left[D^4\bar{w} - \left(2a^2 + \frac{\lambda}{Pr}\right) D^2\bar{w} + \left(a^4 + \frac{a^2\lambda}{Pr}\right)\bar{w}\right] dz = a^2 Ra \int_0^1 \bar{w}^* \bar{T}\, dz. \tag{3.2.23}$$

If we integrate the $D^4\bar{w}$ term by parts two times and the $D^2\bar{w}$ term one time, noting that $\bar{w}^*$ satisfies the same boundary conditions as $\bar{w}$, we have

$$\int_0^1 \left[|D^2\bar{w}|^2 + \left(2a^2 + \frac{\lambda}{Pr}\right)|D\bar{w}|^2 + \left(a^4 + \frac{a^2\lambda}{Pr}\right)|\bar{w}|^2\right] dz = a^2 Ra \int_0^1 \bar{w}^* \bar{T}\, dz. \tag{3.2.24}$$

Similarly, if we multiply equation (3.2.16) by $\bar{T}^*$, the complex conjugate of $\bar{T}$, and integrate with respect to z between $z = 0$ and $z = 1$ we obtain

$$\int_0^1 \bar{T}^* \left(D^2 - a^2 - \lambda\right)\bar{T}\, dz = -\int_0^1 \bar{T}^* \bar{w}\, dz, \tag{3.2.25}$$

which can be rewritten as

$$\int_0^1 \left[\bar{T}^* D^2 \bar{T} - \left(a^2 + \lambda\right)|\bar{T}|^2\right] dz = -\int_0^1 \bar{T}^* \bar{w}\, dz, \tag{3.2.26}$$

since we note that $\bar{T}$ and $\bar{T}^*$ are functions of only z which allows us to treat $(a^2 + \lambda)$ as a constant.

Using that $\bar{T}^*$ satisfies the same boundary conditions as $\bar{T}$, we integrate the $\bar{T}^* D^2 \bar{T}$ term by parts. Therefore, equation (3.2.26) becomes

$$\int_0^1 \left[|D\bar{T}|^2 + (a^2 + \lambda)|\bar{T}|^2\right] dz = \int_0^1 \bar{T}^* \bar{w}\, dz. \tag{3.2.27}$$

Notice that the integrand of the right hand side of equation (3.2.24), $\bar{w}^*\bar{T}$, is the complex conjugate of the integrand of the right hand side of equation (3.2.27), $\bar{T}^*\bar{w}$. Therefore, we can write

$$\begin{aligned} &\int_0^1 \left[|D^2\bar{w}|^2 + \left(2a^2 + \frac{\lambda}{Pr}\right)|D\bar{w}|^2 + \left(a^4 + \frac{a^2\lambda}{Pr}\right)|\bar{w}|^2\right] dz \\ &\qquad = a^2 Ra \left(\int_0^1 \left[|D\bar{T}|^2 + (a^2 + \lambda)|\bar{T}|^2\right] dz\right)^*, \end{aligned} \tag{3.2.28}$$

which for brevity we rewrite as

$$\mathcal{I} + \frac{\lambda}{Pr}\mathcal{J} = a^2 Ra \left(\mathcal{K} + \lambda \mathcal{L}\right)^*, \tag{3.2.29}$$

where we have defined $\mathcal{I} \equiv \int_0^1 \left[|D^2\bar{w}|^2 + 2a^2|D\bar{w}|^2 + a^4|\bar{w}|^2\right] dz$, $\mathcal{J} \equiv \int_0^1 \left[|D\bar{w}|^2 + a^2|\bar{w}|^2\right] dz$, $\mathcal{K} \equiv \int_0^1 \left[|D\bar{w}|^2 + a^2|\bar{w}|^2\right] dz$, and $\mathcal{L} \equiv \int_0^1 |\bar{T}|^2\, dz$.

Since the eigenvalues λ may be complex, let us write $\lambda = \mathrm{Re}(\lambda) + i\,\mathrm{Im}(\lambda)$. Then equation (3.2.29) is only true when we equate the real and imaginary parts of both sides, giving

$$\mathcal{I} + \frac{\mathrm{Re}(\lambda)}{Pr}\mathcal{J} - a^2 Ra(\mathcal{K} + \mathrm{Re}(\lambda)\mathcal{L}) = 0 \tag{3.2.30a}$$

$$\frac{\mathrm{Im}(\lambda)}{Pr}\mathcal{J} + \mathrm{Im}(\lambda)\mathcal{L} = 0. \tag{3.2.30b}$$

Notice that a^2, Ra, Pr and all the integrals $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ and $\mathcal{L}$ cannot be negative, so in order for equation (3.2.30b) to be true, $\mathrm{Im}(\lambda)$ must be zero. This implies that λ is real and that Rayleigh–Bénard convection cannot set in by a Hopf bifurcation. Furthermore, we know that the onset of instability occurs at $\lambda = 0$.

3.2.3 The Growth Rates

We now solve the characteristic equation (3.2.21) for λ_n. Expanding and equating like powers of λ_n we have

$$\lambda_n^2 + (Pr+1)(n^2\pi^2 + a^2)\lambda_n + \left(\left(n^2\pi^2 + a^2\right)^2 Pr - \frac{a^2 Ra Pr}{(n^2\pi^2 + a^2)}\right) = 0. \tag{3.2.31}$$

Therefore, λ_n is given by

$$\lambda_n = \frac{-(Pr+1)(n^2\pi^2 + a^2) \pm \sqrt{(Pr+1)^2(n^2\pi^2 + a^2) - 4\left(Pr(n^2\pi^2 + a^2)^2 - \frac{a^2 Ra Pr}{(n^2\pi^2+a^2)}\right)}}{2}. \tag{3.2.32}$$

From the previous section 3.2.2, we proved that the λ_n are real and the onset of stability occurs at $\lambda = 0$. Substituting this into the above equation, we find that the value of Ra for marginal stability of the nth mode is given by the following marginal curves

$$Ra_n(a) = \frac{(n^2\pi^2 + a^2)^3}{a^2}. \tag{3.2.33}$$

By the Routh–Hurwitz criterion (Gradshteyn and Ryzhik, 2000), we require all the coefficients of the characteristic equation (3.2.31) to be positive for stability. $(Pr+1)(n^2\pi^2+a^2)$, the coefficient of the the λ_n term, is clearly positive, recalling that Pr is a positive constant. For the coefficient of the constant term of the characteristic equation to be positive, we require that Ra is less than the marginal stability value given in equation (3.2.33). Therefore, the critical Rayleigh number Ra_c at which the onset of convection occurs (not to be confused with Ra_C the solutal Rayleigh number) is found by minimising equation (3.2.33) over n and a.

Clearly, $n = 1$ minimises the equation over n which gives us

$$Ra_1 = \frac{\pi^2 + a^2}{a^2}. \tag{3.2.34}$$

Differentiating this with respect to a and equating it to zero we find that the critical wavenumber a_c and Ra_c are given by

$$a_c = \frac{\pi}{\sqrt{2}} \approx 2.221 \tag{3.2.35a}$$

$$Ra_c = Ra(a_c) = \frac{27\pi^4}{4} \approx 657.511. \tag{3.2.35b}$$

Thus, by linear stability theory, if $Ra < Ra_c$ we expect stability. If $Ra > Ra_c$, then we expect instability. If Ra is slightly larger than Ra_c, we would expect the $n = 1$ mode to grow exponentially, but none of the others to grow.

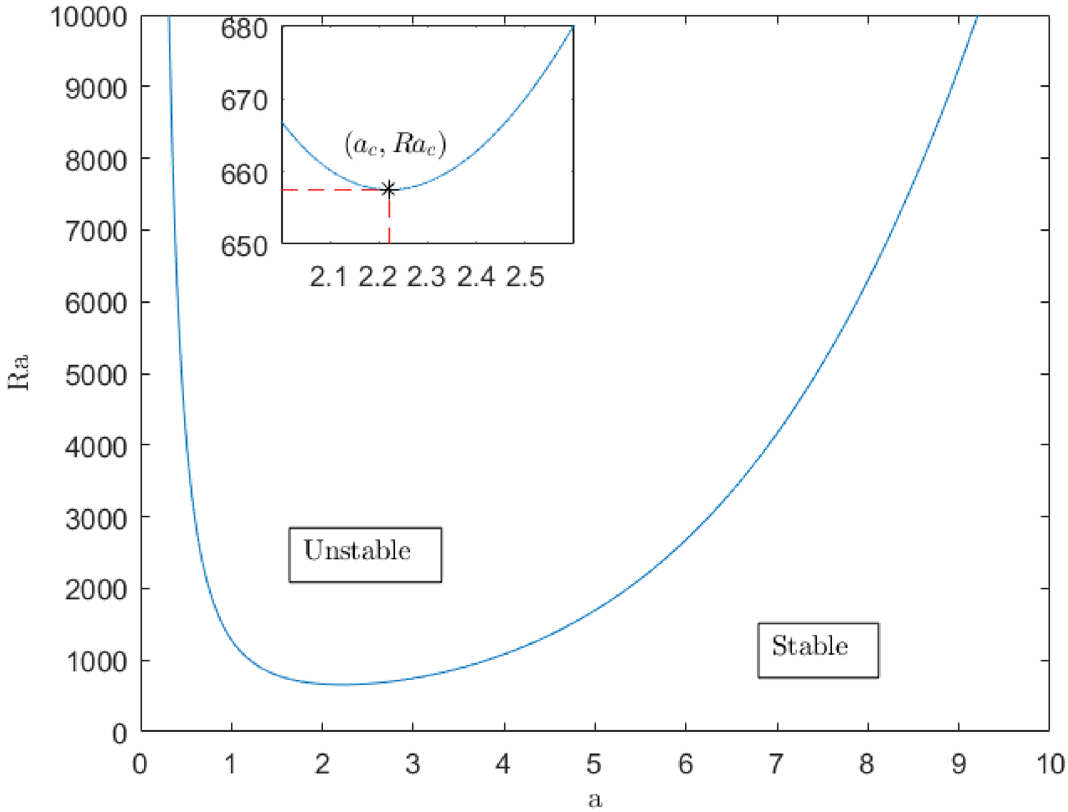

Figure 3.1: Graph showing the marginal curve $R_1(a)$ which gives the regions of growing and decaying modes in the plane of the Rayleigh number Ra and the dimensionless wavenumber in the x–direction, a. The critical wavenumber a_c, and the critical Rayleigh number Ra_c are found at the minimum of this curve.

Notice that the onset of convection did not depend on the Prandtl number Pr, but the actual rates at which the solutions will grow or decay (given by equation (3.2.32)) do depend on Pr.

It is worth reiterating that the way we have formulated the problem is very idealistic. In reality, most fluids are bounded between surfaces with no-slip boundary conditions, but to solve the Rayleigh–Bénard problem with such conditions is much more involved but has been done (see Chandrasekhar, 1961).

We also have not determined the shape of the convection rolls. To do this, we would need to consider solutions to equation (3.2.11). In fact, to be more realistic, we would need to consider the wavenumbers in both the x–direction and the y–direction, so would need to solve $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x, y) + a^2 f(x, y){=}0$, where a would now be the wavenumber in the x, y–plane.

Since we have considered an infinite layer of fluid, we seek solutions to equation (3.2.11) that are periodic. In the simplest case, we can see that $f(x) = \cos(ax)$ is a solution that results in long convection rolls, independent of y–direction. This has a period of $\frac{2\pi}{a}$ in the x–direction. Substituting this form for f into our postulated normal mode form solution

(3.2.6a), and using the continuity equation (3.1.1a), we have

$$u = -\frac{1}{a}\sin(ax)\frac{d\bar{w}}{dz}e^{\lambda t} \quad \text{and} \quad w = \cos(ax)\bar{w}e^{\lambda t}. \tag{3.2.36}$$

From this we can deduce that $u = 0$ when $x = \frac{k\pi}{a}$, for $k \in \mathbb{Z}$, which does indeed characterise 2D convection rolls.

In reality however, fluids do not exist in unbounded infinite layers. We would be required to consider sidewalls that bound the fluid in the x–direction and the y–direction, as well as the already bounded z–direction. In Bénard's initial experiments, he tried to simulate in infinite layer of fluid by studying a fluid with an aspect ratio so large that the layer was theoretically infinite. However, more recent work has been done (Kidachi, 1982; Hirschberg and Knobloch, 1997) to study the effect of side walls in fluid layers. Note that in such cases, the wavenumber a would be a discrete eigenvalue for a given Rayleigh number, rather than a continuous parameter as we have taken here.

3.3 Doubly-Diffusive Convection: A Horizontal Layer Heated and Salted from Below

3.3.1 Normal Mode Analysis

Using the same techniques that we used in the previous section (3.2), we now turn to the doubly-diffusive problem. Let us first consider the configuration that most closely resembles the Rayleigh–Bénard problem — a horizontal fluid layer that is heated and salted from below — given by equations (3.1.2a), (3.1.2b), (3.1.2c) and (3.1.2d).

We again simplify the problem by eliminating the pressure term and assuming that the flow does not vary in the y–direction. Therefore, we now have to solve the following three equations.

$$\frac{\partial}{\partial t}\nabla^2 w = Ra_T Pr \frac{\partial^2}{\partial x^2}(T - NC) + Pr\nabla^4 w, \tag{3.3.1a}$$

$$\frac{\partial T}{\partial t} = w + \nabla^2 T, \tag{3.3.1b}$$

$$\frac{\partial C}{\partial t} = w + \frac{1}{Le}\nabla^2 C. \tag{3.3.1c}$$

We postulate that the above equations have separable normal mode solutions of the form

$$w(x, z, t) = f(x)\bar{w}(z)e^{\lambda t}, \tag{3.3.2a}$$

$$T(x, z, t) = f(x)\bar{T}(z)e^{\lambda t}, \tag{3.3.2b}$$

$$C(x, z, t) = f(x)\bar{C}(z)e^{\lambda t}, \tag{3.3.2c}$$

where λ is again the complex eigenvalue whose real part determines the system's stability.

Substituting equations (3.3.2a) – (3.3.2c) into equations (3.3.1a) – (3.3.1c), we can again show that we may only seek separable solutions when

$$\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = constant \equiv -a^2, \tag{3.3.3}$$

where we can think of a as the horizontal wavenumber of a particular normal mode. Letting $D = \frac{d}{dz}$ and $D^2 = \frac{d^2}{dz^2}$, we now have

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \frac{\lambda}{Pr}\right)\bar{w} = a^2 Ra_T \left(T - NC\right), \tag{3.3.4a}$$

$$\left(D^2 - a^2 - \lambda\right)\bar{T} = -\bar{w}, \tag{3.3.4b}$$

$$\left(D^2 - a^2 - Le\lambda\right)\bar{C} = -Le\bar{w}, \tag{3.3.4c}$$

with free-slip boundary conditions

$$\bar{w} = D^2\bar{w} = \bar{T} = \bar{C} = 0 \quad \text{at} \;\; z = 0, 1. \tag{3.3.5}$$

Evaluating equation (3.3.4a) at the above boundary conditions, we find that

$$D^4\bar{w} = 0 \quad \text{at} \;\; z = 0, 1 \implies D^{(2m)}\bar{w} = 0 \quad \text{at} \;\; z = 0, 1, \tag{3.3.6}$$

where $m \in \mathbb{N}$. This further implies that the solutions $\bar{w}$ are of the form $\bar{w} = \bar{w}_0 \sin(n\pi z)$ for $n \in \mathbb{N}$. Equations (3.3.4b) and (3.3.4c) suggest that solutions $\bar{T}$ and $\bar{C}$ have the same parity as $\bar{w}$ and are of the form $\sin(n\pi z)$ as well.

We now eliminate $\bar{T}$ and $\bar{C}$ to obtain one equation for $\bar{w}$, which is given by

$$\begin{aligned}\left(\left(D^2 - a^2 - \frac{\lambda}{Pr}\right)(D^2 - a^2)(D^2 - a^2 - \lambda)(D^2 - a^2 - Le\lambda)\right)\bar{w} \\ = a^2 Ra_T Le N(D^2 - a^2 - \lambda)\bar{w} - a^2 Ra_T (D^2 - a^2 - Le\lambda)\bar{w}.\end{aligned} \tag{3.3.7}$$

Given the boundary conditions (3.3.5) and (3.3.6), solving the above equation gives us the following characteristic equation

$$\begin{aligned}\left(-n^2\pi^2 - a^2\frac{\lambda_n}{Pr}\right)(-n^2\pi^2 - a^2)(-n^2\pi^2 - a^2 - \lambda_n)(-n^2\pi^2 - a^2 - Le\lambda_n) \\ = a^2 Ra_T (LeN(-n^2\pi^2 - a^2 - \lambda_n) - (-n^2\pi^2 - a^2 - Le\lambda_n)),\end{aligned} \tag{3.3.8}$$

where λ_n are the eigenvalues associated with the eigenfunctions $\bar{w}_n = \bar{w}_0 \sin(n\pi z)$. Note that we have to assume in general that the eigenvalues here are complex as we cannot prove that they are real like we did for Rayleigh–Bénard convection in Section 3.2.2.

3.3.2 The Growth Rates

We now solve the dispersion relation (3.3.8) for λ_n. Expanding and equating like powers of λ_n we have

$$\lambda_n^3 + p_2\lambda_n^2 + p_1\lambda_n + p_0 = 0, \tag{3.3.9}$$

where

$$p_2 = (n^2\pi^2 + a^2)\left(\frac{1}{Le} + Pr + 1\right), \tag{3.3.10a}$$

$$p_1 = (n^2\pi^2 + a^2)^2\left(\frac{Pr}{Le} + \frac{1}{Le} + Pr\right) - \frac{a^2 Pr}{(n^2\pi^2 + a^2)} Ra_T \left(1 - N\right), \tag{3.3.10b}$$

$$p_0 = (n^2\pi^2 + a^2)^3\frac{Pr}{Le} - a^2\frac{Pr}{Le} Ra_T(1 - LeN). \tag{3.3.10c}$$

Since p_0, p_1 and p_2 are all real, we expect the cubic characteristic equation to have either three real roots or one real root and and a pair of complex conjugate roots.

