Supplementary Material for Convectons in a Rotating Fluid Layer
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Appendix B. Derivation of the fifth order Ginzburg-Landau equation

This file provides the pertinent details of the derivation of the nonlocal fifth order Ginzburg-Landau equation

$$\mu A + A_{XX} + i\left(\gamma A_X + a_1|A|^2 A_X + a_2 A^2 A_X^*\right) + b|A|^2 A - |A|^4 A = 0, \tag{1}$$

and of the coefficients

$$\begin{aligned} \mu &= \mu_0 + \mu_1 \left\langle |A|^2 \right\rangle + \mu_2 \left\langle |A|^4 \right\rangle + \mu_3 \mathrm{Im}\left[\langle A A_X^* \rangle\right] + \mu_4 \left\langle |A|^2 \right\rangle^2, \\ \gamma &= \gamma_0 + \gamma_1 \left\langle |A|^2 \right\rangle, \quad b = b_0 + b_1 \left\langle |A|^2 \right\rangle \end{aligned} \tag{2}$$

used in the accompanying paper Convectons in a Rotating Fluid Layer by C. Beaume, A. Bergeon, H.-C. Kao and E. Knobloch, published in the Journal of Fluid Mechanics.

We start with the equations of motion (Veronis 1959)

$$Ra\theta_x - Tv_z + \nabla^4\psi = \sigma^{-1}\left[\nabla^2\psi_t + J(\psi, \nabla^2\psi)\right], \tag{3}$$

$$\psi_x + \nabla^2\theta = \theta_t + J(\psi, \theta), \tag{4}$$

$$T\psi_z + \nabla^2 v = \sigma^{-1}\left[v_t + J(\psi, v)\right] \tag{5}$$

subject to the boundary conditions

$$\psi = \psi_{zz} = \theta = v_z = 0 \text{ at } z \in \{0, 1\}. \tag{6}$$

Equation (1) describes the above problem near the critical Rayleigh number Ra_c for the onset of convection and the critical Taylor number T_c determined by the degeneracy condition $\xi^2 = 1/3$ (Cox & Matthews 2001). Here $\xi \equiv \frac{T\pi^2}{\sqrt{3}pk^2\sigma}$, $p \equiv k^2 + \pi^2$ and the wavenumber k represents the critical wavenumber corresponding to Ra_c. Consequently, we write $Ra = Ra_c + \epsilon^2 r_2 + \epsilon^4 r_4$ and $T = T_c + \epsilon^2\delta$, where $\epsilon \ll 1$. We also introduce a slow spatial scale $X \equiv \epsilon^2 x$ and write

$$\psi \sim \sum_{n=1}^{\infty} \epsilon^n \psi_n(x, X, z), \quad \theta \sim \sum_{n=1}^{\infty} \epsilon^n \theta_n(x, X, z), \quad v \sim v_0(X) + \sum_{n=1}^{\infty} \epsilon^n v_n(x, X, z). \tag{7}$$

The leading order term in the zonal velocity is now of order one instead of being of order ϵ (Cox & Matthews 2001). To simplify the expressions that follow we use the notation

$$\begin{aligned} &\nabla^2 \equiv \partial_{xx} + \partial_{zz}, \qquad J(u, w) \equiv u_x w_z - u_z w_x, \qquad \tilde{J}(u, w) \equiv u_X w_z - u_z w_X, \\ &p_{nm} \equiv -\left(n^2k^2 + m^2\pi^2\right)^3 + Ra_c n^2 k^2 - T_c^2 m^2 \pi^2. \end{aligned}$$

Expansion (7) leads, order by order, to linear inhomogeneous problems of the form

$$\mathbf{M}\boldsymbol{\Psi}_n \equiv \begin{pmatrix} \nabla^4 & Ra_c\partial_x & -T_c\partial_z \\ \partial_x & \nabla^2 & 0 \\ T_c\partial_z & 0 & \nabla^2 \end{pmatrix} \boldsymbol{\Psi}_n = \mathbf{f}_n, \tag{8}$$

where $\boldsymbol{\Psi}_n$ represents $(\psi_n, \theta_n, v_n)^T$ and $\mathbf{f}_n$ is a vector with components that are polynomials in $\psi_1, ..., \psi_{n-1}, \theta_1, ..., \theta_{n-1}, v_0, ..., v_{n-1}$, and their derivatives. Equation (8) can also be written as a single equation with respect to ψ_n,

$$M\psi_n \equiv \left(\nabla^6 - Ra_c\partial_{xx} + T_c^2\partial_{zz}\right)\psi_n = \nabla^2 f_{n1} - Ra_c\partial_x f_{n2} + T_c\partial_z f_{n3}. \tag{9}$$

We solve this equation for ψ_n and determine the corresponding θ_n and v_n from Eq. (8).