By the Routh–Hurwitz criterion (Gradshteyn and Ryzhik, 2000), for stability we require that $p_0 > 0$, $p_1 > 0$, $p_2 > 0$ and $p_1 p_2 > p_0$ are satisfied. For oscillatory instability, or a Hopf bifurcation, we require that $p_1 p_2 = p_0$, which is given by

$$\left(\frac{1}{Le} + Pr + 1\right)\left[\mu^3\left(\frac{Pr}{Le} + \frac{1}{Le} + Pr\right) - a^2 Ra_T(1-N)\right] = \mu^3 \frac{Pr}{Le} - a^2 \frac{Pr}{Le} Ra_T(1 - LeN), \tag{3.3.11}$$

where we have defined $\mu \equiv (n^2\pi^2 + a^2)$ for brevity. Solving this for Ra_T, we have

$$\begin{aligned} Ra_T &= \frac{\mu^3}{a^2} \frac{(Pr+1)(Le+1)(LePr+1)}{LePr(Le(Pr+1) - N(LePr+1))} \\ &= \frac{\mu^3}{a^2}\left(1 + \frac{1}{Le}\right)\left(1 + \frac{1}{LePr}\right)\left(\frac{Le(Pr+1)}{Le(Pr+1) - N(LePr+1)}\right). \end{aligned} \tag{3.3.12}$$

The critical point at which this occurs is given by minimising equation (3.3.12) over n and a, since Pr, Le and N are given constants. Clearly, $n = 1$ minimises the equation over n so substituting this into the equation and differentiating with respect to a, we find that the critical wavenumber a_c and the critical thermal Rayleigh number for oscillatory instability $Ra_T^{(O)}$ are given by

$$a_c = \frac{\pi}{\sqrt{2}} \tag{3.3.13a}$$

$$Ra_T^{(O)} = \frac{27\pi^4}{4}\left(1 + \frac{1}{Le}\right)\left(1 + \frac{1}{LePr}\right)\left(\frac{Le(Pr+1)}{Le(Pr+1) - N(LePr+1)}\right). \tag{3.3.13b}$$

Stationary instability occurs when one of the roots is zero, that is $\lambda = 0$. Substituting this into equation (3.3.9) and rearranging for Ra_T, this occurs when

$$Ra_T = \frac{\mu^3}{a^2(1 - LeN)}. \tag{3.3.14}$$

Minimising this over n and a, we find that the critical wavenumber a_c is again $\frac{\pi}{\sqrt{2}}$, the critical thermal Rayleigh number for stationary instability $Ra_T^{(S)}$ is given by

$$Ra_T^{(S)} = \frac{27\pi^4}{4(1 - LeN)}. \tag{3.3.15}$$

Note that in the limit $N \to 0$, or equivalently when $Ra_C \ll Ra_T$, $Ra_T^{(S)}$ tends to $Ra_c = \frac{24\pi^4}{4}$, the critical Rayleigh number for Rayleigh–Bénard convection, as is expected.

In the plane of Ra_T and N, the linear stability boundary is a combination of both $Ra_T^{(O)}$ and $Ra_T^{(S)}$, which can be determined for fixed values of Pr and Le (see Figure 3.2). The critical value of N at which $Ra_T^{(O)} = Ra_T^{(S)}$ is given by

$$N^{(OS)} = \frac{Pr + 1}{Le(LePr + 1)}. \tag{3.3.16}$$

For $N > N^{(OS)}$, instability arises through an oscillatory instability by the mechanism described in Section 1.3 as Ra_T exceeds $Ra_T^{(O)}$. For $N < N^{(OS)}$, the solutal contribution to the fluid is not large enough to result in growing oscillations by the physical mechanism described in Section 1.3, and instability arises through a stationary instability instead as Ra_T exceeds $Ra_T^{(S)}$.

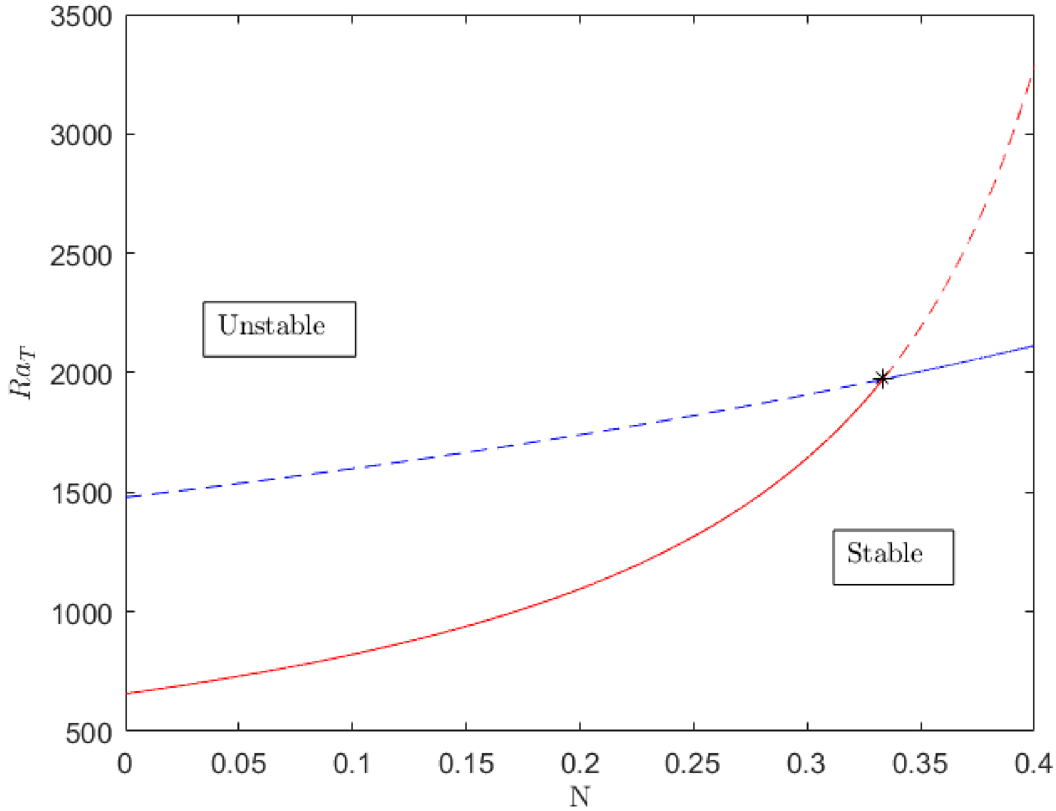

Figure 3.2: Graph showing the regions of growing modes (above the solid line) and decaying modes (below the solid line) of a fluid heated and salted from below in the plane of the thermal Rayleigh number Ra_T and the density ratio N for $Pr = 1$ and $Le = 2$. The blue curve shows the critical Rayleigh number for oscillatory instability $Ra_T^{(O)}$ and the red curve shows the critical Rayleigh number for stationary instability. The critical value $N^{(OS)}$ is the point at which the two curves intersect shown by a black star.

3.4 Doubly-Diffusive Convection: A Horizontal Layer Heated and Salted from Above

3.4.1 Normal Mode Analysis

Let us now consider the problem where the fluid is heated and salted from above, given by equations (3.1.2a), (3.1.2b), (3.1.2c) and (3.1.2d).

We proceed in the same method that we have followed in the previous sections by eliminating the pressure term and assuming that the flow does not vary in the y–direction. We solve the following three equations

$$\frac{\partial}{\partial t}\nabla^2 w = Ra_T Pr \frac{\partial^2}{\partial x^2}(T - NC) + Pr\nabla^4 w, \tag{3.4.1a}$$

$$\frac{\partial T}{\partial t} = -w + \nabla^2 T, \tag{3.4.1b}$$

$$\frac{\partial C}{\partial t} = -w + \frac{1}{Le}\nabla^2 C, \tag{3.4.1c}$$

and postulate that they have separable normal mode solutions of the form

$$w(x,z,t) = f(x)\bar{w}(z)e^{\lambda t}, \tag{3.4.2a}$$

$$T(x,z,t) = f(x)\bar{T}(z)e^{\lambda t}, \tag{3.4.2b}$$

$$C(x,z,t) = f(x)\bar{C}(z)e^{\lambda t}, \tag{3.4.2c}$$

where λ is a complex eigenvalue. By similar reasoning, we let a be the horizontal wavenumber of a particular normal mode, $D = \frac{d}{dz}$ and $D^2 = \frac{d^2}{dz^2}$ and write

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \frac{\lambda}{Pr}\right)\bar{w} = a^2 Ra_T\left(T - NC\right), \tag{3.4.3a}$$

$$\left(D^2 - a^2 - \lambda\right)\bar{T} = \bar{w}, \tag{3.4.3b}$$

$$\left(D^2 - a^2 - Le\lambda\right)\bar{C} = Le\bar{w}, \tag{3.4.3c}$$

with free-slip boundary conditions

$$\bar{w} = D^2\bar{w} = \bar{T} = \bar{C} = 0 \quad \text{at} \;\; z = 0,1. \tag{3.4.4}$$

Evaluating equation (3.4.3a) at the above boundary conditions, we find that

$$D^4\bar{w} = 0 \quad \text{at} \;\; z = 0,1 \implies D^{(2m)}\bar{w} = 0 \quad \text{at} \;\; z = 0,1, \tag{3.4.5}$$

where $m \in \mathbb{N}$. This further implies that the solutions $\bar{w}$ are of the form $\bar{w} = \bar{w}_0 \sin(n\pi z)$ for $n \in \mathbb{N}$. Equations (3.4.3b) and (3.4.3c) suggest that solutions $\bar{T}$ and $\bar{C}$ have the same parity as $\bar{w}$ and are of the form $\sin(n\pi z)$ as well.

We now eliminate $\bar{T}$ and $\bar{C}$ again to obtain one equation for $\bar{w}$, which is given by

$$\begin{aligned}\left(\left(D^2 - a^2 - \frac{\lambda}{Pr}\right)(D^2 - a^2)(D^2 - a^2 - \lambda)(D^2 - a^2 - Le\lambda)\right)\bar{w} \\ = a^2 Ra_T(D^2 - a^2 - Le\lambda)\bar{w} - a^2 Ra_T LeN(D^2 - a^2 - \lambda)\bar{w}.\end{aligned} \tag{3.4.6}$$

Given the boundary conditions (3.4.4) and (3.4.5), solving the above equation gives us the following characteristic equation

$$\begin{aligned}\left(-n^2\pi^2 - a^2\frac{\lambda_n}{Pr}\right)(-n^2\pi^2 - a^2)(-n^2\pi^2 - a^2 - \lambda_n)(-n^2\pi^2 - a^2 - Le\lambda_n) \\ = a^2 Ra_T((-n^2\pi^2 - a^2 - Le\lambda_n) - LeN(-n^2\pi^2 - a^2 - \lambda_n)),\end{aligned} \tag{3.4.7}$$

where λ_n are the eigenvalues associated with the eigenfunctions $\bar{w}_n = \bar{w}_0 \sin(n\pi z)$. Note that this is identical to the dispersion relation given in equation (3.3.8) except for a sign change on the right hand side of the equation.

3.4.2 The Growth Rates

We now solve the dispersion relation (3.4.7) for λ_n. Expanding and equating like powers of λ_n we have

$$\lambda_n^3 + p_2\lambda_n^2 + p_1\lambda_n + p_0 = 0, \tag{3.4.8}$$

where

$$p_2 = (n^2\pi^2 + a^2)\left(\frac{1}{Le} + Pr + 1\right), \tag{3.4.9a}$$

$$p_1 = (n^2\pi^2 + a^2)^2\left(\frac{Pr}{Le} + \frac{1}{Le} + Pr\right) + \frac{a^2 Pr}{(n^2\pi^2 + a^2)} Ra_T\left(1 - N\right), \tag{3.4.9b}$$

$$p_0 = (n^2\pi^2 + a^2)^3\frac{Pr}{Le} + a^2\frac{Pr}{Le} Ra_T(1 - LeN). \tag{3.4.9c}$$

Since p_0, p_1 and p_2 are all real again, we expect the cubic characteristic equation to have either three real roots or one real root and and a pair of complex conjugate roots. However, it can be shown that the discriminant of the dispersion relation, given by

$$\Delta = 18p_2p_1p_0 - 4p_2^3p_0 + p_2^2p_1^3 - 27p_0^2 - 4, \tag{3.4.10}$$

is greater than zero given that all of our variables Pr, Le, Ra_T and N are positive quantities. This implies that the eigenvalus are all real, so instability in this system cannot arise through a Hopf bifurcation. This is in agreement with the physical problem and the mechanism of the salt-fingering instability that we described in Section 1.3.

Instead, instability in the system arises through a stationary instability. This occurs when one of the roots is is zero, or when $\lambda = 0$. This is given by the equation

$$Ra_T = \frac{\mu^3}{a^2(LeN - 1)}, \tag{3.4.11}$$

where $\mu = n^2\pi^2 + a^2$. Minimising this over n and a, we find that the critical wavenumber a_c is again $\frac{\pi}{\sqrt{2}}$, the critical thermal Rayleigh number for stationary instability $Ra_T^{(S)}$ is given by

$$Ra_T^{(S)} = \frac{27\pi^4}{4(LeN - 1)}, \tag{3.4.12}$$

where $N > \frac{1}{Le}$ (see Figure 3.3). Note that $Ra_T^{(S)}$ as a function of N is an asymptotic curve of the form $\frac{1}{N}$ which has asymptote $N = \frac{1}{Le}$, for which the solution is stable for any Ra_T. For $N < \frac{1}{Le}$, Ra_T has negative values, which cannot happen given that Ra_T has been defined with positive physical constants.

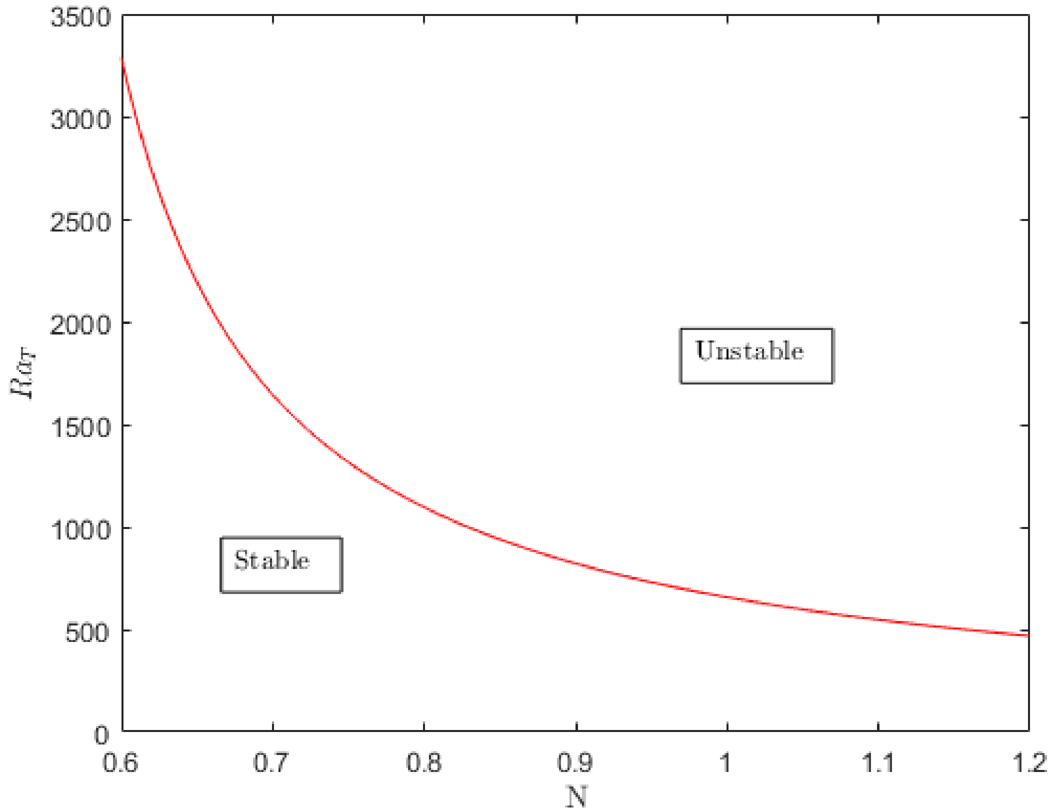

Figure 3.3: Graph showing the regions of growing modes and decaying modes of a fluid layer heated and salted from above in the plane of the thermal Rayleigh number Ra_T and the density ratio N for $Pr = 1$ and $Le = 2$. The red curve shows the critical Rayleigh number for stationary instability, $Ra_T^{(S)}$.

Chapter 4

Weakly Nonlinear Analysis

4.1 Foundations of Weakly Nonlinear Theory

In Chapter 3, we studied the linear stability of the Rayleigh–Bénard problem and the doubly-diffusive problem, which told us when the systems were stable or unstable when subjected to infinitesimally-small perturbations. When the flow is just unstable, it is usually a good enough approximation to only consider the most unstable mode as we do in linear theory. However, to fully characterise a flow, it is necessary to take into account the nonlinear terms. The systems that we are studying here contain quadratic nonlinearities, so the amplitudes of these terms will at first grow in time exponentially, but then will soon dominate the dynamics.

A weakly nonlinear system is one where the amplitude of the perturbations is just large enough for the nonlinear terms to become relevant. In weakly nonlinear theory, we study the dynamics of such a system close to the critical value of the control parameter, which is Ra, or Ra_T for these problems, by creating a reduced set of equations that describes the nonlinear interaction between these few unstable modes. This allows us to have a deeper understanding of the system behaves, and allows us to determine the types of local bifurcations the system undergoes.

4.1.1 Fredholm's Alternative

An important result that is useful for us in weakly nonlinear analysis is *Fredholm's alternative*. (This result is adapted from Fredholm (1903) and Haberman (2004).)

Consider the following ordinary differential equation,

$$\mathcal{L}u(x) = F(x), \tag{4.1.1}$$

on the interval $[a, b]$, with boundary conditions,

$$u = 0 \quad \text{at } x = a \text{ and } x = b, \tag{4.1.2}$$

where the operator $\mathcal{L}$ and the function $F(x)$ are both non-singular.

Now also consider the homogenous problem,

$$\mathcal{L}^\dagger u_h(x) = 0, \tag{4.1.3}$$

subject to the same boundary conditions, where $\mathcal{L}^\dagger$ is the adjoint of the operator $\mathcal{L}$.

Fredholm's alternative states that only one of the following statements are true:

1. $\mathcal{L}u(x) = F(x)$ has a unique solution.

2. $\mathcal{L}^{\dagger}u_h(x) = 0$ has a non-trivial solution.

A corollary of this result is that if the operator $\mathcal{L}$ is self-adjoint and if the homogeneous problem, $\mathcal{L}u_h(x) = 0$, has a non-trivial solution, then the non-homogeneous problem, $\mathcal{L}u(x) = F(x)$, has a solution if and only if,

$$\langle F, u_h \rangle = 0, \tag{4.1.4}$$

where $\langle \cdot, \cdot \rangle$ represents the inner product. Equation (4.1.4) is called the *solvability condition.*

Note that if an operator $\mathcal{L}$ is self-adjoint, then there exists a weight function, $w(x)$ and a corresponding inner product,

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx, \tag{4.1.5}$$

such that,

$$\langle f, \mathcal{L}g \rangle = \langle \mathcal{L}f, g \rangle, \tag{4.1.6}$$

for all functions f and g.

4.1.2 A Simple Example

To illustrate the concept and methods of weakly nonlinear theory, we will now investigate a simple problem, which is a adapted from Matkowsky (1970) and Drazin and Reid (2004). These methods will then be applied to the Rayleigh–Bénard problem and the doubly-diffusive problem in the rest of this chapter.

Consider the following partial differential equation,

$$\frac{\partial u}{\partial t} - \sin(u) = \frac{1}{R}\frac{\partial^2 u}{\partial z^2}, \tag{4.1.7}$$

where u is a function of z and time, t and R is a parameter, with boundary conditions,

$$u = 0 \quad \text{at } z = 0 \text{ and } z = \pi. \tag{4.1.8}$$

This may be thought of as the flow of a fluid with velocity $u(z,t)$ between two parallel planes, $z = 0$ and $z = \pi$, with R imitating the role of a Reynolds number and a forcing term $\sin(u)$.

We see that equation (4.1.7) has null solution, $u_0 = 0$. Therefore, we linearise the perturbations around this solution to obtain the linearised version of the problem, given below,

$$\frac{\partial \hat{u}}{\partial t} - \hat{u} = \frac{1}{R}\frac{\partial^2 \hat{u}}{\partial z^2}, \tag{4.1.9}$$

by noting that $\sin(\tilde{u}) \approx \tilde{u}$ for small $\tilde{u}$.

This linear problem may be solved by supposing the (non-trivial) solution may be written as the product of two functions of a single variable. That is, the solution is of the form,

$$u(z,t) = u_0 + \tilde{u}(z,t) = 0 + \zeta(z)\tau(t). \tag{4.1.10}$$

By substituting equation (4.1.10) into equation (4.1.9), we have,

$$\zeta\frac{\partial \tau}{\partial t} - \zeta\tau = \frac{1}{R}\frac{\partial^2 \zeta}{\partial z}\tau. \tag{4.1.11}$$

Since we are assuming that $u(z, t)$ is not identically zero, we may divide (4.1.11) by $\zeta\tau$ and obtain,

$$\frac{1}{\tau}\frac{\partial \tau}{\partial t} = \frac{1}{R\zeta}\frac{\partial^2 \zeta}{\partial z^2} + 1, \tag{4.1.12}$$

for all z and t.

Since the left hand side of (4.1.12) is a function of only t and the right hand side is function of only z, it follows that each side is equal to a real arbitrary constant, say λ. We therefore have two ordinary differential equations,

$$\frac{d\tau}{dt} = \lambda\tau, \tag{4.1.13}$$

$$\frac{d^2\zeta}{dz^2} = R(\lambda - 1)\zeta. \tag{4.1.14}$$

Since we are assuming that we have a separated solution, the boundary conditions (4.1.8) imply that we have,

$$\zeta = 0 \quad \text{at } z = 0 \text{ and } z = \pi. \tag{4.1.15}$$

If we have $R(\lambda - 1) = k^2 > 0$, where $k \in \mathbb{R}$, we find that equation (4.1.14) has general solution,

$$\zeta(z) = c_1 e^{kz} + c_2 e^{-kz}, \tag{4.1.16}$$

where c_1 and c_2 are constants. The boundary conditions (4.1.15) imply,

$$\zeta(0) = c_1 + c_2 = 0, \quad \zeta(\pi) = c_1 e^{k\pi} + c_2 e^{-k\pi} = 0 \implies c_1 = c_2 = 0, \tag{4.1.17}$$

since $e^{2k} = 1$ has no real solutions for $k \in \mathbb{R}$. Therefore, the only solution in this case is $u = 0$.

If we have $R(\lambda - 1) = 0$, equation (4.1.14) has general solution,

$$\zeta(z) = c_1 + c_2 z, \tag{4.1.18}$$

where c_1 and c_2 are constants. The boundary conditions (4.1.15) imply,

$$\zeta(0) = c_1 = 0, \quad \zeta(\pi) = c_1 + c_2\pi = 0 \implies c_1 = c_2 = 0. \tag{4.1.19}$$

Therefore, again the only solution in this case is $u = 0$.

If we have $R(\lambda - 1) = -k^2 < 0$, where $k \in \mathbb{R}$, equation (4.1.14) has general solution,

$$\zeta(z) = c_1 \cos(kz) + c_2 \sin(kz), \tag{4.1.20}$$

where c_1 and c_2 are constants. The boundary conditions (4.1.15) imply,

$$\zeta(0) = c_1 = 0, \quad \zeta(\pi) c_2 \sin(k\pi) = 0. \tag{4.1.21}$$

To avoid the trivial solution again (that is to avoid having $c_1 = c_2 = 0$) we choose k to satisfy $\sin(k\pi) = 0$, so we have $k = n$ where $n \in \mathbb{Z}$. Therefore equation (4.1.14) has non-trivial solutions for an infinite sequence of values of λ_n that are given by,

$$\lambda_n = 1 - \frac{n^2}{R}, \quad \text{and} \quad \zeta_n(z) = \sin(nz). \tag{4.1.22}$$

Solving equation (4.1.13) for these values of λ_n, we have,

$$\tau_n(t) = e^{\lambda_n t}, \tag{4.1.23}$$

and therefore, we have separated solutions $u_n(z,t)$, given by,

$$\tilde{u}_n(z,t) = e^{\lambda_n t}\sin(nz). \tag{4.1.24}$$

Since equation (4.1.9) is linear, we can take linear combinations of the basic separated solutions to obtain,

$$\tilde{u}(z,t) = \sum_{n=1}^{\infty} a_n e^{\lambda_n t}\sin(nz), \tag{4.1.25}$$

where the a_n are constants determined from the Fourier sine-series representation.

The threshold of instability occurs for the smallest value of R such that $\lambda_n = 0$. This implies that the critical value of R, R_c, is

$$R_c \equiv \min_{n\geq 1} n^2 = 1. \tag{4.1.26}$$

For λ_n to be positive, we require $R > n^2$. Hence the base flow will be stabilising if and only if all the modes are stable as well.

If we assume the flow is just unstable, that is R is slightly greater than R_c or equivalently $0 < R - R_c \ll 1$, then all normal modes decay exponentially in time except the $n = 1$ mode. However this exponentially growing mode in linear theory cannot represent the correct solution to the problem because it quickly grows large enough that the nonlinear terms become significant. Hence, a nonlinear analysis is necessary here.

Let us define a small parameter ϵ by the relationship,

$$\epsilon^2 r = R - R_c, \tag{4.1.27}$$

where r is a scaling parameter of order 1. This choice of ϵ^2 seems arbitrary, but will become apparent throughout the calculation. Then, let us also define the variable T as,

$$T = \epsilon^2 t, \tag{4.1.28}$$

not to be confused with the temperature T in the fluid flow systems that we are studying. This leads us to the assumption that the weakly nonlinear solution is of the form,

$$u(z,T) = u_0(z,T) + \epsilon^{\alpha}u_1(z,T) + \epsilon^{2\alpha}u_2(z,T) + \ldots, \tag{4.1.29}$$

where the value of α is to be determined and recalling that $u_0 = 0$.

Substituting the previous expansion of u into the governing equation (4.1.7), we have,

$$\begin{aligned}\epsilon^{\alpha+2}\frac{\partial u_1}{\partial T} + \epsilon^{2\alpha+2}\frac{\partial u_2}{\partial T} + \ldots - (\epsilon^{\alpha}u_1 + \epsilon^{2\alpha}u_2 + \ldots) + \frac{1}{6}(\epsilon^{\alpha}u_1 + \epsilon^{2\alpha}u_2)^3 + \ldots \\ = (1 - \epsilon^2 r + \epsilon^4 r^2 + \ldots)\left(\epsilon^{\alpha}\frac{\partial^2 u_1}{\partial z^2} + \epsilon^{2\alpha}\frac{\partial^2 u_2}{\partial z^2} + \ldots\right)\end{aligned} \tag{4.1.30}$$

noting that $\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \ldots$, and that $\frac{1}{R} = \frac{1}{R_c+\epsilon^2 r} = \frac{1}{1+\epsilon^2 r} = 1 - \epsilon^2 r + \epsilon^4 r^2 - \epsilon^6 r^3 + \ldots$.

Equating coefficients of order ϵ^{α}, regardless of the value of α, we have,

$$\mathcal{O}(\epsilon^{\alpha}): \quad \frac{\partial^2 u_1}{\partial z^2} + u_1 = 0. \tag{4.1.31}$$

Equation (4.1.31) can be shown (by similar methods we have used earlier in this section) to have a non-zero solution of the form,

$$u_1(z,T) = A_1(T)\sin(z). \tag{4.1.32}$$

In other words, there is a non-zero solution to the equation $\mathcal{L}u_1 = 0$, where $\mathcal{L} = \frac{\partial^2}{\partial z^2} + 1$ is a self-adjoint operator with weight function $w(x) = 1$.

Subtracting equation (4.1.31) from the governing equation (4.1.30) and dividing the remaining terms by ϵ^α, we are left with,

$$\begin{aligned}\epsilon^2\frac{\partial u_1}{\partial T} + \epsilon^{\alpha+2}\frac{\partial u_2}{\partial T} + \ldots - (\epsilon^\alpha u_2 + \epsilon^{2\alpha}u_3 + \ldots) + \frac{1}{6}(\epsilon^{2\alpha}u_1^3 + \epsilon^{5\alpha}u_2^3 + \ldots) + \ldots \\ = -\epsilon^2 r\frac{\partial^2 u_1}{\partial z^2} + \epsilon^\alpha\frac{\partial^2 u_2}{\partial z^2} - \epsilon^{\alpha+2}r\frac{\partial^2 u_2}{\partial z^2} + \epsilon^{2\alpha}\frac{\partial^2 u_3}{\partial z^2} - \epsilon^{2\alpha+2}r\frac{\partial^2 u_3}{\partial z^2} + \ldots.\end{aligned} \tag{4.1.33}$$

If we assume here that $\alpha = 2$, then to the lowest order of ϵ, we have,

$$\mathcal{O}(\epsilon^2): \quad \frac{\partial u_1}{\partial T} - u_2 = -r\frac{\partial^2 u_1}{\partial z^2} + \frac{\partial^2 u_2}{\partial z^2}. \tag{4.1.34}$$

Rearranging equation (4.1.34), and by using equations (4.1.31) and (4.1.32), we can write,

$$\begin{aligned}\frac{\partial^2 u_2}{\partial z^2} + u_2 &= \frac{\partial u_1}{\partial T} + r\frac{\partial^2 u_1}{\partial z^2} \\ &= \frac{\partial u_1}{\partial T} - ru_1 \\ &= \frac{\partial}{\partial T}A_1\sin(z) - rA_1\sin(z) \\ &= \left(\frac{\partial A_1}{\partial T} - rA_1\right)\sin(z).\end{aligned} \tag{4.1.35}$$

This equation for u_2 is not homogeneous, but still satisfies the same boudary conditions that u_1 did. Since there was a non-zero solution (4.1.32) to the homogeneous problem, by Fredholm's alternative (see Section 4.1.1), we know that equation (4.1.35) only has solution if the solvability condition,

$$\left\langle \left(\frac{\partial A_1}{\partial T} - rA_1\right)\sin(z), u_1\right\rangle = 0, \tag{4.1.36}$$

is satisfied. Given that $u_1(z) \propto \sin(z)$ and that $\langle \sin(z), \sin(z)\rangle \neq 0$, equation (4.1.36) is equivalent to requiring,

$$\frac{\partial A_1}{\partial T} - rA_1 = 0, \tag{4.1.37}$$

which has solution,

$$A_1(T) = c_1 e^{rT}, \tag{4.1.38}$$

where c_1 is an integration constant. This essentially recovers linear theory—as long as the amplitude of the perturbation is of the order ϵ^2 or less, A_1 continues to grow exponentially.

In order to account for nonlinear saturation, we want the nonlinear terms of equation (4.1.33) to be of the same order as the $\frac{\partial u_1}{\partial T}$ term. This is why it was convenient to define ϵ as a squared term. Therefore, taking $\alpha = 1$, to the lowest order of ϵ we have,

$$\mathcal{O}(\epsilon): \quad \frac{\partial^2 u_2}{\partial z^2} + u_2 = 0. \tag{4.1.39}$$

Solving this, we find that this has solution,

$$u_2(z, T) = A_2(T)sin(z). \tag{4.1.40}$$

Taking the next order terms, we have,

$$\mathcal{O}(\epsilon^2): \quad \frac{\partial u_1}{\partial T} - u_3 + \frac{1}{6}u_1^3 = -r\frac{\partial^2 u_1}{\partial z^2} + \frac{\partial^2 u_3}{\partial z^2}. \tag{4.1.41}$$

Rearranging equation (4.1.41), we can write,

$$\begin{aligned}\frac{\partial^2 u_3}{\partial z^2} + u_3 &= \frac{\partial u_1}{\partial T} + r\frac{\partial^2 u_1}{\partial z^2} + \frac{1}{6}u_1^3 \\ &= \frac{\partial}{\partial T}A_1\sin(z)\frac{1}{6}A_1^3\sin^3(z) - rA_1\sin(z) \\ &= \left(\frac{\partial A_1}{\partial T} + \frac{1}{8}A_1^3 - rA_1\right)\sin(z) - \frac{1}{24}A_1^3\sin(3z),\end{aligned} \tag{4.1.42}$$

by using equations (4.1.31) and (4.1.32), and the fact that $\sin^3(z) = \frac{3}{4}\sin(z) - \frac{1}{4}\sin(3z)$. By Fredholm's alternative, equation (4.1.42) only has a solution if the solvability condition,

$$\left\langle \left(\frac{\partial A_1}{\partial T} + \frac{1}{8}A_1^3 - rA_1\right)\sin(z) - \frac{1}{24}A_1^3\sin(3z), u_1 \right\rangle = 0, \tag{4.1.43}$$

is satisfied. Since we know that $u_1 \propto \sin(z)$ and that the $\sin(3z)$ term is orthogonal to $\sin(z)$ whilst the $\sin(z)$ term is not, equation (4.1.43) is equivalent to requiring,

$$\frac{\partial A_1}{\partial T} + \frac{1}{8}A_1^3 - rA_1 = 0. \tag{4.1.44}$$

This is the Landau equation for this problem, which we see can be transformed into the normal form for a pitchfork bifurcation. The sign of the parameter r tells us the bifurcation is subcritical (when r is negative) or supercritical (when r is positive). Changing the value of r does not change the solution, but changes the scale of ϵ, since the Landau equation is essentially of the form

$$\epsilon^3\frac{dA_1}{dt} = (R - R_c)A_1 - \frac{1}{8}\epsilon^3 A_1^3. \tag{4.1.45}$$

Equation (4.1.44) is a Bernouilli ordinary differential equation with general solution,

$$A_1(T) = \pm\frac{2\sqrt{2}\sqrt{r}c_1e^{rT}}{\sqrt{c_1e^{2rT} - 1}}, \tag{4.1.46}$$

where c_1 is an integration constant. Since we assumed that $\alpha = 1$, we have that,

$$\begin{aligned}u(z,T) &= \epsilon A_1(T)\sin(z) \\ &= \pm\frac{2\sqrt{2r}c_1e^{rT}}{\sqrt{c_1e^{2rT} - 1}}\sin(z),\end{aligned} \tag{4.1.47}$$

is an approximate solution to the original equation (4.1.7).

Note that we could go to higher order of ϵ and find solutions for u_4, u_5 and so on. We stopped here because we found the leading asymptotic behaviour of the system and determined the primary bifurcation.

4.2 Weakly Nonlinear Analysis of Rayleigh–Bénard Convection

Using the methods in Section 4.1, we will derive weakly nonlinear equations for the Rayleigh–Bénard problem.