First & second order

At $O(\epsilon)$ $\mathbf{f}_1 = 0$. The resulting homogeneous problem has a solution of the form

$$\begin{aligned}
\psi_1 &= \frac{a(X)}{2} e^{ikx} \sin(\pi z) + c.c., \\
\theta_1 &= \frac{ika(X)}{2p} e^{ikx} \sin(\pi z) + c.c., \\
v_1 &= \frac{\pi T_c a(X)}{2p} e^{ikx} \cos(\pi z) + c.c.,
\end{aligned}$$

where k and π are, respectively, the wavenumbers in the x and z directions. The critical value Ra_c and the critical wavenumber k satisfy

$$Ra_c = 3p^2, \qquad p^2 \left(2k^2 - \pi^2\right) = T^2 \pi^2. \tag{10}$$

Note that the linear operator M in Eq. (9) is self-adjoint with kernel $e^{\pm ikx} \sin(\pi z)$. This fact simplifies the solvability condition applied at each subsequent order.

At $O(\epsilon^2)$ $\mathbf{f}_2$ is given by

$$\begin{aligned}
f_{21} &= \sigma^{-1} J(\psi_1, \nabla^2 \psi_1), \\
f_{22} &= -J(\theta_1, \psi_1), \\
f_{23} &= -J(v_1, \psi_1).
\end{aligned}$$

The solvability condition for ψ_2 is always satisfied. Thus ψ_2 may be set to zero and $\mathbf{\Psi}_2$ is given by

$$\begin{aligned}
\psi_2 &= 0, \\
\theta_2 &= -\frac{k^2 |a|^2}{8\pi p} \sin(2\pi z), \\
v_2 &= v_{20}(X) + \frac{iT_c \pi^2 a^2}{16 k p \sigma} e^{2ikx} + c.c.
\end{aligned}$$

The homogeneous term $v_{20}(X)$ plays an important role in what follows.

Third order

At $O(\epsilon^3)$ $\mathbf{f}_3$ is given by

$$\begin{aligned}
f_{31} &= -4\nabla^2 \psi_{1,Xx} - Ra_c \theta_{1,X} + \delta v_{1,z} + \sigma^{-1} \left[J(\psi_2, \nabla^2 \psi_1) + J(\psi_1, \nabla^2 \psi_2)\right] - r_2 \theta_{1,x}, \\
f_{32} &= J\left(\psi_1, \theta_2\right) + J(\psi_2, \theta_1) - \psi_{1,X} - 2\theta_{1,Xx}, \\
f_{33} &= -\delta \psi_{1,z} - 2v_{1,Xx} + \sigma^{-1} \left[J\left(\psi_1, v_2\right) + J(\psi_2, v_1) - \psi_{1,z} v_{0,X}\right].
\end{aligned}$$

The solvability condition at this order gives

$$\left(\frac{k^2 r_2}{2} - T_c \pi^2 \delta\right) a - \frac{T_c \pi^2}{2\sigma} a v_{0,X} + \left(\frac{T_c^2 \pi^4}{16 p \sigma^2} - \frac{3pk^4}{16}\right) |a|^2 a = 0. \tag{11}$$

In traditional approaches which do not include spatial modulation the result (11) with $r_2 = \delta = 0$ reduces to $\xi^2 = 1$, i.e., the relation that determines the location of the codimension two point at which a subcritical periodic wavetrain becomes supercritical (Veronis 1959).

The solution $\mathbf{\Psi}_3$ is

$$\begin{aligned}
\psi_3 &= -\frac{T_c^2\pi^4 a^3}{16p\sigma^2 p_{31}}\, e^{3ikx}\sin(\pi z) - \frac{3pk^4|a|^2 a}{16p_{13}}\, e^{ikx}\sin(3\pi z) + c.c.,\\
\theta_3 &= \frac{ik^3(p_{13}-3p^2k^2)|a|^2 a}{16pp_{13}(k^2+9\pi^2)}\, e^{ikx}\sin(3\pi z) - \frac{3iT_c^2 k\pi^4 a^3}{16pp_{31}\sigma^2(9k^2+\pi^2)}\, e^{3ikx}\sin(\pi z)\\
&\quad + \frac{(\pi^2-k^2)a_X}{2p^2}\, e^{ikx}\sin(\pi z) - \frac{ik^3|a|^2 a}{16p^2}\, e^{ikx}\sin(\pi z) + c.c.,\\
v_3 &= \left[\frac{\pi\left(v_{0,X}+\delta\sigma\right)a}{2\sigma p} + \frac{iT_c k\pi a_X}{p^2} - \frac{T_c\pi^3|a|^2 a}{16p^2\sigma^2}\right] e^{ikx}\cos(\pi z)\\
&\quad - \frac{9pT_c k^4\pi|a|^2 a}{16p_{13}(k^2+9\pi^2)}\, e^{ikx}\cos(3\pi z) - \frac{T_c\pi^3 a^3\left(T_c^2\pi^2+p_{31}\right)}{16\sigma^2 pp_{31}(9k^2+\pi^2)}\, e^{3ikx}\cos(\pi z) + c.c.
\end{aligned}$$