Recall that the behaviour of Rayleigh–Bénard convection for Ra just above onset Ra_c is a rapid transition to another steady state that has 2D convective rolls. We will therefore look for weakly nonlinear solutions of the problem by first defining a small parameter that represents this distance to onset, expand all quantities in powers of that small parameter, and then finding solutions order by order using the solvability condition (4.1.4).

Recall that in Section 2.3.5, we rewrote the governing equations in terms of the vorticity and streamfunction to obtain,

$$\frac{\partial \omega}{\partial t} + J(\psi, \omega) = -RaPr\frac{\partial T}{\partial x} + Pr\nabla^2\omega, \tag{4.2.1}$$

$$\frac{\partial T}{\partial t} + J(\psi, T) = \frac{\partial \psi}{\partial x} + \nabla^2 T, \tag{4.2.2}$$

where $J(\psi, \omega) = \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\omega}{\partial x}$ and $J(\psi, T) = \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial T}{\partial x}$.

Let us define a small parameter ϵ by the relationship,

$$\epsilon^2 r = Ra - Ra_c, \tag{4.2.3}$$

where Ra_c is the critical value at which the first mode becomes unstable, which we found in Section 3.2. Let us also define the variable τ as

$$\tau = \epsilon^2 t. \tag{4.2.4}$$

This leads us to the assumption that the weakly nonlinear solution is of the form

$$\psi(x, z, \tau) = \epsilon^\alpha \psi_1(x, z, \tau) + \epsilon^{2\alpha}\psi_2(x, z, \tau) + \ldots \tag{4.2.5a}$$

$$\omega(x, z, \tau) = \epsilon^\alpha \omega_1(x, z, \tau) + \epsilon^{2\alpha}\omega_2(x, z, \tau) + \ldots \tag{4.2.5b}$$

$$T(x, z, \tau) = \epsilon^\alpha T_1(x, z, \tau) + \epsilon^{2\alpha}T_2(x, z, \tau) + \ldots, \tag{4.2.5c}$$

noticing that these are functions of τ rather than t. Substituing equations (4.2.5) into equation (4.2.1) and dividing by ϵ^α we have

$$\begin{aligned}\left(\epsilon^2\frac{\partial\omega_1}{\partial\tau} + \epsilon^{\alpha+2}\frac{\partial\omega_2}{\partial\tau} + \ldots\right) + \epsilon^\alpha J(\psi_1, \omega_1) + \epsilon^{2\alpha}J(\psi_1, \omega_2) + \epsilon^{2\alpha}J(\psi_2, \omega_1) + \ldots \\ = -(Ra_c + \epsilon^2 r)Pr\left(\frac{\partial T_1}{\partial x} + \epsilon^\alpha\frac{\partial T_2}{\partial x} + \ldots\right) + Pr\nabla^2(\omega_1 + \epsilon^\alpha\omega_2 + \ldots).\end{aligned} \tag{4.2.6}$$

Similarly, equation (4.2.2) becomes

$$\begin{aligned}\left(\epsilon^2\frac{\partial T_1}{\partial\tau} + \epsilon^{\alpha+2}\frac{\partial T_2}{\partial\tau} + \ldots\right) + \epsilon^\alpha J(\psi_1, T_1) + \epsilon^{2\alpha}J(\psi_1, T_2) + \epsilon^{2\alpha}J(\psi_2, T_1) \\ = \left(\frac{\partial\psi_1}{\partial x} + \epsilon^\alpha\frac{\partial\psi_2}{\partial x} + \ldots\right) + \nabla^2(T_1 + \epsilon^\alpha T_2 + \ldots).\end{aligned} \tag{4.2.7}$$

At zeroeth order in ϵ, ($\mathcal{O}(\epsilon^0) = \mathcal{O}(1)$) we have

$$-Ra_c Pr\frac{\partial T_1}{\partial x} - Pr\nabla^4\psi_1 = 0, \tag{4.2.8a}$$

$$\frac{\partial\psi_1}{\partial x} + \nabla^2 T_1 = 0. \tag{4.2.8b}$$

There is a trivial solution to the above equations, namely

$$\psi_1 = T_1 = 0. \tag{4.2.9}$$

However, we know that at $Ra = Ra_c$ there is a non-trival solution since linear theory predicted that this is the point convection is just becoming linearly unstable.

Let us define $\Psi_1 \equiv (\psi_1, T_1)^T$. Then we can write equations (4.2.8a) and (4.2.5) as

$$\mathcal{L}\Psi_1 = 0, \tag{4.2.10}$$

where $\mathcal{L}$ is a 2×2 matrix operator given by

$$\mathcal{L} = \begin{pmatrix} -Pr\nabla^4 & -Ra_c Pr\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \nabla^2 \end{pmatrix}. \tag{4.2.11}$$

It can be shown that $\mathcal{L}$ is self-adjoint with respect to the inner product

$$\langle \Psi_m, \Psi_n \rangle = \iint (\psi_m \psi_n + Ra_c Pr T_m T_n)\, dzdx, \tag{4.2.12}$$

assuming periodicity in x, for any two functions Ψ_m and Ψ_n. Indeed, by integration by parts we have

$$\begin{aligned}
\langle \Psi_m, \mathcal{L}\Psi_n \rangle &= \int_0^L \int_0^1 \left[\psi_m \left(-Pr\nabla^4 \psi_n - Ra_c Pr \frac{\partial T_n}{\partial x} \right) + Ra_c Pr T_m \left(\frac{\partial \psi_n}{\partial x} + \nabla^2 T_n \right) \right] dzdx \\
&= \int_0^L \int_0^1 \left[-Pr\nabla^2\psi_m \nabla^2 \psi_n + Ra_c Pr T_n \frac{\partial \psi_m}{\partial x} - Ra_c Pr \psi_n \frac{\partial T_m}{\partial x} + Ra_c Pr \nabla T_m \nabla T_n \right] dzdx \\
&= \int_0^L \int_0^1 \left[\psi_n \left(-Pr\nabla^4 \psi_m - Ra_c Pr \frac{\partial T_m}{\partial x} \right) + Ra_c Pr T_n \left(\frac{\partial \psi_m}{\partial x} + \nabla^2 T_m \right) \right] dzdx \\
&= \langle \Psi_n, \mathcal{L}\Psi_m \rangle \\
&= \langle \mathcal{L}\Psi_n, \Psi_m \rangle,
\end{aligned} \tag{4.2.13}$$

where the last equality holds since the inner product is symmetric for real functions. Recall that we found the critical wavenumber at $Ra = Ra_c$ to be $a_c = \frac{\pi}{\sqrt{2}}$. Therefore, if we assume periodicity in x, then we must have $L = \frac{2\pi}{a_c}$.

Recall that in Section 3.2 we assumed that solutions were of the form

$$w = f(x)\bar{w}(z)e^{\lambda t}, \tag{4.2.14}$$

and that

$$\bar{w}_n(z) \propto \sin(n\pi z). \tag{4.2.15}$$

Note that in general, the eigenvalue λ is complex-valued, but for Rayleigh–Bénrad convection, we showed that λ is real. Therefore, we define

$$\psi_1(x, z, \tau) = \hat{\psi} \sin(\pi z) \cos(a_c x) A_1(\tau), \tag{4.2.16}$$

where we arbitrarily pick the cosine mode in the x–direction, recalling that in Section 3.2.3, we deduced that 2D convection rolls arise when choosing $f(x)$ to be of the form $\cos(ax)$. Then we have

$$\begin{aligned}
\omega_1 &= -\nabla^2 \psi \\
&= (a_c^2 + \pi^2)(\hat{\psi} \sin(\pi z) \cos(a_c x)) A_1(\tau),
\end{aligned} \tag{4.2.17}$$

and

$$\nabla^2 T_1 = -\frac{\partial \psi_1}{\partial x}, \tag{4.2.18}$$

which implies that

$$T_1 = -\frac{a_c \hat{\psi} \sin(\pi z) \sin(a_c x) A_1(\tau)}{a_c^2 + \pi^2}. \tag{4.2.19}$$

Returning now to equation (4.2.6), we see that in order for the nonlinear terms (the Jacobian terms) to be of the same order as the $\epsilon^2 r Pr \frac{\partial T_1}{\partial x}$ term, we choose $\alpha = 1$. Note that we chose to define ϵ in the way we did in order to be able to choose $\alpha = 1$. We now rewrite equation (4.2.6) as

$$\begin{aligned} \left(\epsilon^2 \frac{\partial \omega_1}{\partial \tau} + \epsilon^3 \frac{\partial \omega_2}{\partial \tau} + \ldots\right) + \epsilon J(\psi_1, \omega_1) + \epsilon^2 J(\psi_1, \omega_2) + \epsilon^2 J(\psi_2, \omega_1) + \ldots \\ = -(Ra_c + \epsilon^2 r) Pr \left(\frac{\partial T_1}{\partial x} + \epsilon \frac{\partial T_2}{\partial x} + \ldots\right) + Pr \nabla^2(\omega_1 + \epsilon \omega_2 + \ldots), \end{aligned} \tag{4.2.20}$$

and equation (4.2.7) as

$$\begin{aligned} \left(\epsilon^2 \frac{\partial T_1}{\partial \tau} + \epsilon^3 \frac{\partial T_2}{\partial \tau} + \ldots\right) + \epsilon J(\psi_1, T_1) + \epsilon^2 J(\psi_1, T_2) + \epsilon^2 J(\psi_2, T_1) \\ = \left(\frac{\partial \psi_1}{\partial x} + \epsilon \frac{\partial \psi_2}{\partial x} + \ldots\right) + \nabla^2(T_1 + \epsilon T_2 + \ldots). \end{aligned} \tag{4.2.21}$$

Now, to the next order in ϵ, $\mathcal{O}(\epsilon)$, we have

$$J(\psi_1, \omega_1) = -Ra_c Pr \frac{\partial T_2}{\partial x} - Pr \nabla^4 \psi_2, \tag{4.2.22a}$$

$$J(\psi_1, T_1) = \frac{\partial \psi_2}{\partial x} + \nabla^2 T_2. \tag{4.2.22b}$$

However, since $J(\psi_1, \omega_1) = -J(\psi_1, \nabla^2 \psi_1) = -(a_c^2 + \pi^2) J(\psi_1, \psi_1) = 0$, we have

$$\mathcal{L}\Psi_2 = \begin{pmatrix} -Pr\nabla^4 & -Ra_c Pr \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \nabla^2 \end{pmatrix} \begin{pmatrix} \psi_2 \\ T_2 \end{pmatrix} = \begin{pmatrix} 0 \\ J(\psi_1, T_1) \end{pmatrix}, \tag{4.2.23}$$

where we have defined $\Psi_2 \equiv (\psi_2, T_2)^T$. Since we already know that $\mathcal{L}$ is self-adjoint and that $\mathcal{L}\Psi_1 = 0$ has a non-trivial solution, then by Fredholm's alternative, the solvability condition is given by

$$\left\langle \begin{pmatrix} \psi_1 \\ T_1 \end{pmatrix}, \begin{pmatrix} 0 \\ J(\psi_1, T_1) \end{pmatrix} \right\rangle = 0, \tag{4.2.24}$$

or equivalently,

$$Ra_c Pr \iint T_1 J(\psi_1, T_1)\, dz dx = 0. \tag{4.2.25}$$

Substituing expressions for ψ_1 and T_1, $J(\psi_1, T_1)$ is given by

$$J(\psi_1, T_1) = \hat{\psi}^2 \frac{a_c^2 \pi}{a_c^2 + \pi^2} \frac{\sin(2\pi z)}{2} A_1^2, \tag{4.2.26}$$

which is independent of x. Since $T_1 \propto \sin(a_c x)$, we know that equation (4.2.25) is true, and so we can expect a solution to equations (4.2.22a) and (4.2.23).

Since the left hand side of equations (4.2.22a)) and (4.2.23) are independent of x, let us posit that the solution is also independent of x. We try $\psi_2 = \psi_2(z,\tau)$ and $T_2 = T_2(z,\tau)$. This gives us

$$-Pr\frac{\partial^4\psi_2}{\partial z^4} = 0, \tag{4.2.27}$$

which, when subject to the boundary conditions we have defined for this problem, has solution

$$\psi_2 = 0, \tag{4.2.28}$$

and

$$\frac{\partial^2 T_2}{\partial z^2} = \hat{\psi}^2\frac{a_c^2\pi}{a_c^2+\pi^2}\frac{\sin(2\pi z)}{2}A_1^2, \tag{4.2.29}$$

which has solution

$$T_2 = -\frac{\hat{\psi}a_c^2}{a_c^2+\pi^2}\frac{\sin(2\pi z)}{8\pi}A_1^2. \tag{4.2.30}$$

Returning to equations (4.2.20) and (4.2.21), equating coefficients of $\mathcal{O}(\epsilon^2)$, we have

$$\frac{\partial\omega_1}{\partial\tau} + J(\psi_1,\omega_2) + J(\psi_2,\omega_1) = -rPr\frac{\partial T_1}{\partial x} - Ra_cPr\frac{\partial T_3}{\partial x} + Pr\nabla^2\omega_3, \tag{4.2.31a}$$

$$\frac{\partial T_1}{\partial\tau} + J(\psi_1,T_2) + J(\psi_2,T_1) = \frac{\partial\psi_3}{\partial x} + \nabla^2 T_3. \tag{4.2.31b}$$

However, $\psi_2 = \omega_2 = 0$ implies that $J(\psi_1,\omega_2) = J(\psi_2,\omega_1) = J(\psi_2,T_1) = 0$. Therefore, we write

$$\frac{\partial\omega_1}{\partial\tau} + rPr\frac{\partial T_1}{\partial x} = -Ra_cPr\frac{\partial T_3}{\partial x} - Pr\nabla^4\omega_3, \tag{4.2.32a}$$

$$\frac{\partial T_1}{\partial\tau} + J(\psi_1,T_2) = \frac{\partial\psi_3}{\partial x} + \nabla^2 T_3, \tag{4.2.32b}$$

which is equivalent to

$$\mathcal{L}\Psi_3 = \begin{pmatrix} -Pr\nabla^4 & -Ra_cPr\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \nabla^2 \end{pmatrix}\begin{pmatrix}\psi_3 \\ T_3\end{pmatrix} = N_3, \tag{4.2.33}$$

where we have defined $\Psi \equiv (\psi_3, T_3)$ and

$$N_3 \equiv \begin{pmatrix} \frac{\partial\omega_1}{\partial\tau} + rPr\frac{\partial T_1}{\partial x} \\ \frac{\partial T_1}{\partial\tau} + J(\psi_1,T_2) \end{pmatrix}. \tag{4.2.34}$$

By Fredholm's alternative, equation (4.2.33) only has solution when

$$\langle\Psi_1, N_3\rangle = 0. \tag{4.2.35}$$

Through a lengthy process, we solve equation (4.2.35), which gives us the Landau equation

$$\left[\left((a_c^2+\pi^2) + \frac{Ra_cPra_c^2}{a_c^2+\pi^2}\right)\frac{\partial A_1}{\partial\tau} - \frac{rPra_c^2}{a_c^2+\pi^2}A_1 + \frac{Ra_cPra_c^4\hat{\psi}^2}{8(a_c^2+\pi^2)^2}A_1^3\right] = 0. \tag{4.2.36}$$

This tells us that the primary bifurcation of Rayleigh–Bénard convection is a pitchfork bifurcation. It is subcritical when r is negative and supercritical when r is positive. However, since this analysis was done by assuming that the fluid is just unstable, so that $0 < Ra - Ra_c \ll 1$, we have that r is positive.

Using a continuation and bifurcation package for MATLAB called pde2path (Uecker et al., 2014; Dohnal et al., 2014; deWitt et al., 2017), we can plot a bifurcation diagram showing the primary bifurcation. Figure 4.1 shows bifurcation diagram for $Pr = 1$ with a supercritical bifurcation. We can see a bifurcation branch at $\psi = 0$ that is stable for $Ra < Ra_c$ and unstable for $Ra > Ra_c$. At Ra_c, a stable branch of solutions emerge. In Figure 4.2, we can see how solutions of the fluid at the stable bifurcation branch form convection rolls.

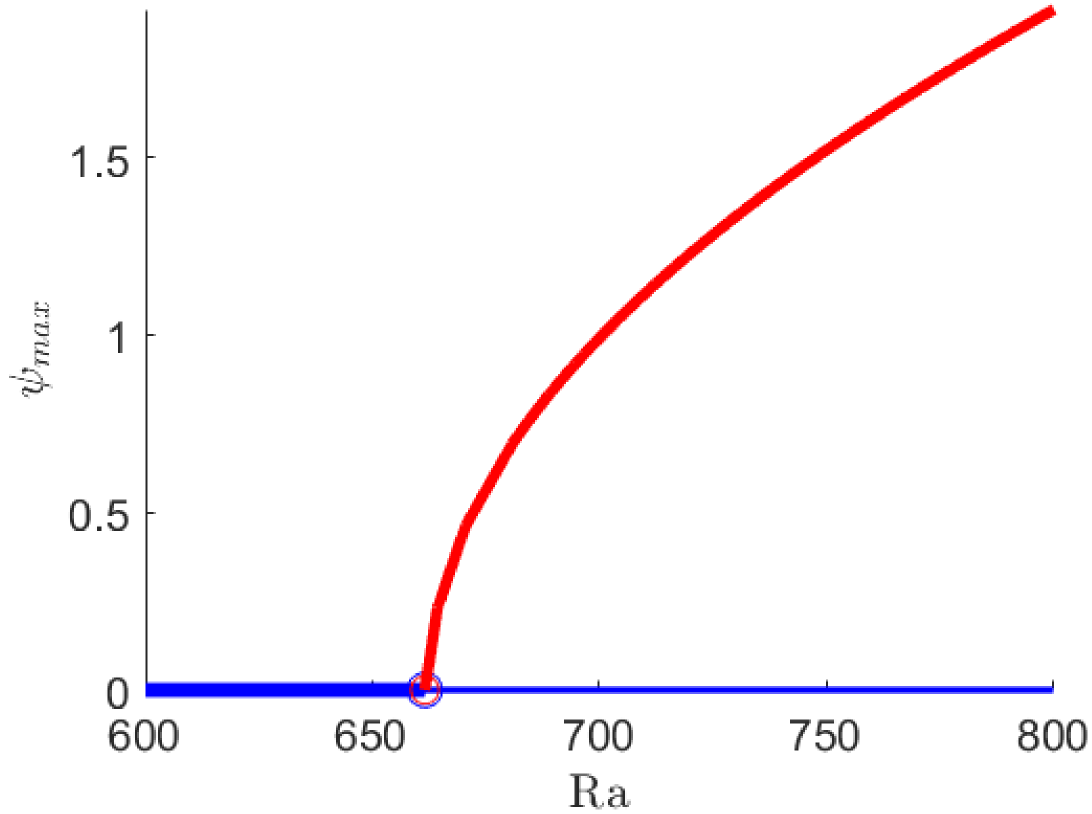

Figure 4.1: Bifurcation diagram showing the primary supercritical pitchfork bifurcation in Rayleigh–Bénard convection for $Pr = 1$. Thicker lines represent stable solutions and thinner lines represent unstable solutions. The circle represents the bifurcation point Ra_c.

Note that these figures were calculated for a fluid layer rescaled to be between $z = -0.5$ and $z = 0.5$ with free-slip horizontal boundary conditions at $x = \pm\frac{2\pi}{a_c}$ so that we have periodic boundary conditions that emulate a fluid in an infinite layer. Note also that this was computed on a rather coarse mesh of 100×25 grid points due to computational constraints. For a more accurate diagram, a more refined mesh should be used, however, this is good enough here to qualitatively visualise the bifurcation diagram.

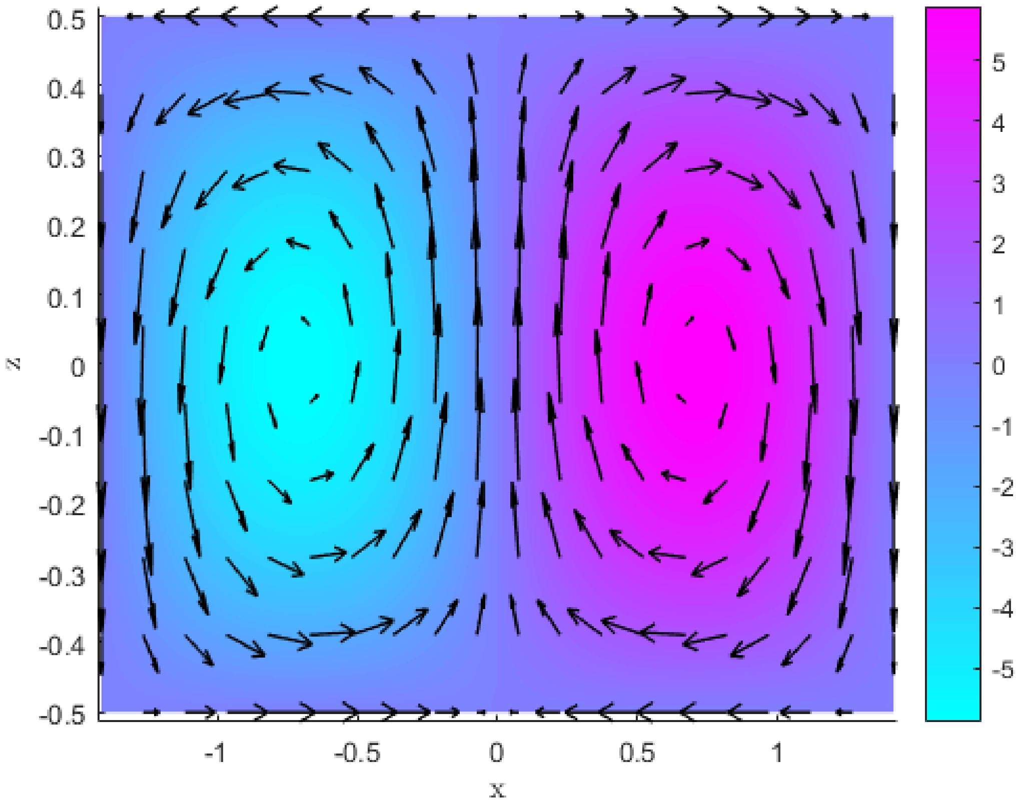

Figure 4.2: Sample solution of Rayleigh–Bénard convection at the stable branch shown in red in Figure 4.1. This is in terms of the streamfunction ψ, where the arrows indicate the direction of the fluid flow.

4.3 Weakly Nonlinear Analysis of Doubly-Diffusive Convection: A Horizontal Layer Heated and Salted from Below

Following a similar procedure as we used in the previous sections, we now derive weakly nonlinear equations for a fluid in a horizontal layer that is heated and salted from below.

Recall that for $N < N^{(OS)}$ instability arises through a stationary instability, and for $N > N^{(OS)}$ instability arises through an oscillatory instability, where $N^{(OS)}$ is given by

$$N^{(OS)} = \frac{Pr + 1}{Le(LePr + 1)}. \tag{4.3.1}$$

We will look for weakly nonlinear solutions to the problem by defining a small parameter that represents a small distance of Ra_T to the critical value of the onset of convection, which is given by $Ra_T^{(O)}$ or $Ra_T^{(S)}$ depending on the value of N, where

$$Ra_T^{(O)} = \frac{27\pi^4}{4}\left(1 + \frac{1}{Le}\right)\left(1 + \frac{1}{LePr}\right)\left(\frac{Le(Pr + 1)}{Le(Pr + 1) - N(LePr + 1)}\right), \tag{4.3.2}$$

and

$$Ra_T^{(S)} = \frac{27\pi^4}{4(1 - LeN)}. \tag{4.3.3}$$

Recall that we may write the governing equations of the problem in terms of the streamfunction and the vorticity to obtain

$$\frac{\partial \omega}{\partial t} + J(\psi, \omega) = -Ra_T Pr \frac{\partial}{\partial x}(T - NC) + Pr\nabla^2 \omega, \tag{4.3.4a}$$

$$\frac{\partial T}{\partial t} + J(\psi, T) = \frac{\partial \psi}{\partial x} + \nabla^2 T, \tag{4.3.4b}$$

$$\frac{\partial C}{\partial t} + J(\psi, C) = \frac{\partial \psi}{\partial x} + \frac{1}{Le}\nabla^2 C, \tag{4.3.4c}$$

where $J(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}$.

Let ϵ be a small parameter defined by the relationship

$$\epsilon^2 r = Ra_T - Ra_{Tc}, \tag{4.3.5}$$

where Ra_{TC} is the critical value at which the first mode of the normal mode solution becomes unstable. Let us also define the variable τ as

$$\tau = \epsilon^2 t. \tag{4.3.6}$$

We will now assume that the weakly nonlinear solution is of the form

$$\psi(x,z,\tau) = \epsilon^{\alpha}\psi_1(x,z,\tau) + \epsilon^{2\alpha}\psi_2(x,z,\tau) + \dots \tag{4.3.7a}$$

$$\omega(x,z,\tau) = \epsilon^{\alpha}\omega_1(x,z,\tau) + \epsilon^{2\alpha}\omega_2(x,z,\tau) + \dots \tag{4.3.7b}$$

$$T(x,z,\tau) = \epsilon^{\alpha}T_1(x,z,\tau) + \epsilon^{2\alpha}T_2(x,z,\tau) + \dots, \tag{4.3.7c}$$

$$C(x,z,\tau) = \epsilon^{\alpha}C_1(x,z,\tau) + \epsilon^{2\alpha}C_2(x,z,\tau) + \dots. \tag{4.3.7d}$$

Substituting the above equations into equations (4.3.4a), (4.3.4b) and (4.3.4c) and dividing by ϵ^{α}, we obtain the following equations:

$$\begin{aligned}\left(\epsilon^2\frac{\partial \omega_1}{\partial \tau} + \epsilon^{\alpha+2}\frac{\partial \omega_2}{\partial \tau} + \dots\right) + \epsilon^{\alpha}J(\psi_1,\omega_1) + \epsilon^{2\alpha}J(\psi_1,\omega_2) + \epsilon^{2\alpha}J(\psi_2,\omega_1) + \dots \\ = -(Ra_{Tc} + \epsilon^2 r)Pr\left(\frac{\partial T_1}{\partial x} + \epsilon^{\alpha}\frac{\partial T_2}{\partial x} + \dots\right) + (Ra_{Tc} + \epsilon^2 r)PrN\left(\frac{\partial C_1}{\partial x} + \epsilon^{\alpha}\frac{\partial C_2}{\partial x} + \dots\right) \\ + Pr\nabla^2(\omega_1 + \epsilon^{\alpha}\omega_2 + \dots),\end{aligned} \tag{4.3.8}$$

$$\begin{aligned}\left(\epsilon^2\frac{\partial T_1}{\partial \tau} + \epsilon^{\alpha+2}\frac{\partial T_2}{\partial \tau} + \dots\right) + \epsilon^{\alpha}J(\psi_1,T_1) + \epsilon^{2\alpha}J(\psi_1,T_2) + \epsilon^{2\alpha}J(\psi_2,T_1) \\ = \left(\frac{\partial \psi_1}{\partial x} + \epsilon^{\alpha}\frac{\partial \psi_2}{\partial x} + \dots\right) + \nabla^2(T_1 + \epsilon^{\alpha}T_2 + \dots),\end{aligned} \tag{4.3.9}$$

$$\begin{aligned}\left(\epsilon^2\frac{\partial C_1}{\partial \tau} + \epsilon^{\alpha+2}\frac{\partial C_2}{\partial \tau} + \dots\right) + \epsilon^{\alpha}J(\psi_1,C_1) + \epsilon^{2\alpha}J(\psi_1,C_2) + \epsilon^{2\alpha}J(\psi_2,C_1) \\ = \left(\frac{\partial \psi_1}{\partial x} + \epsilon^{\alpha}\frac{\partial \psi_2}{\partial x} + \dots\right) + \frac{1}{Le}\nabla^2(C_1 + \epsilon^{\alpha}C_2 + \dots).\end{aligned} \tag{4.3.10}$$

At zeroeth order in ϵ, or $\mathcal{O}(1)$, we have

$$0 = -Ra_{Tc}Pr\frac{\partial T_1}{\partial x} + Ra_{Tc}\frac{\partial T_1}{\partial x} - Pr\nabla^4\psi_1, \tag{4.3.11a}$$

$$0 = \frac{\partial \psi_1}{\partial x} + \nabla^2 T_1, \tag{4.3.11b}$$

$$0 = \frac{\partial \psi_1}{\partial x} + \frac{1}{Le}\nabla^2 C_1. \tag{4.3.11c}$$

We can see that there is a trivial solution to these equations, which is given by

$$\psi_1 = T_1 = C_1 = 0, \tag{4.3.12}$$

however, from linear theory, we know that there is a non-trivial solution at the point $Ra = Ra_{Tc}$. If we define $\Psi_1 \equiv (\psi_1, T_1, C_1)^T$, then we can write equations (4.3.11a), (4.3.11b) and (4.3.11c) as

$$\mathcal{L}\Psi_1 = 0, \tag{4.3.13}$$

where $\mathcal{L}$ is now a 3×3 operator given by

$$\mathcal{L} = \begin{pmatrix} -Pr\nabla^4 & -Ra_{Tc}Pr\frac{\partial}{\partial x} & Ra_{Tc}PrN\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \nabla^2 & 0 \\ \frac{\partial}{\partial x} & 0 & \frac{1}{Le}\nabla^2 \end{pmatrix}. \tag{4.3.14}$$

It can be shown by repeated integration by parts that $\mathcal{L}$ is self-adjoint with respect to the inner product

$$\langle \Psi_m, \Psi_n \rangle = \iint (\psi_m\psi_n + Ra_{Tc}PrT_mT_n - Ra_{Tc}PrC_mC_n)\, dzdx, \tag{4.3.15}$$

when we assume periodicity in the x–direction, with period $\frac{2\pi}{a_c}$.

Recall now that we wanted to solve for

$$w = f(x)\bar{w}e^{\lambda t} \tag{4.3.16}$$

where λ is a complex eigenvalue. We deduced that $\bar{w}_n$ was of the form $\sin(n\pi z)$, hence, we choose

$$\psi_1(x, z, \tau) = \hat{\psi}\sin(\pi z)cos(a_c x)A(\tau), \tag{4.3.17}$$

where we again arbitrarily choose the cosine mode in the x-direction. Then we have

$$\omega_1 = (a_c^2 + \pi^2)\hat{\psi}\sin(\pi z)cos(a_c x)A(\tau), \tag{4.3.18a}$$

$$T_1 = -\frac{a_c\hat{\psi}\sin(\pi z)\sin(a_c x)A(\tau)}{a_c^2 + \pi^2} \tag{4.3.18b}$$

$$C_1 = -\frac{Lea_c\hat{\psi}\sin(\pi z)\sin(a_c x)A(\tau)}{a_c^2 + \pi^2}. \tag{4.3.18c}$$

Now, if we want the nonlinear terms and the $\epsilon^2 rPr(\frac{\partial T_1}{\partial x} - N\frac{\partial C_1}{\partial x})$ term of equation (4.3.8) to be of the same order, then we choose $\alpha = 1$.

Therefore, we now rewrite equations (4.3.8), (4.3.9) and (4.3.10) as

$$\begin{aligned} &\left(\epsilon^2\frac{\partial\omega_1}{\partial\tau} + \epsilon^3\frac{\partial\omega_2}{\partial\tau} + \ldots\right) + \epsilon J(\psi_1, \omega_1) + \epsilon^2 J(\psi_1, \omega_2) + \epsilon^2 J(\psi_2, \omega_1) + \ldots \\ &= -(Ra_{Tc} + \epsilon^2 r)Pr\left(\frac{\partial T_1}{\partial x} + \epsilon\frac{\partial T_2}{\partial x} + \ldots\right) + (Ra_{Tc} + \epsilon^2 r)PrN\left(\frac{\partial C_1}{\partial x} + \epsilon\frac{\partial C_2}{\partial x} + \ldots\right) \\ &\qquad + Pr\nabla^2(\omega_1 + \epsilon\omega_2 + \ldots), \end{aligned} \tag{4.3.19}$$

$$\begin{aligned} &\left(\epsilon^2\frac{\partial T_1}{\partial\tau} + \epsilon^3\frac{\partial T_2}{\partial\tau} + \ldots\right) + \epsilon J(\psi_1, T_1) + \epsilon^2 J(\psi_1, T_2) + \epsilon^2 J(\psi_2, T_1) \\ &\qquad = \left(\frac{\partial\psi_1}{\partial x} + \epsilon\frac{\partial\psi_2}{\partial x} + \ldots\right) + \nabla^2(T_1 + \epsilon T_2 + \ldots), \end{aligned} \tag{4.3.20}$$

$$\left(\epsilon^2\frac{\partial C_1}{\partial \tau}+\epsilon^3\frac{\partial C_2}{\partial \tau}+\ldots\right)+\epsilon J(\psi_1,C_1)+\epsilon^2 J(\psi_1,C_2)+\epsilon^2 J(\psi_2,C_1)$$
$$=\left(\frac{\partial \psi_1}{\partial x}+\epsilon\frac{\partial \psi_2}{\partial x}+\ldots\right)+\frac{1}{Le}\nabla^2(C_1+\epsilon C_2+\ldots). \tag{4.3.21}$$

Equating coefficients of $\mathcal{O}(\epsilon)$, we have

$$J(\psi_1,\omega_1)=-Ra_{Tc}Pr\frac{\partial T_2}{\partial x}+Ra_{Tc}PrN\frac{\partial C_2}{\partial x}-Pr\nabla^4\psi_2, \tag{4.3.22a}$$
$$J(\psi_1,T_1)=\frac{\partial \psi_2}{\partial x}+\nabla^2 T_2, \tag{4.3.22b}$$
$$J(\psi_1,C_1)=\frac{\partial \psi_2}{\partial x}+\frac{1}{Le}\nabla^2 C_2. \tag{4.3.22c}$$

However, since $J(\psi_1,\omega_1)=-J(\psi_1,\nabla^2\psi_1)=-(a_c^2+\pi^2)J(\psi_1,\psi_1)=0$, we have

$$\mathcal{L}\Psi_2=\begin{pmatrix}-Pr\nabla^4 & -Ra_{Tc}Pr\frac{\partial}{\partial x} & Ra_{Tc}PrN\frac{\partial}{\partial x}\\ \frac{\partial}{\partial x} & \nabla^2 & 0\\ \frac{\partial}{\partial x} & 0 & \frac{1}{Le}\nabla^2\end{pmatrix}\begin{pmatrix}\psi_2\\ T_2\\ C_2\end{pmatrix}=\begin{pmatrix}0\\ J(\psi_1,T_1)\\ J(\psi_1,C_1)\end{pmatrix}, \tag{4.3.23}$$

where $\Psi_2\equiv(\psi_T,T_2,C_2)^T$. Since $\mathcal{L}$ is self-adjoint and we know that $\mathcal{L}\Psi_1=0$ has a non-trivial solution, then Fredholm's alternative tells us that equation (4.3.23) has a solution if

$$\left\langle\begin{pmatrix}\psi_1\\ T_1\\ C_1\end{pmatrix},\begin{pmatrix}0\\ J(\psi_1,T_1)\\ J(\psi_1,C_1)\end{pmatrix}\right\rangle=0, \tag{4.3.24}$$

is satisfied. The above inner product is equivalent to

$$Ra_{Tc}Pr\iint T_1 J(\psi_1,T_1)-NC_1J(\psi_1,C_1)\,dzdx=0, \tag{4.3.25}$$

which is always satisfied since $J(\psi_1,T_1)$ and $J(\psi_1,C_1)$ depend only on z and τ, whilst T_1 and C_1 are proportional to $\sin(a_cx)$. Therefore, we expect a solution to equations (4.3.22a), (4.3.22b) and (4.3.22c).