Fourth order

At $O(\epsilon^4)$ $\mathbf{f}_4$ is given by

$$\begin{aligned}
f_{41} &= \sigma^{-1}\Big[2J\left(\psi_1,\psi_{1,Xx}\right) + \tilde{J}\left(\psi_1,\nabla^2\psi_1\right) + J\left(\psi_1,\nabla^2\psi_3\right) + J\left(\psi_3,\nabla^2\psi_1\right)\\
&\qquad + J(\psi_2,\nabla^2\psi_2)\Big] - r_2\theta_{2,x} - Ra_c\theta_{2,X} + \delta v_{2,z} - 4\nabla^2\psi_{2,Xx},\\
f_{42} &= -\psi_{2,X} - 2\theta_{2,Xx} - \tilde{J}\left(\theta_1,\psi_1\right) - J\left(\theta_3,\psi_1\right) - J\left(\theta_2,\psi_2\right) - J\left(\theta_1,\psi_3\right),\\
f_{43} &= -\sigma^{-1}\Big[v_{0,X}\psi_{2,z} + \tilde{J}\left(v_1,\psi_1\right) + J\left(v_1,\psi_3\right) + J\left(v_2,\psi_2\right) + J\left(v_3,\psi_1\right)\Big]\\
&\qquad - 2v_{2,Xx} - v_{0,XX} - \delta\psi_{2,z}.
\end{aligned}$$

The solvability condition for ψ_4 is always satisfied at this order. However, the Laplace operator appearing in the equation for v_4 has a nonempty kernel spanned by multiples of $v_4 = 1$, a fact that is responsible for the presence of a second solvability condition, viz.,

$$v_{0,XX} + \frac{T_c\pi^2}{4p\sigma}(|a|^2)_X = 0. \tag{12}$$

From Eqs. (11) and (12) we obtain the condition $\xi^2 = 1/3$ that defines the critical Taylor number T_c:

$$T_c = T_{\pm}^{mod} \equiv \frac{\sigma\pi^2(2\pm\sqrt{1-\sigma^2})}{\left(1\pm\sqrt{1-\sigma^2}\right)^2}. \tag{13}$$

Note that two such critical values of T are present.

Now that all solvability conditions have been imposed we can proceed to solve the $O(\epsilon^4)$ problem. Here we only list the terms which enter into the computation of the solvability conditions arising at $O(\epsilon^5)$ and $O(\epsilon^6)$. These are

$$\begin{aligned}
\psi_4 &= -\frac{k^2\pi(2T_c^2\pi^2+4p^2\pi^2+3\sigma p^3)(|a|^2)_X}{2p^2p_{02}\sigma}\sin(2\pi z)+...,\\
\theta_4 &= \frac{k^4(p_{13}-3p^2k^2)(k^2+5\pi^2)|a|^4}{32\pi p^2p_{13}(k^2+9\pi^2)}\sin(2\pi z)+\left\{\frac{ik\pi a^* a_X}{8p^2}\sin(2\pi z)+c.c.\right\}+...,\\
v_4 &= -\frac{k^4(4T_c^2\pi^2+8p^2\pi^2+6\sigma p^3+p_{02})(|a|^2)_X}{8pp_{02}\pi^2}\cos(2\pi z)\\
&\quad+\left\{-\frac{ik\pi^2(T_c^2\pi^2+p_{31})(5k^2+\pi^2)|a|^2a^2}{64pp_{31}\sigma^2(9k^2+\pi^2)}e^{2ikx}+\frac{i\pi^2(v_{0,X}+\delta\sigma)a^2}{16kp\sigma^2}e^{2ikx}\right.\\
&\quad\left.-\frac{(3k^2+\pi^2)aa_X}{16p}e^{2ikx}+c.c.\right\}+....
\end{aligned}$$