Noting that the left hand side of equations (4.3.22a), (4.3.22b) and (4.3.22c) are independent of x, we expect the solution to the equations to also be independent of x. Letting $\psi_2=\psi(z,\tau)$, $T_2=T_2(z,\tau)$ and $C_2=C_2(z,\tau)$, we have

$$-Pr\frac{\partial^4\psi_2}{\partial z^4}=0 \tag{4.3.26a}$$
$$\frac{\partial^2 T_2}{\partial z^2}=J(\psi_1,T_1) \tag{4.3.26b}$$
$$\frac{\partial^2 C_2}{\partial z^2}=LeJ(\psi_1,C_1), \tag{4.3.26c}$$

which has solutions

$$\psi_2=0, \tag{4.3.27a}$$
$$T_2=-\frac{a_c^2\hat{\psi}^2A^2}{8\pi(a_c^2+\pi^2)}\sin(2\pi z), \tag{4.3.27b}$$
$$C_2=-\frac{Le^2a_c^2\hat{\psi}^2A^2}{8\pi(a_c^2+\pi^2)}\in(2\pi z). \tag{4.3.27c}$$

Equating coefficients of $\mathcal{O}(\epsilon^2)$, we have

$$\frac{\partial \omega_1}{\partial \tau} + rPr\frac{\partial T_1}{\partial x} - rPrN\frac{\partial C_1}{\partial x} = -Ra_{Tc}Pr\frac{\partial T_3}{\partial x} + Ra_{Tc}PrN\frac{\partial C_3}{\partial x} - Pr\nabla^4\omega_3, \tag{4.3.28a}$$

$$\frac{\partial T_1}{\partial \tau} + J(\psi_1, T_2) = \frac{\partial \psi_3}{\partial x} + \nabla^2 T_3, \tag{4.3.28b}$$

$$\frac{\partial C_1}{\partial \tau} + J(\psi_1, C_2) = \frac{\partial \psi_3}{\partial x} + \frac{1}{Le}\nabla^2 C_3, \tag{4.3.28c}$$

where we have used the fact that $J(\psi_1, \omega_2) = J(\psi_2, \omega_1) = J(\psi_2, T_1) = J(\psi_2, C_1) = 0$. Therefore, by similar reasoning as before, the solvability condition is

$$\langle \Psi_3, N_3 \rangle = 0, \tag{4.3.29}$$

where $\Psi_3 = (\psi_3, T_3, C_3)^T$ and

$$N_3 = \begin{pmatrix} \frac{\partial \omega_1}{\partial \tau} + rPr\frac{\partial T_1}{\partial x} - rPrN\frac{\partial C_1}{\partial x} \\ \frac{\partial T_1}{\partial \tau} + J(\psi_1, T_2) \\ \frac{\partial C_1}{\partial \tau} + J(\psi_1, C_2) \end{pmatrix}. \tag{4.3.30}$$

Solving equation (4.3.29) gives us

$$\begin{aligned} &(a_c^2 + \pi^2) + (1 - NLe^2)\frac{Ra_{Tc}Pra_c^2}{(\pi^2 + a_c^2)^2}\frac{\partial A}{\partial \tau} \\ -(1 - NLe)&\frac{rPra_c^2}{\pi^2 + a_c^2}A + (1 - NLe^2)\frac{Ra_{Tc}Pra_c^4\hat{\psi}^2}{8(\pi^2 + a_c^2)}A^3 = 0. \end{aligned} \tag{4.3.31}$$

Notice here, that the subcriticality, or the supercriticality of the bifurcation depends on the value of NLe and NLe^2. If we are in the steady state, then the system, then we can say that the primary bifurcation is subcritical for $1 - NLe^2 < 0$ and $1 - NLe > 0$, or $\frac{1}{Le^2} < N < \frac{1}{Le}$.

Using pde2path, we plot bifurcation diagrams in Figure 4.3 and Figure 4.4 for fixed values of Pr, Le and for different values of N to show the supercriticality or the subcriticality of the primary pitchfork bifurcation, respectively. We have rescaled the configuration of the problem such that the fluid layer is bounded between $z = -0.5$ and $z = 0.5$, and we define free-slip horizontal boundary conditions at $x = \pm\frac{2\pi}{a_c}$ so that we can emulate a fluid in an infinite layer.

4.4 Weakly Nonlinear Analysis of Doubly-Diffusive Convection: A Horizontal Layer Heated and Salted from Above

We will now derive weakly nonlinear equations for a fluid in a horizontal layer that is heated and salted from above, by following the same methods.

Recall that in Section 3.4 we found that in this configuration, instability arises through a stationary bifurcation point, which was given by

$$Ra_T^{(S)} = \frac{27\pi^4}{4(LeN - 1)} \tag{4.4.1}$$

We look for weakly nonlinear solutions to this problem by defining a small parameter that represents a small distance of Ra_T to the critical value $Ra_T^{(S)}$.

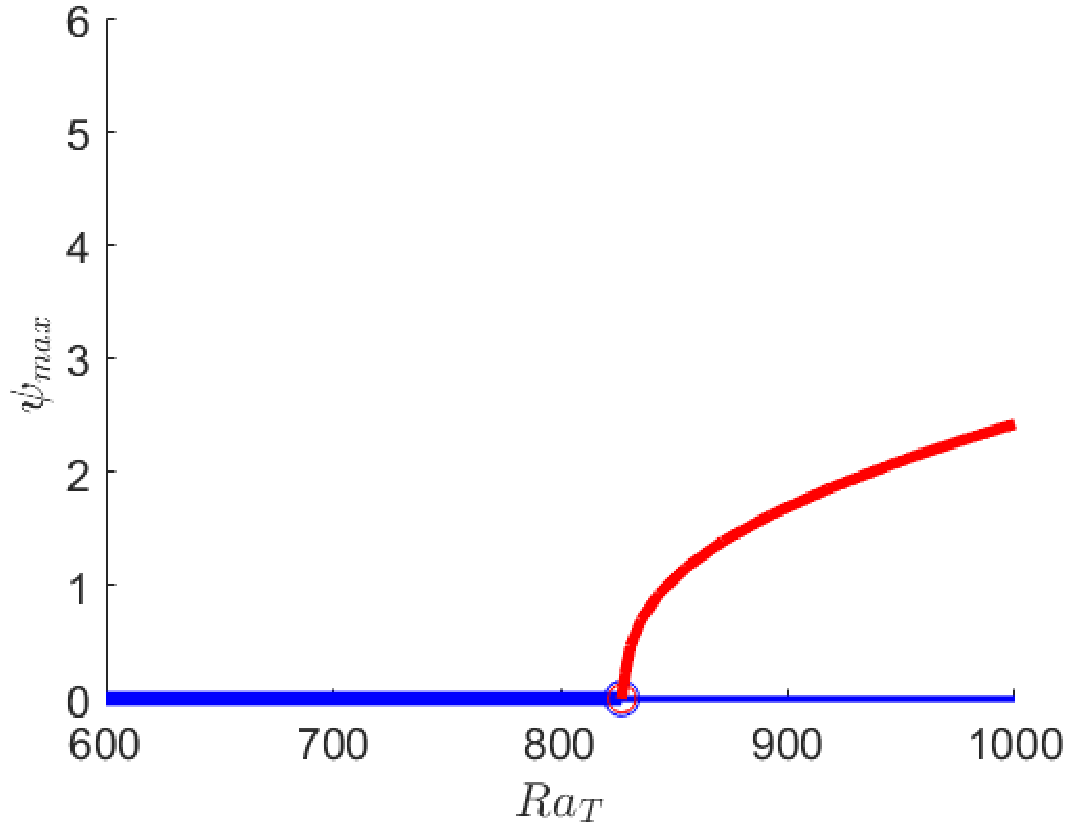

Figure 4.3: Bifurcation diagram showing the primary supercritical pitchfork bifurcation in a horizontal fluid heated and salted from below for $Pr = 1$, $Le = 2$ and $N = 0.1$. Thicker lines represent stable solutions and thinner lines represent unstable solutions. The circle represents the bifurcation point $Ra_T^{(S)}$.

Recall that we may write the governing equations of the problem in terms of the stream-function and the vorticity to obtain

$$\frac{\partial \omega}{\partial t} + J(\psi, \omega) = -Ra_T Pr \frac{\partial}{\partial x}(T - NC) + Pr \nabla^2 \omega, \tag{4.4.2a}$$

$$\frac{\partial T}{\partial t} + J(\psi, T) = -\frac{\partial \psi}{\partial x} + \nabla^2 T, \tag{4.4.2b}$$

$$\frac{\partial C}{\partial t} + J(\psi, C) = -\frac{\partial \psi}{\partial x} + \frac{1}{Le}\nabla^2 C, \tag{4.4.2c}$$

where $J(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}$. We reiterate that these equations are very similar to the governing equations of the other configuration of doubly-diffusive convection that we are studying—the only difference is a sign change in the $\frac{\partial \psi}{\partial x}$ terms.

Now, let ϵ be a small parameter defined by the relationship

$$\epsilon^2 r = Ra_T - Ra_T^{(S)}, \tag{4.4.3}$$

where $Ra_T^{(S)}$ is the critical value at which the first mode of the normal mode solution becomes unstable. Then let us also define the variable τ as

$$\tau = \epsilon^2 t. \tag{4.4.4}$$

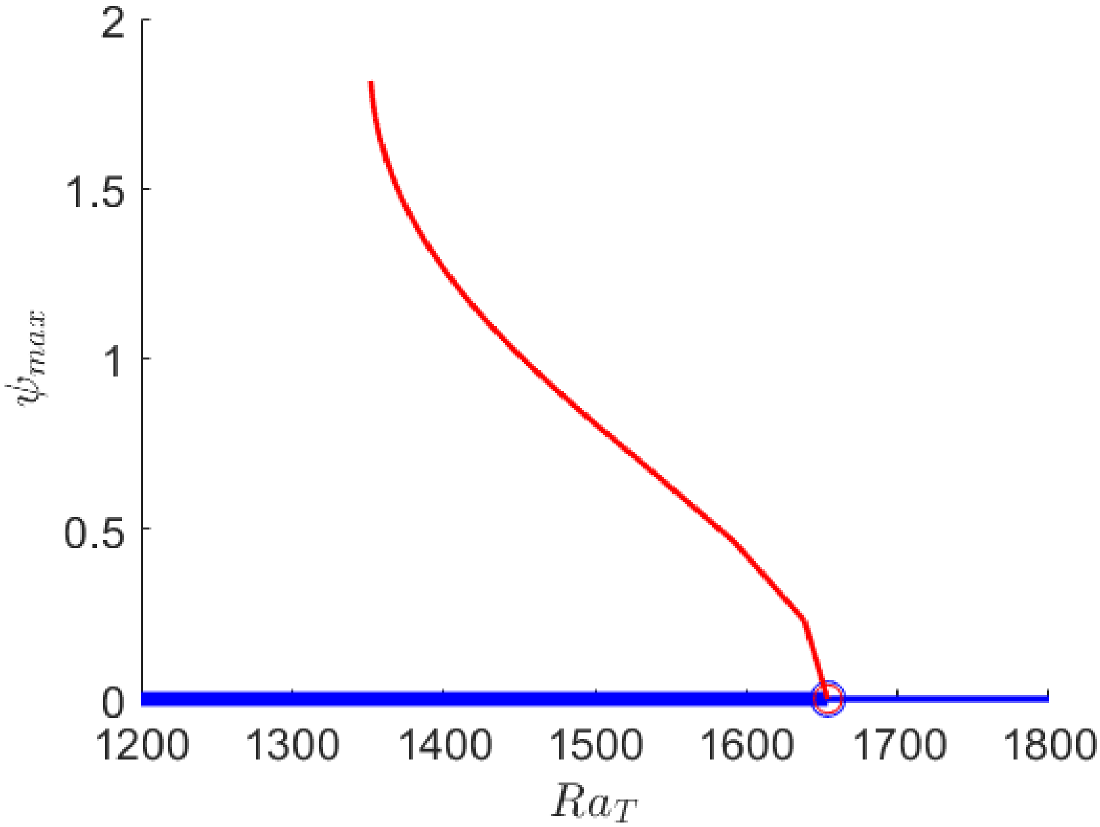

Figure 4.4: Bifurcation diagram showing the primary subcritical pitchfork bifurcation in a horizontal fluid heated and salted from below for $Pr = 1$, $Le = 2$ and $N = 0.3$. Thicker lines represent stable solutions and thinner lines represent unstable solutions. The circle represents the bifurcation point $Ra_T^{(S)}$.

We postulate that the weakly nonlinear solution is of the form

$$\psi(x,z,\tau) = \epsilon^{\alpha}\psi_1(x,z,\tau) + \epsilon^{2\alpha}\psi_2(x,z,\tau) + \dots \tag{4.4.5a}$$

$$\omega(x,z,\tau) = \epsilon^{\alpha}\omega_1(x,z,\tau) + \epsilon^{2\alpha}\omega_2(x,z,\tau) + \dots \tag{4.4.5b}$$

$$T(x,z,\tau) = \epsilon^{\alpha}T_1(x,z,\tau) + \epsilon^{2\alpha}T_2(x,z,\tau) + \dots, \tag{4.4.5c}$$

$$C(x,z,\tau) = \epsilon^{\alpha}C_1(x,z,\tau) + \epsilon^{2\alpha}C_2(x,z,\tau) + \dots. \tag{4.4.5d}$$

Substituting the above equations into equations (4.4.2a), (4.4.2b) and (4.4.2c) and dividing by ϵ^{α}, we obtain the following equations:

$$\begin{aligned}\left(\epsilon^2\frac{\partial\omega_1}{\partial\tau} + \epsilon^{\alpha+2}\frac{\partial\omega_2}{\partial\tau} + \dots\right) + \epsilon^{\alpha}J(\psi_1,\omega_1) + \epsilon^{2\alpha}J(\psi_1,\omega_2) + \epsilon^{2\alpha}J(\psi_2,\omega_1) + \dots \\ = -(Ra_T^{(S)} + \epsilon^2 r)Pr\left(\frac{\partial T_1}{\partial x} + \epsilon^{\alpha}\frac{\partial T_2}{\partial x} + \dots\right) + (Ra_T^{(S)} + \epsilon^2 r)PrN\left(\frac{\partial C_1}{\partial x} + \epsilon^{\alpha}\frac{\partial C_2}{\partial x} + \dots\right) \\ + Pr\nabla^2(\omega_1 + \epsilon^{\alpha}\omega_2 + \dots),\end{aligned} \tag{4.4.6}$$

$$\left(\epsilon^2\frac{\partial T_1}{\partial\tau}+\epsilon^{\alpha+2}\frac{\partial T_2}{\partial\tau}+\ldots\right)+\epsilon^{\alpha}J(\psi_1,T_1)+\epsilon^{2\alpha}J(\psi_1,T_2)+\epsilon^{2\alpha}J(\psi_2,T_1)$$
$$=-\left(\frac{\partial\psi_1}{\partial x}+\epsilon^{\alpha}\frac{\partial\psi_2}{\partial x}+\ldots\right)+\nabla^2(T_1+\epsilon^{\alpha}T_2+\ldots), \quad (4.4.7)$$

$$\left(\epsilon^2\frac{\partial C_1}{\partial\tau}+\epsilon^{\alpha+2}\frac{\partial C_2}{\partial\tau}+\ldots\right)+\epsilon^{\alpha}J(\psi_1,C_1)+\epsilon^{2\alpha}J(\psi_1,C_2)+\epsilon^{2\alpha}J(\psi_2,C_1)$$
$$=-\left(\frac{\partial\psi_1}{\partial x}+\epsilon^{\alpha}\frac{\partial\psi_2}{\partial x}+\ldots\right)+\frac{1}{Le}\nabla^2(C_1+\epsilon^{\alpha}C_2+\ldots). \quad (4.4.8)$$

At zeroeth order in ϵ, or $\mathcal{O}(1)$, we have

$$0=-Ra_T^{(S)}Pr\frac{\partial T_1}{\partial x}+Ra_T^{(S)}\frac{\partial T_1}{\partial x}-Pr\nabla^4\psi_1, \quad (4.4.9a)$$
$$0=-\frac{\partial\psi_1}{\partial x}+\nabla^2 T_1, \quad (4.4.9b)$$
$$0=-\frac{\partial\psi_1}{\partial x}+\frac{1}{Le}\nabla^2 C_1. \quad (4.4.9c)$$

We can clearly see that a trivial solution to these equations is given by

$$\psi_1=T_1=C_1=0, \quad (4.4.10)$$

however, from linear theory, we know that there is a non-trivial solution at the point $Ra = Ra_T^{(S)}$. If we define $\Psi_1 \equiv (\psi_1, T_1, C_1)^T$, then we can write equations (4.4.9a), (4.4.9b) and (4.4.9c) as

$$\mathcal{L}\Psi_1=0, \quad (4.4.11)$$

where $\mathcal{L}$ is a 3×3 matrix operator given by

$$\mathcal{L}=\begin{pmatrix}-Pr\nabla^4 & -Ra_T^{(S)}Pr\frac{\partial}{\partial x} & Ra_T^{(S)}PrN\frac{\partial}{\partial x}\\ -\frac{\partial}{\partial x} & \nabla^2 & 0\\ -\frac{\partial}{\partial x} & 0 & \frac{1}{Le}\nabla^2\end{pmatrix}. \quad (4.4.12)$$

It can be shown by successive integration by parts that $\mathcal{L}$ is self-adjoint with respect to the inner product

$$\langle\Psi_m,\Psi_n\rangle=\iint(\psi_m\psi_n+Ra_T^{(S)}PrT_mT_n-Ra_T^{(S)}PrC_mC_n)\,dzdx, \quad (4.4.13)$$

when we assume periodicity in the x–direction, with period $\frac{2\pi}{a_c}$.

Recall now that in linear theory we wanted to solve for

$$w=f(x)\bar{w}e^{\lambda t} \quad (4.4.14)$$

where λ is a complex eigenvalue. However, we deduced that for this problem, λ was real, so we may also take λ real here. We also deduced that $\bar{w}_n$ was of the form $\sin(n\pi z)$, hence, we choose

$$\psi_1(x,z,\tau)=\hat{\psi}\sin(\pi z)cos(a_cx)A(\tau), \quad (4.4.15)$$

where we again arbitrarily choose the cosine mode in the x-direction. Then we have

$$\omega_1 = (a_c^2 + \pi^2)\hat{\psi}\sin(\pi z)cos(a_c x)A(\tau), \tag{4.4.16a}$$

$$T_1 = \frac{a_c\hat{\psi}\sin(\pi z)\sin(a_c x)A(\tau)}{a_c^2 + \pi^2} \tag{4.4.16b}$$

$$C_1 = \frac{Lea_c\hat{\psi}\sin(\pi z)\sin(a_c x)A(\tau)}{a_c^2 + \pi^2}. \tag{4.4.16c}$$

In order for the nonlinear terms and the $\epsilon^2 rPr(\frac{\partial T_1}{\partial x} - N\frac{\partial C_1}{\partial x})$ term of equation (4.3.8) to be of the same order, then we choose $\alpha = 1$.

Therefore, we now rewrite equations (4.4.6), (4.4.7) and (4.4.8) as

$$\begin{aligned}\left(\epsilon^2\frac{\partial\omega_1}{\partial\tau} + \epsilon^3\frac{\partial\omega_2}{\partial\tau} + \ldots\right) + \epsilon J(\psi_1,\omega_1) + \epsilon^2 J(\psi_1,\omega_2) + \epsilon^2 J(\psi_2,\omega_1) + \ldots \\ = -(Ra_T^{(S)} + \epsilon^2 r)Pr\left(\frac{\partial T_1}{\partial x} + \epsilon\frac{\partial T_2}{\partial x} + \ldots\right) + (Ra_T^{(S)} + \epsilon^2 r)PrN\left(\frac{\partial C_1}{\partial x} + \epsilon\frac{\partial C_2}{\partial x} + \ldots\right) \\ + Pr\nabla^2(\omega_1 + \epsilon\omega_2 + \ldots),\end{aligned} \tag{4.4.17}$$

$$\begin{aligned}\left(\epsilon^2\frac{\partial T_1}{\partial\tau} + \epsilon^3\frac{\partial T_2}{\partial\tau} + \ldots\right) + \epsilon J(\psi_1,T_1) + \epsilon^2 J(\psi_1,T_2) + \epsilon^2 J(\psi_2,T_1) \\ = -\left(\frac{\partial\psi_1}{\partial x} + \epsilon\frac{\partial\psi_2}{\partial x} + \ldots\right) + \nabla^2(T_1 + \epsilon T_2 + \ldots),\end{aligned} \tag{4.4.18}$$

$$\begin{aligned}\left(\epsilon^2\frac{\partial C_1}{\partial\tau} + \epsilon^3\frac{\partial C_2}{\partial\tau} + \ldots\right) + \epsilon J(\psi_1,C_1) + \epsilon^2 J(\psi_1,C_2) + \epsilon^2 J(\psi_2,C_1) \\ = -\left(\frac{\partial\psi_1}{\partial x} + \epsilon\frac{\partial\psi_2}{\partial x} + \ldots\right) + \frac{1}{Le}\nabla^2(C_1 + \epsilon C_2 + \ldots).\end{aligned} \tag{4.4.19}$$

Equating coefficients of $\mathcal{O}(\epsilon)$, we have

$$J(\psi_1,\omega_1) = -Ra_T^{(S)}Pr\frac{\partial T_2}{\partial x} + Ra_T^{(S)}PrN\frac{\partial C_2}{\partial x} - Pr\nabla^4\psi_2, \tag{4.4.20a}$$

$$J(\psi_1,T_1) = -\frac{\partial\psi_2}{\partial x} + \nabla^2 T_2, \tag{4.4.20b}$$

$$J(\psi_1,C_1) = -\frac{\partial\psi_2}{\partial x} + \frac{1}{Le}\nabla^2 C_2. \tag{4.4.20c}$$

However, since $J(\psi_1,\omega_1) = -J(\psi_1,\nabla^2\psi_1) = -(a_c^2+\pi^2)J(\psi_1,\psi_1) = 0$, we write this as

$$\mathcal{L}\Psi_2 = \begin{pmatrix} -Pr\nabla^4 & -Ra_T^{(S)}Pr\frac{\partial}{\partial x} & Ra_T^{(S)}PrN\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x} & \nabla^2 & 0 \\ -\frac{\partial}{\partial x} & 0 & \frac{1}{Le}\nabla^2 \end{pmatrix}\begin{pmatrix}\psi_2 \\ T_2 \\ C_2\end{pmatrix} = \begin{pmatrix} 0 \\ J(\psi_1,T_1) \\ J(\psi_1,C_1)\end{pmatrix}, \tag{4.4.21}$$

where $\Psi_2 \equiv (\psi_T, T_2, C_2)^T$. Since $\mathcal{L}$ is self-adjoint and we know that the homogenous problem $\mathcal{L}\Psi_1 = 0$ has a non-trivial solution, then Fredholm's alternative tells us that equation (4.4.21) has a solution if

$$\left\langle \begin{pmatrix}\psi_1 \\ T_1 \\ C_1\end{pmatrix}, \begin{pmatrix} 0 \\ J(\psi_1,T_1) \\ J(\psi_1,C_1)\end{pmatrix}\right\rangle = 0, \tag{4.4.22}$$

is satisfied. The above inner product is equivalent to

$$Ra_T^{(S)}Pr\iint T_1J(\psi_1,T_1)-NC_1J(\psi_1,C_1)\,dzdx=0, \tag{4.4.23}$$

which we can show is always satisfied since $J(\psi_1,T_1)$ and $J(\psi_1,C_1)$ depend only on z and τ, whilst T_1 and C_1 are proportional to $\sin(a_cx)$. Therefore, we expect a solution to equations (4.4.20a), (4.4.20b) and (4.4.20c).

Observe that left hand side of equations (4.4.20a), (4.4.20b) and (4.4.20c) are independent of x. Therefore, we may reasonably expect the solution to the equations to also be independent of x. Letting $\psi_2=\psi(z,\tau)$, $T_2=T_2(z,\tau)$ and $C_2=C_2(z,\tau)$, we have

$$-Pr\frac{\partial^4\psi_2}{\partial z^4}=0 \tag{4.4.24a}$$

$$\frac{\partial^2T_2}{\partial z^2}=J(\psi_1,T_1) \tag{4.4.24b}$$

$$\frac{\partial^2C_2}{\partial z^2}=LeJ(\psi_1,C_1), \tag{4.4.24c}$$

which has solutions

$$\psi_2=0, \tag{4.4.25a}$$

$$T_2=\frac{a_c^2\hat{\psi}^2A^2}{8\pi(a_c^2+\pi^2)}\sin(2\pi z), \tag{4.4.25b}$$

$$C_2=\frac{Le^2a_c^2\hat{\psi}^2A^2}{8\pi(a_c^2+\pi^2)}\sin(2\pi z). \tag{4.4.25c}$$

Equating coefficients of $\mathcal{O}(\epsilon^2)$, we have

$$\frac{\partial\omega_1}{\partial\tau}+rPr\frac{\partial T_1}{\partial x}-rPrN\frac{\partial C_1}{\partial x}=-Ra_T^{(S)}Pr\frac{\partial T_3}{\partial x}+Ra_T^{(S)}PrN\frac{\partial C_3}{\partial x}-Pr\nabla^4\omega_3, \tag{4.4.26a}$$

$$\frac{\partial T_1}{\partial\tau}+J(\psi_1,T_2)=-\frac{\partial\psi_3}{\partial x}+\nabla^2T_3, \tag{4.4.26b}$$

$$\frac{\partial C_1}{\partial\tau}+J(\psi_1,C_2)=-\frac{\partial\psi_3}{\partial x}+\frac{1}{Le}\nabla^2C_3, \tag{4.4.26c}$$

where we have used the fact that $J(\psi_1,\omega_2)=J(\psi_2,\omega_1)=J(\psi_2,T_1)=J(\psi_2,C_1)=0$. Therefore, by similar reasoning as before, the solvability condition is

$$\langle\Psi_3,N_3\rangle=0, \tag{4.4.27}$$

where $\Psi_3=(\psi_3,T_3,C_3)^T$ and

$$N_3=\begin{pmatrix}\frac{\partial\omega_1}{\partial\tau}+rPr\frac{\partial T_1}{\partial x}-rPrN\frac{\partial C_1}{\partial x}\\ \frac{\partial T_1}{\partial\tau}+J(\psi_1,T_2)\\ \frac{\partial C_1}{\partial\tau}+J(\psi_1,C_2)\end{pmatrix}. \tag{4.4.28}$$

Solving equation (4.4.27) gives us the Landau equation

$$\begin{aligned}(a_c^2+\pi^2)+(NLe^2-1)\frac{Ra_T^{(S)}Pra_c^2}{(\pi^2+a_c^2)^2}\frac{\partial A}{\partial\tau}\\ -(LeN-1)\frac{rPra_c^2}{\pi^2+a_c^2}A+(NLe^2-1)\frac{Ra_T^{(S)}Pra_c^4\hat{\psi}^2}{8(\pi^2+a_c^2)}A^3=0.\end{aligned} \tag{4.4.29}$$

Notice the similarities between equation (4.3.31) and (4.4.29). This is expected because notice that the configurations of both of these problems are essentially the same—they are reflections of each other across the horizontal axis. This is also evident in the forms of the stationary instability: for a fluid heated and salted from below, this is given by

$$Ra_T^{(S)} = \frac{27\pi^4}{4(1 - LeN)}, \tag{4.4.30}$$

whilst for a fluid heated and salted from above, this is given by

$$Ra_T^{(S)} = \frac{27\pi^4}{4(LeN - 1)}. \tag{4.4.31}$$

If we are in the steady state, we see that in order for the primary bifurcation to be subcritical, we require that $\frac{1}{Le^2} < N\frac{1}{Le}$. However, this means that the bifurcation occurs at negative values of Ra_T, which is not physically possible. Nevertheless, using pde2path, we do plot a bifurcation diagram of a subcritical bifurcation in Figure 4.6, alongside a diagram showing a supercritical bifurcation in Figure 4.5 for fixed values of Pr and Le. We have also rescaled the configuration of the problem such that the fluid layer is bounded between $z = -0.5$ and $z = 0.5$, and define free-slip horizontal boundary conditions at $x = \pm\frac{2\pi}{a_c}$ so that we can have periodic solutions.

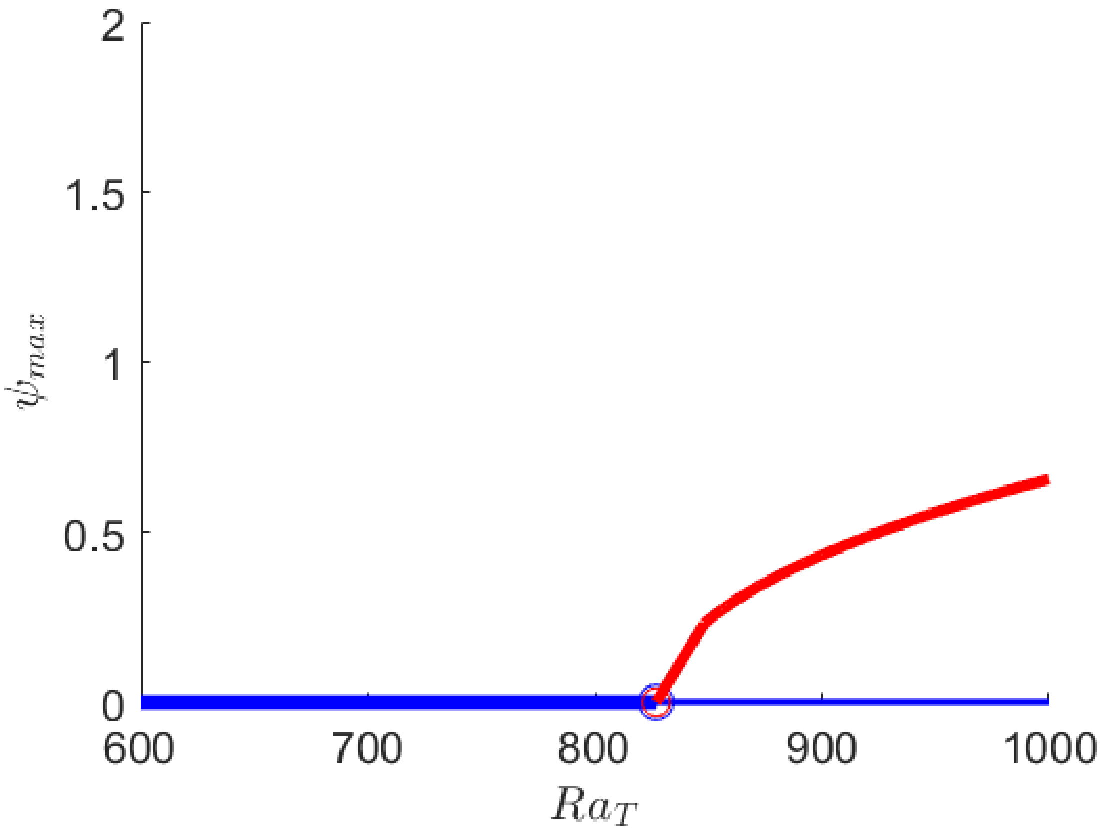

Figure 4.5: Bifurcation diagram showing the primary supercritical pitchfork bifurcation in a horizontal fluid heated and salted from above for $Pr = 1$, $Le = 2$ and $N = 0.9$. Thicker lines represent stable solutions and thinner lines represent unstable solutions. The circle represents the bifurcation point $Ra_T^{(S)}$.

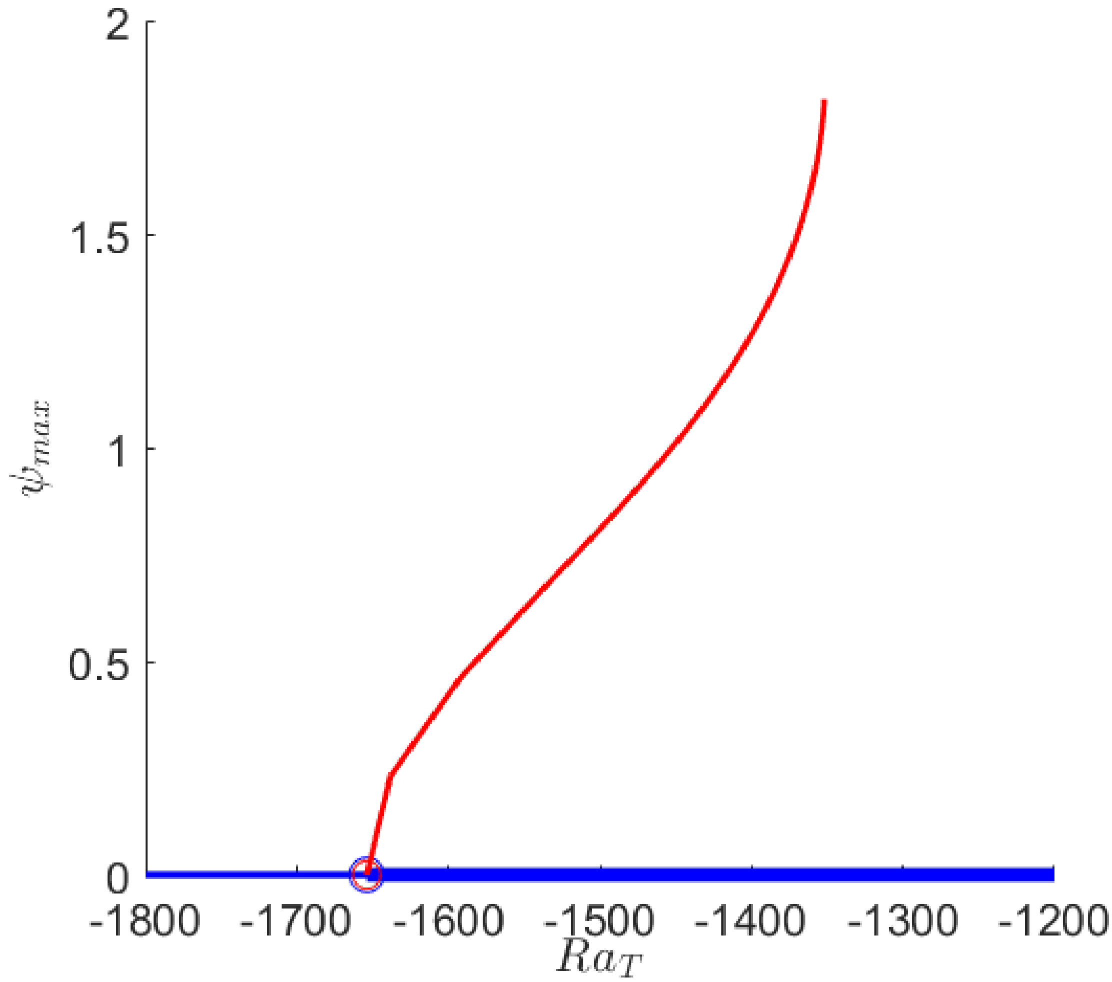

Figure 4.6: Bifurcation diagram showing the primary subcritical pitchfork bifurcation in a horizontal fluid heated and salted from above for $Pr = 1$, $Le = 2$ and $N = 0.3$. Thicker lines represent stable solutions and thinner lines represent unstable solutions. The circle represents the bifurcation point $Ra_T^{(S)}$.