Fifth & sixth order

At $O(\epsilon^5)$ and $O(\epsilon^6)$, $\mathbf{f}_5$ and f_{63} are given by

$$\begin{aligned}
f_{51} &= \sigma^{-1}\left[\sum_{n=1}^{2}\Big(2J(\psi_n,\psi_{3-n,Xx})+\tilde{J}(\psi_n,\nabla^2\psi_{3-n})\Big)+\sum_{n=1}^{4}J(\psi_n,\nabla^2\psi_{5-n})\right]-4\psi_{1,XXxx}\\
&\quad-2\nabla^2\psi_{1,XX}-4\nabla^2\psi_{3,Xx}-Ra_c\theta_{3,X}-r_4\theta_{1,x}+\delta v_{3,z}-r_2\theta_{1,X}-r_2\theta_{3,x},\\
f_{52} &= -\psi_{3,X}-\theta_{1,XX}-2\theta_{3,Xx}-\sum_{n=1}^{4}J(\theta_n,\psi_{5-n})-\sum_{n=1}^{2}\tilde{J}(\theta_n,\psi_{3-n}),\\
f_{53} &= -\sigma^{-1}\left[\sum_{n=1}^{4}J(v_n,\psi_{5-n})+\sum_{n=1}^{2}\tilde{J}(v_n,\psi_{3-n})+v_{0,X}\psi_{3,z}\right]-v_{1,XX}-2v_{3,Xx}-\delta\psi_{3,z},\\
f_{63} &= -\sigma^{-1}\left[\sum_{n=1}^{3}\tilde{J}(v_n,\psi_{4-n})+\sum_{n=1}^{5}J(v_n,\psi_{6-n})+v_{0,X}\psi_{4,z}\right]-v_{2,XX}-2v_{4,Xx}-\delta\psi_{4,z}.
\end{aligned}$$

The solvability condition for ψ_5 yields

$$\begin{aligned}
&(\tilde{\mu}_0+\tilde{\mu}_1v_{0,X}+i\tilde{\mu}_2v_{0,XX})\,a+\tilde{d}a_{XX}+i\left[(\tilde{\gamma}_0+\tilde{\gamma}_1v_{0,X})\,a_X+\tilde{a}_{10}|a|^2a_X+\tilde{a}_{20}a^2\bar{a}_X\right]\\
&+\Big(\tilde{b}_0+\tilde{b}_1v_{0,X}\Big)|a|^2a-\tilde{c}_0|a|^4a-\frac{pk^2}{2}av_{20,X}=0 \qquad (14)
\end{aligned}$$

with coefficients given by

$$\begin{aligned}
\tilde{\mu}_0 &= \frac{k^2r_4-\delta^2\pi^2}{2},\quad \tilde{\mu}_1=-\frac{\delta\pi^2}{2\sigma},\quad \tilde{d}=6k^2p,\quad \tilde{\gamma}_0=-\frac{k\pi^2(r_2+2T_c\delta)}{p},\\
\tilde{\mu}_2 &= \tilde{\gamma}_1=-k^3,\quad \tilde{b}_0=\frac{2p\delta\pi^2-k^2r_2\sigma}{16p\sigma}k^2,\quad \tilde{b}_1=\frac{k^2\pi^2}{16\sigma^2},\\
\tilde{a}_{10} &= \frac{k^3p(4k^2+2\pi^2+3p\sigma)(2k^4-9\pi^4+7k^2\pi^2+6k^2\pi^2\sigma)}{8p_{02}\sigma^2}+\frac{9k^3(3\pi^2+k^2)}{32}+\frac{k^7}{16\pi^2},\\
\tilde{a}_{20} &= \frac{k^3p(4k^2+2\pi^2+3p\sigma)(2k^4-9\pi^4+7k^2\pi^2+6k^2\pi^2\sigma)}{8p_{02}\sigma^2}+\frac{k^3(3\pi^2+k^2)}{32}+\frac{k^7}{16\pi^2},\\
\tilde{c}_0 &= \frac{p^2k^8(19k^2+3\pi^2)}{128p_{31}(9k^2+\pi^2)}+\frac{k^4\pi^2(5k^2+\pi^2)}{64\sigma^2(9k^2+\pi^2)}+\frac{9k^8p^2}{128p_{13}}-\frac{3k^6(k^2+5\pi^2)(p_{13}-3k^2p^2)}{64p_{13}(k^2+9\pi^2)}.
\end{aligned}$$

The solvability condition for v_6 yields

$$v_{20,XX} + \left(\frac{k^3}{2p} \mathrm{Im}[a a_X^*] + \frac{\pi^2 |a|^2 v_{0,X}}{4p\sigma^2} + \frac{\delta \pi^2 |a|^2}{4p\sigma} - \frac{k^2 \pi^2 |a|^4}{32 p \sigma^2} \right)_X = 0. \qquad (15)$$

Equation (1) follows from (14) on integrating conditions (12) and (15) with respect to X.

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