Chapter 5

Conclusion

5.1 Discussion

In this report, we analysed the behaviour of Rayleigh–Bénard convection and doubly-diffusive convection in infinite horizontal layers of fluid.

In Chapter 2, we formulated the equations governing Rayleigh–Bénard convection and doubly-diffusive convection from physical laws governing the motion of fluids, and then expressed them in non-dimensional quantities. We therefore saw that the dynamics of Rayleigh–Bénard convection is controlled by non-dimensional parameters Pr and Ra. Similarly, the dynamics of doubly-diffusive convection is controlled by the non-dimensional parameters Pr, Le and N. The governing equations were then expressed in two different ways: in terms of the fluid velocity $\mathbf{u}$ and in terms of the streamfunction ψ, and the vorticity ω. The distinction between the two formulations was stressed because they were be used independently in the subsequent chapters.

In Chapter 3, we performed linear stability calculations to find the critical values at which instability arises in the Rayleigh–Bénard and the doubly-diffusive problems. This involved subjecting the base state of the fluid to small infinitesimal perturbations, and then neglecting the nonlinear perturbations since these are even smaller than the linear perturbations.

We found that Rayleigh–Bénard convection emerges through a stationary instability at a critical wavenumber a_c and critical Rayleigh number Ra_c that do not depend on any other parameters such as Pr. For an infinite horizontal layer, the $n = 1$ mode in our normal mode expansion corresponded to the most unstable mode. For $Ra > Ra_c$, linear theory predicts that the solutions will have growing modes, corresponding to the instability or convection cells that can be observed experimentally and in natural phenomenon (Bénard and Avsec, 1938; Mishra et al., 1999).

Doubly-diffusive convection arises at different critical values depending on the configuration of the problem. For an infinite horizontal fluid layer that is heated and salted from below, instability arises through a stationary instability for N smaler than the criticalv value $N^{(OS)}$, and arises through an oscillatory instability, or Hopf bifurcation, for N larger than the critical value $N^{(OS)}$ that depends on Le and Pr. For a horizontal fluid layer that is heated and salted from above, instability arises through a stationary instability, because we restrict our parameters to be positive valued. The critical value $Ra_T^{(S)}$ at which this occurs depends on Le and N. These calculations are in agreement with the observed phenomena that motivated this study. Growing oscillations can be seen in fluids that are heated and salted from below (Garaud, 2018; Huppert and Moore, 1976) when they exceed a certain threshold. Similarly, salt fingers is a well-known oceanographical phenomena that occurs at the critical values that we calculated Stern (1960).

While linear stability analyses are useful in determining when instabilities will occur, or at what parameter values they occur, they do not offer us any more information than this. In order to know how the fluid behaves, we needed to consider the nonlinear terms and how they contribute to the system. Therefore we turned to weakly nonlinear theory in Chapter 4.

In Chapter 4, we outlined the underlying principles of weakly nonlinear theory by first explaining Fredholm's alternative and the solvability condition, and then using the idea to solve a 1D toy problem. Using this as a foundation, weakly nonlinear solutions were found for the Rayleigh–Bénard problem and the doubly-diffusive problem. This involved expanding the solutions in terms of a small parameter ϵ, and then successively finding solutions using the solvability condition for increasingly smaller terms. This then allowed us to write Landau equations, which we in turn used to find the form of the primary bifurcations of each problem and the parameter values for which they were subcritical or supercritical. We supplemented this knowledge with bifurcation diagrams generated using the continuation and bifurcation package for MATLAB called pde2path. Note that due to computational constraints, these bifurcation diagrams serve more of a qualitative purpose in this analysis. For better results, a less coarse mesh grid should be used.

We reiterate the fact that we greatly simplified the physical problem in order to perform these analytic calculations. When formulating the problem, we stated that many of the assumptions that we made, such as restricting fluid motion to only two directions and assuming that we have an infinite layer in the horizontal direction with free-slip boundary conditions. We should also note that linear and weakly nonlinear analyses are theoretically only valid in the small regime around the critical value we study. Despite this, we find that our calculations, explain the physical phenomena well, as evidenced by experiments done by Bénard and Avsec (1938); Huppert and Moore (1976); Stern (1960); Veronis (1968), and also help our understanding of the full problems.

5.2 Further Work

Doubly-diffusive convection, in recent years, has garnered a lot of attention. We have laid a foundation for natural extensions of the problem, such as studying the fluid in an inclined cavity, not parallel to the horizontal axis. We chose to study the problem posed in a horizontal layer because this is the physical configuration that motivated the study, and was a natural extension of the simpler Rayleigh–Bénard problem. In recent publications (Gobin and Bennacer, 1994; Xin et al., 1998; Beaume et al., 2013), interesting phenomena such as localised states have been found in vertical fluid layers, that is when a fluid is bounded between vertical plates. Extending the analysis done here to the vertical case and other configurations is worthwhile doing.

Kidachi (1982) and Hirschberg and Knobloch (1997) investigated the effects of side walls in Rayleigh–Bénard convection. While we can theoretically investigate how a fluid will behave in an infinite layer by assuming that the solutions are periodic, in reality, infinite layers do not exist. Even fluid that is bounded between a very long layer (that is with large aspect ratio) will be affected by distant sidewalls due to symmetry breaking. Modifying the governing equations to take into account sidewalls in 3D would present a much more realistic picture.

As most of the work presented in this report was analytic, more numerical simulations would provide a deeper understanding of the dynamics of these systems. This, in conjunction with performing weakly nonlinear analyses to higher orders ($\mathcal{O}(\epsilon^3)$ and so on) will then allow us to be able to investigate beyond the primary bifurcations that we studied here. This might then lead to a full characterisation of the dynamics of the system for the parameters Ra and a in Rayleigh–Bénard convection, and Ra_T, Pr, Le and N in doubly-diffusive convection.

Appendix A

Vector Theorems and Identities

A.1 Vector Calculus Theorems

- The mathematical statement of the divergence theorem is given by,

$$\iiint_V (\nabla \cdot \mathbf{F}) dV = \iint_S (\mathbf{F} \cdot \hat{\mathbf{n}}) dS, \tag{A.1.1}$$

  where V is a volume in $\mathbb{R}^3$, with surface S and outward normal $\hat{\mathbf{n}}$ and $\mathbf{F}$ is a continuously differentiable vector field.

- The mathematical statement of Green's theorem is given by,

$$\oint_C (Ldx + Mdy) = \iint_A \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) \tag{A.1.2}$$

  where C is a positively oriented, piecewise smooth, simple closed curve in a plane, A is the area bounded by C, and L and M are functions of (x, y) defined on an open region containing A with continuous partial derivatives, with anticlockwise path of integration along C.

- The mathematical statement of Stokes' theorem is given by

A.2 Vector Identities

For any vector fields $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ and for any scalar fields f and g, we have the following.

- Addition and Multiplication Identities

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{A.2.1}$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{A.2.2}$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{A.2.3}$$

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C} \tag{A.2.4}$$

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C} \tag{A.2.5}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{A.2.6}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{A.2.7}$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A} \tag{A.2.8}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} + \mathbf{B} \times (\mathbf{A} \times \mathbf{C}) \tag{A.2.9}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) = 0 \tag{A.2.10}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D}) \tag{A.2.11}$$

$$(\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}))\mathbf{D} = (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \times \mathbf{C}) + (\mathbf{B} \cdot \mathbf{D})(\mathbf{C} \times \mathbf{A}) + (\mathbf{C} \cdot \mathbf{D})(\mathbf{A} \times \mathbf{B}) \tag{A.2.12}$$

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot (\mathbf{B} \times \mathbf{D}))\mathbf{C} - (\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}))\mathbf{D} \tag{A.2.13}$$

- Gradient Identities

$$\nabla(f+g) = \nabla f + \nabla g \tag{A.2.14}$$

$$\nabla(fg) = g\nabla f + f\nabla g \tag{A.2.15}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \tag{A.2.16}$$

- Divergence Identities

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \tag{A.2.17}$$

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f \tag{A.2.18}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \tag{A.2.19}$$

- Curl Identities

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \tag{A.2.20}$$

$$\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + \nabla f \times \mathbf{A} \tag{A.2.21}$$

$$\nabla \times (f\nabla g) = \nabla f \times \nabla g \tag{A.2.22}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{A.2.23}$$

- Second Derivative Identities

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{A.2.24}$$

$$\nabla \times (\nabla f) = \mathbf{0} \tag{A.2.25}$$

$$\nabla \cdot (\nabla f) = \nabla^2 f \tag{A.2.26}$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) = \nabla^2 \mathbf{A} \tag{A.2.27}$$

$$\nabla \cdot (f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g \tag{A.2.28}$$

$$f\nabla^2 g - g\nabla^2 f = \nabla \cdot (f\nabla g - g\nabla f) \tag{A.2.29}$$

$$\nabla^2(fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f \tag{A.2.30}$$

$$\nabla^2(f\mathbf{A}) = \mathbf{A}\nabla^2 f + 2(\nabla f \cdot \nabla)\mathbf{A} + f\nabla^2 \mathbf{A} \tag{A.2.31}$$

$$\nabla^2(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \nabla^2 \mathbf{B} - \mathbf{B} \cdot \nabla^2 \mathbf{A} + 2\nabla \cdot ((\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{B} \times \nabla \times \mathbf{A}) \tag{A.2.32}$$

- Third Derivative Identities

$$\nabla^2(\nabla f) = \nabla(\nabla \cdot (\nabla f)) = \nabla(\nabla^2 f) \tag{A.2.33}$$

$$\nabla^2(\nabla \cdot \mathbf{A}) = \nabla \cdot (\nabla(\nabla \cdot \mathbf{A})) = \nabla \cdot (\nabla^2 \mathbf{A}) \tag{A.2.34}$$

$$\nabla^2(\nabla \times \mathbf{A}) = -\nabla \times (\nabla \times (\nabla \times \mathbf{A})) = \nabla \times (\nabla^2 \mathbf{A}) \tag{A.2.35}$$

Reference List

Beaume, C., Bergeon, A., and Knobloch, E. (2013). Convectons and secondary snaking in three-dimensional natural doubly diffusive convection. *Phys. of Fluids*, 25(2):024105.

Bénard, H. (1900). Les tourbillons cellulaires dans une nappe liquide. *Rev. Gén. Sci. Pures Appl.*, 11:1261–1271 and 1309–1328.

Bénard, H. and Avsec, D. (1938). Travaux récents sur les tourbillons cellulaires et les tourbillons en bandes. Applications à l'astrophysique et à la météorologie. *J. Phys. Radium*, 9(11):486–500.

Boussinesq, J. (1903). *Théorie analytique de la chaleur: mise en harmonie avec la thermodynamique et avec la théorie mécanique de la lumière*, volume 2. Gauthier-Villars.

Chandrasekhar, S. (1961). *Hydrodynamic and hydromagnetic stability.* Cover Publications Inc.

deWitt, H., Dohnal, T., Rademacher, J., Uecker, H., and Wetzel, D. (2017). pde2path - a MATLAB package for continuation and bifurcation in systems of pdes, v2.3. `http://www.staff.uni-oldenburg.de/hannes.uecker/pde2path/`.

Dohnal, T., Rademacher, J., Uecker, H., and Wetzel, D. (2014). pde2path 2.0: multi-parameter continuation and periodic domains. In Ecker, H., Steindl, A., and Jakubek, S., editors, *ENOC 2014 - Proceedings of 8th European Nonlinear Dynamics Conference.* Vienna University of Technology.

Drazin, P. G. and Reid, W. H. (2004). *Hydrodynamic Stability.* Cambridge Mathematical Library. Cambridge University Press, 2 edition.

Francis, M. (2011). Physics quanta: Pendulums revisited. `https://galileospendulum.org/2011/05/31/physics-quanta-pendulums-revisited/`.

Fredholm, I. (1903). Sur une classe d'équations fonctionnelles. *Acta Math.*, 27:365–390.

Garaud, P. (2018). Double-diffusive convection at low prandtl number. *Annu. Rev. Fluid Mech.*, 50(1):275–298.

Gobin, D. and Bennacer, R. (1994). Double diffusion in a vertical fluid layer: Onset of the convective regime. *Phys. Fluids*, 6(1):59–67.

Gradshteyn, I. S. and Ryzhik, I. M. (2000). *Table of integrals, series, and products, 6th ed.* Academic Press, 6 edition.

Haberman, R. (2004). *Applied Partial Differential Equations: With Fourier Series and Boundary Value Problems.* Pearson Prentice Hall.

Hirschberg, P. and Knobloch, E. (1997). Mode interactions in large aspect ratio convection. *J. Nonlinear Sci.*, 7(6):537–556.

Huppert, H. E. and Moore, D. R. (1976). Nonlinear double-diffusive convection. *J. Fluid Mech.*, 78(4):821–854.

Kidachi, H. (1982). Side wall effect on the pattern formation of the rayleigh-bénard convection. *Progr. Theor. Phys*, 68(1):49–63.

Koschmieder, E. (1993). *Bénard Cells and Taylor Vortices.* Cambridge Monographs on Mechanics. Cambridge University Press.

Lord Rayleigh (1916). On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side. *London, Edinburgh Dublin Philos. Mag. J. Sci.*, 32(192):529–546.

Matkowsky, B. J. (1970). A simple nonlinear dynamic stability problem. *Bull. Amer. Math. Soc.*, 76(3):620–625.

Mishra, D., Muralidhar, K., and Munshi, P. (1999). Experimental study of rayleighbenard convection at intermediate rayleigh numbers using interferometric tomography. *Fluid Dynamics Res.*, 25(5):231 – 255.

Ovchinnikov, I. V. and Enßlin, T. A. (2016). Kinematic dynamo, supersymmetry breaking, and chaos. *Phys. Rev. D*, 93:085023.

Oxburgh, E. R. and Turcotte, D. L. (1970). Thermal structure of island arcs. *GSA Bull.*, 81(6):1665.

Parker, E. (1955). Hydromagnetic Dynamo Models. *Astrophys. J.*, 122:293.

Plait, P. (2009). How far away is the horizon? `http://blogs.discovermagazine.com/badastronomy/2009/01/15/how-far-away-is-the-horizon/`.

Rahmstorf, S. (2003). Thermohaline circulation: The current climate. *Nature*, 421:699.

Reynolds, O., Brightmore, A. W., and Moorby, W. H. (1903). *Papers on Mechanical and Physical Subjects: The sub-mechanics of the universe*, volume 3. The University Press.

Sani, R. (1965). On finite amplitude roll cell disturbances in a fluid subjected to heat and mass transfer. *AIChE J.*, 11:971 – 980.

Saunders, O. A., Fishenden, M., and Mansion, H. D. (1935). Some measurements of convection by an optical method. *Engineering*, 139:483–485.

Schmidt, R. J. and Milverton, S. W. (1935). On the instability of a fluid when heated from below. *Proc. Royal Soc. A*, 152(877):586–594.

Schmidt, R. J. and Saunders, O. A. (1938). On the motion of a fluid heated from below. *Proc. Royal Soc. A*, 165(921):216–228.

Spiegel, E. and Veronis, G. (1960). On the boussinesq approximation for a compressible fluid. *Astrophys. J.*, 131:442.

Stern, M. E. (1960). The “salt-fountain” and thermohaline convection. *Tellus*, 12(2):172–175.

Stommel, H. and Arons, A. (1959a). On the abyssal circulation of the world ocean—I. stationary planetary flow patterns on a sphere. *Deep Sea Res.*, 6:140–154.

Stommel, H. and Arons, A. (1959b). On the abyssal circulation of the world ocean—II. an idealized model of the circulation pattern and amplitude in oceanic basins. *Deep Sea Res.*, 6:217–233.

Stommel, H., Arons, A. B., and Blanchard, D. (1956). An oceanographical curiosity: the perpetual salt fountain. *Deep Sea Res.*, 3(2):152–153.

Thorpe, S. A., Hutt, P. K., and Soulsby, R. (1969). The effect of horizontal gradients on thermohaline convection. *J. Fluid Mech.*, 38(2):375400.

Uecker, H., Wetzel, D., and Rademacher, J. (2014). pde2path a matlab package for continuation and bifurcation in 2D elliptic systems. *Numer. Math. Theor. Meth. Appl.*, 7:58–106.

Veronis, G. (1965). On finite amplitude instability in thermohaline convection. *J. Mar. Res*, 23(1):1–17.

Veronis, G. (1968). Effect of a stabilizing gradient of solute on thermal convection. *J. Fluid Mech.*, 34(2):315–336.

Weertman, J. (1978). Creep laws for the mantle of the earth. *Philos. Trans. Royal Soc. A*, 288(1350):9–26.

Wilson, J. T. (1963). Evidence from islands on the spreading of ocean floors. *Nature*, 197:536–538.

Xin, S., Quéré, P. L., and Tuckerman, L. S. (1998). Bifurcation analysis of double-diffusive convection with opposing horizontal thermal and solutal gradients. *Phys. Fluids*, 10(4):850–858.

Yang, Y., Verzicco, R., and Lohse, D. (2016). Vertically bounded double diffusive convection in the finger regime: Comparing no-slip versus free-slip boundary conditions. *Phys. Rev. Lett.*, 117:184501.